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THE DERIVED LENGTH OF A SOLUBLE SUBGROUP OF A FINITE-DIMENSIONAL DIVISION ALGEBRA

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Abstract. We determine for all d and p the maximal derived length of a soluble subgroup of the multiplicative group of a division ring of finite degree d and characteristic $p \geq 0$ to within one.

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To state our conclusions precisely we need to introduce some notation. The derived length of a soluble group G we denote by $dl(G)$. For any positive integer n let $dl(n)$ denote the maximal derived length of a soluble group of order n. For $n > 1$ set

$$cl(n) = \max\{dl(c) : c \mid n \text{ and } c \neq n\}$$

and put $cl(1) = -1$. If c divides n then $dl(c) \leq dl(n)$, so $cl(n) \leq dl(n)$, even if $n = 1$. Let G be a soluble group of order $n > 1$ and derived length $dl(n)$. Set $c = |G'|$. Then $c \mid n$, $c \neq n$ and

$$dl(c) \geq dl(G') = dl(G) - 1 = dl(n) - 1.$$

Hence $cl(n) \geq dl(n) - 1$ for $n > 1$. Trivially this holds if $n = 1$. Thus the following is true.

LEMMA 1. *For all positive integers n we have* $cl(n) = dl(n) - 1$ *or* $cl(n) = dl(n)$.

Throughout D will denote a central division F-algebra of finite degree d and characteristic $p \geq 0$ and G will be a soluble subgroup of $D^* = D\backslash\{0\}$. Let $Dl(d, p)$ denote the maximal derived length of a soluble subgroup G of D^* over all possible choices of D and G, but with fixed d and p. Our aim is to obtain good bounds for $Dl(d, p)$ for all d and p. The papers [**4**] and [**7**] give good bounds for the index of some abelian normal subgroup of such a group G. These give reasonable bounds for the derived length of G, but not the best possible. In fact they are about double what is possible. Alternatively we can regard such a group G as a linear group of degree d over a maximal subfield of D with trivial unipotent radical. This leads via [**3**] to the bound $Dl(d, p) < 3.4 + 5(\log_9 d)$. The following is the main result of this paper.

THEOREM. *If d is even with* $cl(d) \leq 1$, *then* $Dl(d, 0) = 4$. *In all other cases*

$$1 + dl(d) \leq Dl(d, p) \leq 2 + cl(d).$$

The even integers d with $cl(d) \leq 1$ are precisely 2, $2q$ *for any prime q, and* 8. *Also, if d is even with $cl(d) = 2$, then $Dl(d, 0) = 4$.*

The anomalous cases with d even and $p = 0$ are caused by the existence of the quaternions. Apart from these cases, where we know the answer is 4, we have pinned $Dl(d, p)$ to within 1; namely such a $Dl(d, p)$ is either $1 + dl(d)$ or $2 + dl(d)$. Determination of the function $dl(n)$ is finite soluble group theory. Both the possibilities for $Dl(d, p)$ above occur, and occur infinitely often. It is difficult to formulate a general rule that determines when each case occurs as it seems to involve as much number theory as group theory.

There are actually three possible situations.

Type (a). $1 + dl(d) = Dl(d, p) = 2 + cl(d) < 2 + dl(d)$.

Type (b). $1 + dl(d) = Dl(d, p) < 2 + cl(d) = 2 + dl(d)$.

Type (c). $1 + dl(d) < Dl(d, p) = 2 + cl(d) = 2 + dl(d)$.

Each of (a), (b) and (c) in each characteristic occur infinitely often. Firstly we have the following.

Proposition 1. *If $d = q^m$ is a power of a prime q, then $Dl(d, p) = 1 + dl(d)$, except that $Dl(d, 0) = 4$ for $q = 2$ and $1 \leq m \leq 6$.*

Now fix the prime q. The integer sequence $\{dl(q^m)\}$ is monotonically increasing to infinity, goes up by at most 1 at each step and goes up much slower than m (in fact something like $\log_2 m$). Thus there is a strictly increasing sequence

$$1 = m_0 < m_1 < \cdots < m_i < \cdots.$$

of positive integers such that $dl(q^m) = dl(q^{m_i})$ for $m_i \leq m < m_{i+1}$ and $dl(q^{m_i}) + 1 = dl(q^{m_{i+1}})$. Thus $cl(q^m) = dl(q^m) - 1$ if $m = m_i$ for some i and $cl(q^m) = dl(q^m)$ otherwise. In view of Proposition 1, for each p, this gives infinitely many examples of type (a) above and infinitely many of type (b). It also suggests that pinning down $Dl(d, p)$ further is likely to involve much number theory. Here are some more examples that further illustrate this last point. Note also that (c) of Proposition 2, for each p, gives infinitely many examples of type (c) above.

