A BANACH SPACE WHICH IS NOT EQUIVALENT TO AN ADJOINT SPACE

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A BANACH SPACE which is not reflexive may or may not be equivalent (in Banach's sense) to an adjoint space. For example, it is an elementary fact that the space (l), though not reflexive, is equivalent to $(c_0)^*$, where (c_0) is the space of all sequences that converge to zero, normed in the usual way. On the other hand, (c_0) itself is not equivalent to any adjoint space : this can be proved by means of the Krein-Milman theorem, but here we obtain the result by an elementary argument which is scarcely more complicated than the standard proof that (c_0) is not reflexive.

Any bounded linear functional z on (c_0) corresponds to a sequence $\{\zeta_n\}$, with $\sum_n |\zeta_n| = \| z \|$, such that

$$z(y) = \sum_n \zeta_n \eta_n, \quad \text{all } y \in (c_0),$$

where $y = \{\eta_n\}$ (and $\| y \| = \sup_n |\eta_n|$). Thus if (c_0) were equivalent, under a correspondence $y \sim \hat{y}$, to some adjoint space X^*, we should have an equivalence, $x \sim \{\xi_n\}$, defined by

$$\hat{y}(x) = \sum_n \xi_n \eta_n, \quad \text{all } y \in (c_0),$$

between X and a subspace, $\hat{X}$, of (l). Assuming this to be the case, let $y_k = \{\delta_n^k\}$ for $k = 1, 2, \ldots$, so that, for any $x \in X$, $\hat{y}_k(x) = \xi_k$. If $\epsilon > 0$ then, for each k, there is an $x \in X$ such that $\| x \| = 1$ and $|\hat{y}_k(x)| > \| \hat{y}_k \| - \epsilon$ (because $\| \hat{y}_k \| = \sup_{\| x \| = 1} |\hat{y}_k(x)|$) ; and we may suppose x chosen so that $\hat{y}_k(x)$ is real and positive. Now $\| x \| = \sum_n |\xi_n|$ and $\| \hat{y}_k \| = \| y_k \| = 1$; thus $\sum_n |\xi_n| = 1$ and $\xi_k > 1 - \epsilon$, so that

$$\| \{ \xi_n \} - \{\delta_n^k\} \| = 1 - |\xi_k| + |\xi_k - 1| < 2\epsilon.$$

Since the sequences y_k generate (l), it follows that $\hat{X}$ is dense in (l). Hence X^* is equivalent to $(l)^*$, that is, to the Banach space (m) of all bounded sequences. But (c_0), being separable, is not equivalent to (m) : it is therefore not equivalent to any adjoint space.

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