BULL. AUSTRAL. MATH. SOC. 28A10, 28A05
VOL. 31 (1985), 325-328.

WILANSKY'S QUERY ON OUTER MEASURES

CHOO-WHAN KIM

On a set X, let μ^* be an outer measure and μ the measure induced by μ^*. We show that if X is a finite set, then the measure μ is saturated. We give two examples of non-regular outer measures on an infinite set X which induce non-saturated and saturated measures, respectively. These answer a query posed by Wilansky.

1. Introduction and preliminaries

Let X be an arbitrary non-empty set and $P(X)$ its power set. On the set X, let μ^* be an outer measure, μ the measure induced by μ^*, μ^+ the outer measure induced by μ, and $\bar{\mu}$ the measure induced by μ^+. Recently Wilansky posed the following query [3]: must every μ^+-measurable set be μ^*-measurable?

Let M and M^+ be the σ-algebras of μ^*-measurable and μ^+-measurable sets, respectively. It is plain that $M \subset M^+$, $\mu^* \leq \mu^+$ on $P(X)$, and $\mu^* = \mu^+$ on M.

Let $(X, \mathcal{B}, \lambda)$ be any measure space. Following Royden [2] we shall say that a subset E of X is locally measurable (with respect to $\mathcal{B}$ and λ), if $E \cap B \in \mathcal{B}$ for each $B \in \mathcal{B}$ with $\lambda(B) < \infty$. Then the family $\mathcal{B}^\wedge$ of all locally measurable sets is a σ-algebra containing $\mathcal{B}$. The measure λ is called saturated (or a saturated measure on $\mathcal{B}$), if $\mathcal{B} = \mathcal{B}^\wedge$. If λ

Received 24 October 1984.

https://doi.org/10.1017/S0004972700009278 Published online by Cambridge University Press

is a σ-finite measure, then it is saturated. For any subset E of X, let $E' = X - E$.

LEMMA 1. *If* $E \in M^+$ *with* $\bar{\mu}(E) < \infty$, *then* $E \in M$.

Proof. Suppose $E \in M^+$ with $\bar{\mu}(E) < \infty$. Let $A \in M$ be such that $E \subset A$ and $\bar{\mu}(E) = \mu(A)$. Then $A - E \in M^+$ and $\bar{\mu}(A-E) = 0$, so that $\mu(A-E) = 0$. Since μ is complete, we have $A - E \in M$, so that $E \in M$. □

LEMMA 2. $M^\wedge = (M^+)^\wedge$.

Proof. Suppose $E \in (M^+)^\wedge$ and $B \in M$ with $\mu(B) < \infty$. Then we have $E \cap B \in M^+$ and $\bar{\mu}(E \cap B) \le \mu(B) < \infty$, so, by Lemma 1, $E \cap B \in M$. Thus $(M^+)^\wedge \subset M^\wedge$. Similarly we obtain $M^\wedge \subset (M^+)^\wedge$. □

PROPOSITION 1. $\bar{\mu}$ *is a saturated measure on* M^+, *and* $M^+ = M^\wedge$.

Proof. Since $M^+ \subset (M^+)^\wedge = M^\wedge$, it remains to show $M^\wedge \subset M^+$. Suppose $E \in M^\wedge$ and $A \subset X$ with $\mu^+(A) < \infty$. Let $B \in M$ be such that $A \subset B$ and $\mu^+(A) = \mu(B)$. Then both $E \cap B$ and $E' \cap B$ are in M, and

$$\mu^+(A) = \mu(B) = \mu(B \cap E) + \mu(B \cap E') \ge \mu^+(A \cap E) + \mu^+(A \cap E'),$$

so that $E \in M^+$. □

In view of Proposition 1, the query may be stated as follows: must the measure μ induced by an outer measure μ^* be saturated?

2. Results

We state without proof the following well-known result ([1], [2]).