Proposition 2.

(a) *If $d = 2q$ with q an odd prime, then $cl(d) = 1 < 2 = dl(d)$ and $Dl(d, p) = 3 = 1 + dl(d) = 2 + cl(d)$ if $p > 0$ and is* 4 *otherwise.*

(b) *If $d = 3q$ with $q \equiv 1 \bmod 3$ and a prime, then $cl(d) = 1 < 2 = dl(d)$ and $Dl(d, p) = 3 = 1 + dl(d) = 2 + cl(d)$.*

(c) *If $d = 3q$ with $q \equiv 2 \bmod 3$ and an odd prime not equal to p, then $cl(d) = 1 = dl(d)$ and $Dl(d, p) = 3 = 2 + cl(d) = 2 + dl(d)$.*

Let T be a periodic soluble subgroup of D^*. The possible structures for T are given by **[5]** 2.1.1 if $p > 0$ and by **[5]** 2.5.9 if $p = 0$. In the latter case there are seven possibilities (the eighth, namely (c) of **[5]** 2.5.9, being insoluble). From this it is easy to derive the following result.

Lemma 2. *Let T be a periodic soluble subgroup of D^*.*

(a) *If $p > 0$, then $Aut T$ is abelian.*

(b) *If T satisfies* (ai), (aii), (aiii) *or* (bii) *of* **[5]** 2.5.9*, then $Aut T$ is metabelian.*

(c) *If T satisfies* (bi), (biii) *or* (biv) *of* **[5]** 2.5.9*, then $Aut T$ is soluble of derived length exactly* 3.

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LEMMA 3. *Let $G \leq D^*$ be soluble. Then $dl(G) \leq 2 + cl(d)$, unless $p = 0$ and d is even, when $dl(G) \leq max\{4, 2 + cl(d)\}$.*

Proof. If $d = 1$ the result is trivial, so assume otherwise. It is easy to see that we may also assume that $D = F[G]$, the F-subalgebra of D generated by G, as this change replaces d by some divisor of d. Let A be a maximal abelian normal subgroup of G. Then the index $(G : A)$ is finite (just treat G as a soluble linear group over F with trivial unipotent radical acting on the F-space D). Set $e = \dim_F F[A]$. Then e divides d. Also $C_{F[A]}(G) = F$. Set $H = C_G(A)$. By Galois theory $(G : H) = \dim_F F[A] = e$. We now break the remainder of the proof of Lemma 3 into five steps.

(a) If $e = d$, then $dl(G) \leq 1 + dl(d) \leq 2 + cl(d)$.

For in this case $F[A]$ is a maximal subfield of D and hence also of $F[H]$. But it is also central in $F[H]$. Thus $F[H] = F[A]$, H is abelian and $H = A$. Therefore $dl(G) \leq 1 + dl(d) \leq 2 + cl(d)$, the latter by Lemma 1.

(b) Let $e = 1$. Then either $dl(G) \leq 2 = 2 + dl(e) \leq 2 + cl(d)$, or $p = 0$, d is even, G has a normal quaternion subgroup of order 8 and $dl(G) \leq 4$.

Here $A \leq F^*$, so A is a central maximal abelian normal subgroup of G of finite index. By a theorem of Schur $T = G'$ is periodic. By **[5]** 2.1.1 and 2.5.9 there is a characteristic subgroup S of T with an ascending characteristic cyclic series such that T/S is isomorphic to $\langle 1 \rangle$, Sym(4), the Klein 4-group or Alt(4) and if $T \neq S$, then $p = 0$, d is even and T has a characteristic quaternion subgroup of order 8.

Clearly $S \leq A$, so if $T = S$, then $G'' = T' \leq A' = \langle 1 \rangle$ and $dl(G) \leq 2$. Suppose $T \neq S$. In all these remaining cases Aut$T$ is soluble of derived length 3 by Lemma 2. Hence $[T, T''] = [T, G'''] = \langle 1 \rangle$. But then T/S cannot be Sym(4). Consequently $G''' = T'' \leq S \leq A$ and so $dl(G) \leq 4$ as claimed.

