DUAL INTEGRAL EQUATIONS WITH TRIGONOMETRICAL KERNELS†

by IAN N. SNEDDON

(Received 26 August, 1961)

1. In the analysis of mixed boundary value problems in the plane, we encounter dual integral equations of the type

$$\left.\begin{aligned} \int_0^\infty \xi^{-1}\psi(\xi)\cos(x\xi)\,d\xi &= f(x) \quad (0 \leqq x \leqq 1), \\ \int_0^\infty \psi(\xi)\cos(x\xi)\,d\xi &= 0 \quad (x > 1). \end{aligned}\right\} \tag{1}$$

If we make the substitutions $\cos(x\xi) = (\frac{1}{2}\pi\xi x)^{\frac{1}{2}}J_{-\frac{1}{2}}(\xi x)$, $\psi(\xi) = (\frac{1}{2}\pi\xi)^{-\frac{1}{2}}\phi(\xi)$, $f(x) = x^{\frac{1}{2}}g(x)$, we obtain a pair of dual integral equations of the Titchmarsh type [**1**, p. 334] with $\alpha = -1$, $\nu = -\frac{1}{2}$ (in Titchmarsh's notation). This is a particular case which is not covered by Busbridge's general solution [**2**], so that special methods have to be derived for the solution.

A simple procedure for dealing with the difficulty when $f(x)$ is a simple polynomial was given by Chong [**3**]. Chong's method consists in using an entirely different method to solve the boundary value problem which corresponds to the case in which $f(x)$ is a constant and then to use the formal solution of the dual integral equations to construct the solution appropriate to the other terms of the polynomial.

More recently, Fredricks [**4**] has given a direct method of solving the equations (1) for the case in which the function $f(x)$ can be represented by the half-range cosine series

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(n\pi x) \quad (0 \leqq x \leqq 1). \tag{2}$$

Fredricks' solution can be written in the form

$$\psi(\xi) = \xi J_1(\xi) \sum_{n=0}^{\infty} a_n J_0(n\pi) + 4 \sum_{n=0}^{\infty} \sum_{q=1}^{\infty} q a_n J_{2q}(n\pi) J_{2q}(\xi). \tag{3}$$

The object of the present note is to show that, by making use of a procedure similar to that of Fredricks, it is possible to derive a simple solution of the pair (1).

2. As a special case of the Weber-Schaftheitlin discontinuous integral [**5**, pp. 398–404], we find that

$$\int_0^\infty J_1(\xi)\cos(x\xi)\,d\xi = 1 \quad (0 \leqq x \leqq 1), \qquad \int_0^\infty \xi J_1(\xi)\cos(x\xi)\,d\xi = 0 \quad (x > 1),$$

† The work described in this paper was done in the the Department of Mathematics, Duke University, North Carolina, and was supported by the U.S. Air Force Office of Scientific Research, A.R.D.C. under contract No. AF 18 (600)–1341. The present paper is a revised version of a part of report AFOSR–TN–60–51 issued by the Department of Mathematics, Duke University.

and that, if $q \geqq 1$,

$$\int_0^\infty \xi^{-1} J_{2q}(\xi) \cos(x\xi)\, d\xi = (2q)^{-1} \mathfrak{F}_q(0, \tfrac{1}{2}, x^2) \quad (0 \leqq x < 1),$$

$$\int_0^\infty J_{2q}(\xi) \cos(x\xi)\, d\xi = 0 \quad (x > 1),$$

where $\mathfrak{F}_q(0, \frac{1}{2}, u)$ is the Jacobi polynomial defined by the equations

$$\mathfrak{F}_q(0, \tfrac{1}{2}, u) = {}_2F_1(-q, q\,;\tfrac{1}{2}\,; u) = \frac{\Gamma(\frac{1}{2})}{\Gamma(q+\frac{1}{2})} u^{\frac{1}{2}}(1-u)^{\frac{1}{2}} D^q[u^{q-\frac{1}{2}}(1-u)^{q-\frac{1}{2}}]$$