PROPOSITION 2. *The following assertions are equivalent:*

(i) $\mu^* = \mu^+$ *on* $P(X)$;

(ii) *for each* $E \subset X$, *there is* $A \in M$ *such that* $E \subset A$ *and* $\mu^*(E) = \mu^*(A)$;

(iii) μ^* *is induced by a measure on an algebra.*

An outer measure μ^* is called regular, if any one of the assertions of Proposition 2 holds. By a minor modification of the proof of Proposition 1 we obtain:

THEOREM 1. *If an outer measure* μ^* *is regular, then the induced measure* μ *is saturated.*

THEOREM 2. *For each outer measure* μ^* *on a finite set* X *, the induced measure* μ *is saturated.*

Proof. It is enough to prove the theorem in the case in which μ^* is not regular, $\mu^*(X) = \infty$, and X has at least three points. Let $X = \{1, 2, \ldots, n\}$ $(n \geq 3)$, $Y = \{i \mid i \in X, \mu^*(i) < \infty\}$, and $Z = \{i \mid i \in X, \mu^*(i) = \infty\}$. By our assumption, Y must contain at least two points and Z is not empty. It follows at once that every subset of Z is μ^*-measurable. Since Y also is μ^*-measurable, we have

$$M = \{E \cup F \mid E \in Y \cap M, F \subset Z\} .$$

If $A \in M^\wedge = M^+$, then $A \cap Z \in M$ and $A \cap Y \in M^+$. Since $\bar{\mu}(A \cap Y) \leq \mu(Y) < \infty$, it follows from Lemma 1 that $A \cap Y \in M$. Thus $A \in M$. □

3. Examples

Here we give two examples of non-regular outer measures on an infinite set which induce non-saturated and saturated measures, respectively.

EXAMPLE 1. Let X be an infinite set. Define the outer measure μ^* by

$$\mu^*(A) = 1 - 2^{-n} \text{ , if } A \text{ contains } n \text{ points,}$$

$$\mu^*(A) = \infty \text{ , if } A \text{ is infinite.}$$

For each non-empty proper subset E of X , let $A = \{x, y\}$, where $x \in E$ and $y \in E'$. Since

$$\mu^*(A \cap E) + \mu^*(A \cap E') = \mu^*(x) + \mu^*(y) = 1 > \mu^*(A) = 3/4 ,$$

the set E is not in M . Thus $M = \{\emptyset, X\}$.

It follows readily that $\mu^+(\emptyset) = 0$, and $\mu^+(E) = \mu(X) = \infty$ for each $E \neq \emptyset$. That is, μ^+ is the "infinite" measure on $P(X)$. It is plain that $M^+ = P(X)$ so that the induced measure μ is not saturated.

EXAMPLE 2. Let X be an infinite set, $Y = \{a, b\}$, where a and b are distinct points of X , and $Z = X - Y$. Define the outer measure μ^* by

$$\mu^*(\emptyset) = 0 \text{ , } \mu^*(a) = \mu^*(b) = 1 \text{ , } \mu^*(Y) = 1.5 \text{ , } \mu^*(A) = \infty \text{ , if } A \cap Z \neq \emptyset .$$

https://doi.org/10.1017/S0004972700009278 Published online by Cambridge University Press

Note that both $\{a\}$ and $\{b\}$ are not in M. It is straightforward to show that $Y \in M$, and $E \in M$ for all $E \subset Z$. Thus we obtain

$$M = \{\emptyset, Y, E, Y \cup E \mid \emptyset \neq E \subset Z\} .$$

Since $\mu^+(a) = \mu^+(b) = \mu(Y) = 1.5$, we have

$$\mu^+(Y \cap \{a\}) + \mu^+(Y \cap \{b\}) = 2\mu(Y) > \mu(Y) ,$$

so that both $\{a\}$ and $\{b\}$ are not in M^+. Thus $M = M^+$, and the measure μ is saturated.

References

[1] Paul R. Halmos, *Measure theory* (Van Nostrand, New York, 1954).

[2] H.L. Royden, *Real analysis*, 2nd ed. (Macmillan, New York, 1968).

[3] A. Wilansky, "Query 305", *Notices Amer. Math. Soc.* 31 (1984), 376.

Department of Mathematics and Statistics,
Simon Fraser University,
Burnaby,
British Columbia V5A 1S6,
Canada.