From now on assume that $1 < e < d$. Set $K = G^{(dl(e))} \leq H \cap G'$, where $G^{(n)}$ denotes the n-th derived subgroup of G. Then $T = K'$ is periodic.

(c) If Aut$T$ is metabelian, or if $dl(\text{Aut}T) = 3$ and $dl(e) \geq 2$, then

$$dl(G) \leq 2 + dl(e) \leq 2 + cl(d).$$

In the first case $T = K' \leq G''$ and $[T, G''] = \langle 1 \rangle$. In the second case $K \leq G''$, $T \leq G'''$ and $[T, G'''] = \langle 1 \rangle$. In both cases T is abelian, K is metabelian and

$$dl(G) \leq 2 + dl(e) \leq 2 + cl(d).$$

(d) Suppose $dl(G) > 2 + dl(e)$. Then $p = 0$, $dl(\text{Aut}T) = 3$, $dl(e) = 1$, d is even, T has a characteristic quaternion subgroup of order 8 and $dl(G) = 4$.

By (c) and Lemma 2 (coupled with **[5]** 2.5.9) we have all of (d) except for the derived length of G. Since $dl(e) = 1$ we have $K = G'$ and $T = G''$. Since $dl(\text{Aut}T) = 3$ we have $[T, G'''] = \langle 1 \rangle$. Hence T' is abelian and so $dl(G) \leq 4$. By hypothesis $dl(G) > 2 + dl(e) = 3$. Therefore $dl(G) = 4$.

The following summarises the main conclusions above and Lemma 3 is immediate from it.

(e) Either $e = d$ and $dl(G) \leq 1 + dl(e)$,
or $e < d$ and $dl(G) \leq 2 + dl(e)$,
or $e < d, p = 0, d$ is even and $dl(G) \leq 4$. □

LEMMA 4. (a) $1 + dl(d) \leq Dl(d, p)$ *for all* d *and* p.
(b) *If* d *is even, then* $Dl(d, 0) \geq 4$.

Proof. (a) We need to construct a division ring D of degree d and characteristic p and a soluble subgroup G of D^* with $dl(G) = 1 + dl(d)$.

Choose a finite soluble group H of order d and derived length $dl(d)$. Take a free presentation L/R of H, where L is free of finite rank at least 2. Set $G = L/R'$ and $A = R/R'$. Then G is finitely generated and A is an abelian normal subgroup of G with $G/A \cong H$. Also G is torsion-free (G. Higman), see [**5**] 1.4.7 and $A = C_G(A)$, see Auslander & Lyndon [**1**].

Clearly G is polycyclic. Let P be any field of characteristic p. The group algebra PG is an Ore domain ([**5**] 1.4.8 or [**2**]); let D be its division ring of quotients. Denote the centre of D by F. Clearly $C_{F(A)}(G) = F$, so Galois theory yields

$$\dim_F F(A) = (G : C_G(A)) = (G : A) = d.$$

Also $D = F(A)[G]$, so $\dim_{F(A)} D \leq (G : A) = d$. Hence $\deg D \leq d$. But $F(A)$ is a subfield of D of dimension d over F, so $\deg D \geq d$. Therefore $\deg D = d$. Finally

$$dl(G) = 1 + dl(H) = 1 + dl(d).$$

This is presumably well known, but see Lemma 5 below.

(b) The quaternion algebra $(-1, -1/\mathbf{Q}(\sqrt{2}))$, where $\mathbf{Q}$ denotes the rationals, contains a copy of the binary octahedral group BO_{48} of order 48 and derived length 4. Here $d = 2$, of course. We can avoid the $\sqrt{2}$. The infinite soluble group

$$K = \langle i, j, i + j, -(1 + i + j + ij)/2\rangle$$

has derived length 4 and lies in the multiplicative group of $D_0 = (-1, -1/\mathbf{Q})$. Again $d = 2$. In particular $Dl(2, 0) \geq 4$.

Suppose $d = 2c$ for some $c \geq 2$. Let H be a cyclic group of order c. With this H let L, R, G and A be as in the proof of part (a). With $D_0 = (-1, -1/\mathbf{Q})$ again, the group ring D_0G is an Ore domain ([**2**]); let D denote its division ring of quotients and F the centre of D. Clearly the soluble group K of derived length 4 embeds into D^*. It remains to check that D has degree $d = 2c$, for if so we will have $Dl(d, 0) \geq 4$.