$(D = d/du)$. Hence

$$\psi(\xi) = p_0 \xi J_1(\xi) + 2 \sum_{q=1}^{\infty} q p_q J_{2q}(\xi) \tag{4}$$

will be a solution of the pair of equations (1) provided that the constants p_q are chosen so that

$$f(x) = p_0 + \sum_{q=1}^{\infty} p_q \mathfrak{F}_q(0, \tfrac{1}{2}, x^2) \quad (0 \leqq x < 1). \tag{5}$$

Using the orthogonality relation

$$\int_0^1 u^{-\frac{1}{2}}(1-u)^{-\frac{1}{2}} \mathfrak{F}_q(0, \tfrac{1}{2}, u) \mathfrak{F}_{q'}(0, \tfrac{1}{2}, u)\, du = \begin{cases} \frac{1}{2}\pi \delta_{qq'}, & \text{if} \quad q \neq 0, \\ \pi \delta_{qq'}, & \text{if} \quad q = 0, \end{cases}$$

for the Jacobi polynomial, we find that

$$p_0 = \frac{1}{\pi} \int_0^1 u^{-\frac{1}{2}}(1-u)^{-\frac{1}{2}} f(u^{\frac{1}{2}})\, du = \frac{2}{\pi} \int_0^1 \frac{f(x)\, dx}{\sqrt{(1-x^2)}}, \tag{6}$$

and that, when $q \geqq 1$,

$$\begin{aligned} p_q &= \frac{2}{\Gamma(\frac{1}{2})\Gamma(q+\frac{1}{2})} \int_0^1 f(u^{\frac{1}{2}}) D^q[u^{q-\frac{1}{2}}(1-u)^{q-\frac{1}{2}}]\, du \\ &= \frac{2(-1)^q}{\Gamma(\frac{1}{2})\Gamma(q+\frac{1}{2})} \int_0^1 u^{q-\frac{1}{2}}(1-u)^{q-\frac{1}{2}}[D^q f(u^{\frac{1}{2}})]\, du. \end{aligned} \tag{7}$$

Since the integral

$$\int_0^\infty \xi J_1(\xi) \cos \xi\, d\xi$$

is divergent and the integral

$$\int_0^\infty J_{2q}(\xi) \cos \xi\, d\xi \quad (q \geqq 1),$$

is convergent, we see from equation (4) that if we impose the additional requirement on our solution $\psi(\xi)$ that

$$G(x) = \int_0^\infty \psi(\xi)\cos(\xi x)\,d\xi \tag{8}$$

must remain finite as $x \to 1$, then the function $f(x)$ must be such that $p_0 = 0$, and we see from equation (6) that this is equivalent to the condition

$$\int_0^1 \frac{f(x)\,dx}{\sqrt{(1-x^2)}} = 0. \tag{9}$$

Especially in the case in which $f(x)$ is a polynomial of low degree in x, these formulae are much more manageable than those derived by Fredricks.

In physical applications it is often desirable to know the form of the function

$$F(x) = \int_0^\infty \xi^{-1}\psi(\xi)\cos(\xi x)\,d\xi$$

when $x > 1$. If $x > 1$,

$$\int_0^\infty \xi^{-1} J_{2q}(\xi)\cos(x\xi)\,d\xi = (-r)^q/(2q),$$

where

$$r = [x+\sqrt{(x^2-1)}]^{-2}. \tag{10}$$

If, therefore, we substitute the value (5) for the function $\psi(\xi)$, we find that, in the case in which $f(x)$ satisfies the condition (9),

$$F(x) = \sum_{q=1}^\infty (-r)^q p_q \quad (x > 1), \tag{11}$$

where r is defined by equation (10).