Set $C = C_{F(A)}(G)$. By Galois theory $\dim_C F(A) = (G : C_G(A)) = (G : A) = c$. Also $F(A)$ centralizes D_0, so $C = F$ and $R = F(A)[D_0]$ is a non-commutative division ring of dimension at most 4 over its centre. Thus its degree is 2 and $R = (-1, -1/F(A))$. Now $R[G]$ has finite dimension over F, so $D = R[G]$ and consequently $\dim_F D \leq 2^2c^2$. But $F(A, i) \leq R$ is a field of dimension $2c$ over F. Therefore $\dim_F D \geq (2c)^2$. Thus $\deg D = 2c = d$ and the proof of the lemma is complete. □

LEMMA 5. *Let* L/R *be a free presentation of the soluble group* H, *where* L *is free of rank at least* 2. *Then* $dl(L/R') = 1 + dl(H)$.

Proof. Let $n = dl(H)$. If $n = 0$ the claim is obvious, so assume $n \geq 1$. Clearly $dl(L/R') \leq 1 + n$. Suppose $dl(L/R') < 1 + n$ and set $K = L^{(n-1)}$. Then $K' \leq R'$, so by

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Theorem 3 of [**1**] we have $K \leq R$. But then $dl(H) < n$, a contradiction. Consequently $dl(L/R') = 1 + n$. □

The Proof of the Theorem. Putting together Lemmas 3 and 4 we have the following. If d is even, then

$$\max\{4, 1 + dl(d)\} \leq Dl(d, 0) \leq \max\{4, 2 + cl(d)\}.$$

In all other cases

$$1 + dl(d) \leq Dl(d, p) \leq 2 + cl(d).$$

Suppose d is even. If $cl(d) \geq 2$, then $1 + dl(d) \leq Dl(d, 0) \leq 2 + cl(d)$. If $cl(d) \leq 2$, then $Dl(d, 0) = 4$. It remains to determine those even d with $cl(d) < 2$.

If q is an odd prime there is a non-abelian, metabelian group of order $2q$. There is also such a group of order 8. Hence if either $2q$ or 8 is a proper divisor of d, then $cl(d) \geq 2$. Suppose d is even with $cl(d) < 2$. Then $d = 2$, 4, 8 or $2q$ for q some odd prime. Moreover in all these cases d is even with $cl(d) < 2$. The proof of the Theorem is complete. □

The Proof of Proposition 1. The existence of the bound is immediate from Theorem 2 of [**6**], except where $p = 0$, $q = 2$, $d > 1$ and G has normal subgroups $A \leq H = C_G(A)$ with $(G : H)$ dividing $d/2$ and $H/A \cong \mathrm{Sym}(4)$ or $\mathrm{Alt}(4)$. In this case set $C = C_G(H/A)$. Since $\mathrm{Aut}(H/A) \cong \mathrm{Sym}(4)$, so $dl(G/C) \leq 3$. Clearly $dl(G/H) \leq dl(d/2)$ and $H \cap C = A$. Therefore

$$dl(G/A) \leq \max\{3, dl(d/2)\} \text{ and } dl(G) \leq \max\{4, 1 + dl(d/2)\}.$$

The only powers d of 2 with $dl(d) \leq 2$, are 1, 2, 4, 8, 16, 32, and 64. The proposition follows. □

The Proof of Proposition 2.

(a) Clearly $cl(2q) = 1$. The dihedral group of order $2q$ shows that $dl(2q) \geq 2$. Thus $dl(2q) = 2$ and $Dl(2q, p) = 3$ if $p > 0$ and 4 otherwise by the Theorem.

(b) Clearly $cl(2q) = 1$ and $dl(3q) \leq 2$. Since 3 divides $(q - 1)$, there is a soluble group of order $3q$ and derived length 2. Thus $dl(3q) = 2$ and $Dl(3q, p) = 3$.

(c) Again $cl(3q) = 1$. Let G be soluble of order $3q$. Then G has a normal subgroup N of order 3 or q. But 3 does not divide $(q - 1)$ and q does not divide $(3 - 1)$. Therefore N is central, G is abelian and $dl(3q) = 1$. The group G of Example 10(i) of [**7**] has derived length 3. By the Theorem $Dl(3q, p) \leq 2 + 1$. Thus $Dl(3q, p) = 3$. □

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