Similarly, using the result

$$\int_0^\infty J_{2q}(\xi)\cos(\xi x)\,d\xi = (1-x^2)^{-\frac{1}{2}}\mathfrak{F}_q(0, \tfrac{1}{2}, x^2) \quad (0 < x < 1),$$

we find that, when $f(x)$ satisfies the condition (9),

$$G(x) = \frac{2}{\sqrt{(1-x^2)}}\sum_{q=1}^\infty q p_q \mathfrak{F}_q(0, \tfrac{1}{2}, x^2) \quad (0 < x < 1). \tag{12}$$

3. Chong [3] considered the case in which $f(x)$ is the polynomial

$$f(x) = \sum_{r=0}^m c_r x^r. \tag{13}$$

If we substitute this expression in equation (9) we obtain the condition

$$\sum_{r=0}^m \frac{\Gamma(\frac{1}{2}r+\frac{1}{2})}{\Gamma(\frac{1}{2}r+1)} c_r = 0, \tag{14}$$

https://doi.org/10.1017/S2040618500034481 Published online by Cambridge University Press

which must be satisfied if $G(x)$ is to remain finite as $x \to 1$. This is the criterion derived otherwise by Chong.

If $f(x)$ is a polynomial containing only even powers of x, the results are much simpler. Suppose, for example, that

$$f(x) = \sum_{r=0}^{n} c_{2r} x^{2r}; \tag{15}$$

then it is easily shown that

$$D^q f(u^{\frac{1}{2}}) = \begin{cases} \sum_{s=0}^{n-q} \frac{(q+s)!}{s!} c_{2q+2s} u^s & (q \leqq n), \\ 0 & (q > n), \end{cases}$$

and hence from equation (7) that $p_q = 0$ if $q > n$ and that

$$p_q = \frac{2(-1)^q}{\sqrt{\pi}} \sum_{s=0}^{n-q} \frac{(q+s)!\,\Gamma(q+s+\frac{1}{2})}{s!\,\Gamma(2q+s+1)} c_{2q+2s} \quad (q \leqq n). \tag{16}$$

4. We can easily derive Fredricks' results from our solution. If $f(x) = \cos(n\pi x)$, then it is easily shown that

$$D^q f(u^{\frac{1}{2}}) = \Gamma(\tfrac{1}{2}) \sum_{s=0} \frac{(-\frac{1}{4}\pi^2 n^2)^{s+q} u^s}{s!\,\Gamma(s+q+\frac{1}{2})},$$

and it follows from equation (7) that

$$p_q = 2(\tfrac{1}{2}\pi n)^{2q} \sum_{s=0}^{\infty} \frac{(-\frac{1}{4}\pi^2 n^2)^s}{s!(s+2q)!} = 2J_{2q}(n\pi).$$

Also

$$p_0 = \frac{2}{\pi} \int_0^1 \frac{\cos(n\pi v)\, dv}{\sqrt{(1-v^2)}} = J_0(n\pi).$$

Hence the solution corresponding to $f(x) = \cos(n\pi x)$ is

$$\psi(\xi) = \xi J_1(\xi) J_0(n\pi) + 4 \sum_{q=1}^{\infty} q J_{2q}(n\pi) J_{2q}(\xi). \tag{17}$$

Fredricks' solution (3) is then obtained by a simple summation process.

If $G(x)$ is to remain finite when $x \to 1$, then the coefficients a_n must satisfy the linear relation

$$a_0 + \sum_{n=1}^{\infty} a_n J_0(n\pi) = 0. \tag{18}$$

5. In a similar way it can be shown that, if $f(0) = 0$, the solution of the dual integral equations

$$\left.\begin{aligned} \int_0^\infty \xi^{-1}\psi(\xi)\sin(x\xi)\,d\xi &= f(x) \quad (0 \leqq x \leqq 1), \\ \int_0^\infty \psi(\xi)\sin(x\xi)\,d\xi &= 0 \qquad\quad (x > 1), \end{aligned}\right\} \tag{19}$$

can be expressed in the form

$$\psi(\xi) = \sum_{q=0}^{\infty} (2q+1)p_q J_{2q+1}(\xi), \tag{20}$$

where, because of the relations

$$\int_0^\infty \xi^{-1} J_{2q+1}(\xi)\sin(x\xi)\,d\xi = x\mathfrak{F}_q(1, \tfrac{3}{2}, x^2) \quad (0 \leqq x \leqq 1),$$

$$\int_0^\infty J_{2q+1}(\xi)\sin(x\xi)\,d\xi = 0 \quad (x > 1),$$

the coefficients p_q must be chosen so that

$$x^{-1}f(x) = \sum_{q=0}^{\infty} (2q+1)p_q\mathfrak{F}_q(1, \tfrac{3}{2}, x^2).$$

Using the orthogonality relation for the Jacobi polynomials, we obtain the equation

$$\begin{aligned} p_q &= \frac{2}{\Gamma(\frac{1}{2})\Gamma(q+\frac{1}{2})}\int_0^1 u^{-\frac{1}{2}}f(u^{-\frac{1}{2}})D^q[u^{q+\frac{1}{2}}(1-u)^{q-\frac{1}{2}}]\,du \\ &= \frac{2(-1)^q}{\Gamma(\frac{1}{2})\Gamma(q+\frac{1}{2})}\int_0^1 u^{q+\frac{1}{2}}(1-u)^{q-\frac{1}{2}}D^q[u^{-\frac{1}{2}}f(u^{\frac{1}{2}})]\,du, \end{aligned} \tag{21}$$

by means of which the coefficients of the solution (20) can be found.

For instance, if $f(x) = \sin(n\pi x)$,

$$D^q[u^{-\frac{1}{2}}f(u^{\frac{1}{2}})] = (-1)^q\Gamma(\tfrac{1}{2})(\tfrac{1}{2}n\pi)^{2q+1}\sum_{s=0}^{\infty}\frac{(-\frac{1}{4}n^2\pi^2)^s u^s}{s!\,\Gamma(q+\frac{3}{2}+s)},$$

and so

$$p_q = (\tfrac{1}{2}n\pi)^{2q+1}\sum_{s=0}^{\infty}\frac{(-\frac{1}{4}n^2\pi^2)^s}{s!(2q+1+s)!} = J_{2q+1}(n\pi),$$

giving

$$\psi(\xi) = 2\sum_{q=0}^{\infty}(2q+1)J_{2q+1}(n\pi).$$

https://doi.org/10.1017/S2040618500034481 Published online by Cambridge University Press

Hence, if $f(x)$ is given by the half-range sine series

$$f(x) = \sum_{n=0}^{\infty} b_n \sin(n\pi x) \quad (0 \leqq x \leqq 1),$$

the solution of the pair of dual integral equations (19) is

$$\psi(\xi) = 2 \sum_{n=0}^{\infty} \sum_{q=0}^{\infty} (2q+1) b_n J_{2q+1}(n\pi) J_{2q+1}(\xi), \tag{22}$$

in agreement with Fredricks' result.

REFERENCES

1. E. C. Titchmarsh, *Introduction to the theory of Fourier integrals* (Clarendon Press, Oxford, 1937).

2. I. W. Busbridge, Dual integral equations, *Proc. London Math. Soc.* **44** (1938), 115–129.

3. F. Chong, Solution by dual integral equations of a plane strain Boussinesq problem for an orthotropic medium, *Iowa State Coll. Jour. of Sci.* **27** (1953), 321–334.

4. R. W. Fredricks, Solution of a pair of integral equations from electrostatics, *Proc. Nat. Acad. Sci.* **44** (1958), 309–312.

5. G. N. Watson, *A treatise on the theory of Bessel functions*, 2nd edn, (Cambridge Univ. Press, 1944).

THE UNIVERSITY
GLASGOW