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CONTRACTION PROPERTY OF THE OPERATOR OF INTEGRATION

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ABSTRACT. It is shown that the operator of integration $Fy(x)=\int_0^x y(t)\,dt$ defined on the space $C(-\infty, \infty)$ of all continuous real valued functions on $(-\infty, \infty)$ is a contraction relative to a certain family of seminorms generating the topology of uniform convergence on compacta. However, as a contrast to this it is proved that F is not contractive with respect to any metric on $C(-\infty, \infty)$ inducing the above topology on $C(-\infty, \infty)$.

1. **Introduction.** Let X be a metrizable topological space and $F:X\to X$ a continuous selfmapping of X into itself. We say F is a *topological contraction* if there is a suitable metric ρ on X inducing the topology of X and a constant $q\in(0, 1)$ such that $\rho(Fx, Fy)\le q\rho(x, y)$ for all $x, y\in X$.

Assume now X is a Fréchet linear topological space and $F: X\to X$ a linear operator on X satisfying the following condition:

There exists a sequence of seminorms $\{p_n \mid n\ge 1\}$ on X inducing the topology of X and a number $q\in(0, 1)$ such that $p_n(Fx)\le qp_n(x)$ for all $x\in X$ and all $n=1, 2, \ldots$ It is natural to call such a linear operator F a *generalized contraction* on X. In [1] has been investigated a more general case where X is a completely regular not necessarily metrizable topological space and $F: X\to X$ a contraction with respect to a suitable family of pseudometrics inducing the topology of X.

Since the Fréchet space X is a metrizable topological space a question arises whether a generalized contraction on X is also a topological contraction in the sense of the first definition. The main purpose of this note is to show that the answer is "no", exhibiting at the same time a contraction property of the operator of integration $y(x)\to\int_0^x y(t)\,dt$ in the Fréchet space $C(-\infty, \infty)$. We prove the following.

THEOREM. *Let $C=C(-\infty, \infty)$ denote the linear space of all continuous real valued functions on $(-\infty, \infty)$ endowed with the topology of uniform convergence on compacta, and let $F: C\to C$ be defined by $Fy(x)=\int_0^x y(t)\,dt$ for $y\in C$. Then the operator F is a generalized contraction on C but it is not a topological contraction on C.*

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2. **Proof of the theorem.**

LEMMA. *Let X be a metrizable topological space and $F:X\to X$ a self-mapping on X such that the following conditions are satisfied:*

(i) *there is a fixed point $x_0 \in X$ of F*, i.e., $F(x_0)=x_0$

(ii) *there is a metric ρ on X inducing the topology of X relative to which F is a contraction*, i.e., *there exists a constant $q \in (0, 1)$ such that $\rho(Fx, Fy)\le q\rho(x, y)$ for all $x, y \in X$.*

Then there exists an open neighbourhood $U(x_0)$ of x_0 such that for any neighbourhood $V(x_0)$ of x_0 there is an integer $k_0\ge 1$ for which the following implication holds: $k\ge k_0 \Rightarrow F^k(U(x_0))\subset V(x_0)$, showing that the iterated images $F^k(U(x_0))$ of $U(x_0)$ under F shrink into any prescribed neighbourhood $V(x_0)$ of x_0 for sufficiently large values of k.

Proof. This is a standard argument.

We are now in the position to prove our theorem. First of all we observe that the topology of C can be induced by the sequence of seminorms defined by

$$\sup_{-n\le x\le n} |f(x)| \quad \text{for any } n = 1, 2, \ldots, \quad \text{and} \quad f\in C.$$

However, the operator F is not contractive with respect to this family. As was done by S. C. Chu and J. B. Diaz in [2] in a different setting, we achieve our end by an elementary modification of the seminorms. Indeed one finds easily that the equivalent family $\{p_n \mid n\ge 1\}$ of seminorms defined by

$$p_n(f) = \sup_{-n\le x\le n} e^{-2|x|}\,|f(x)|$$

for $f\in C$ and $n=1, 2, \ldots$ satisfies the relations

$$p_n(Fy) \le \tfrac{1}{2}p_n(y)$$

for all $n=1, 2, \ldots$ and $y \in C$, proving thus that F is a generalized contraction.

Suppose now that our operator $F:C\to C$ is a topological contraction. As the constant $0 \in C$ is the fixed point of F it follows that F would satisfy the conditions of our Lemma for some metric ρ inducing the topology of C. Let $U(0)$ be the neighbourhood of $\{0\}$ in C existing according to the Lemma and consider the fundamental system of neighbourhoods $\{U(n, a) \mid n\ge 1, a>0\}$ of $\{0\}$ defined by

$$U(n, a) = \{f\in C: p_n(f) < a\}.$$

It follows that there is some $n\ge 1$ and $a>0$ such that $U(n, a)\subset U(0)$ so that the neighbourhood $U(n, a)$ also would satisfy the conclusion of our Lemma. Choosing $V(0)$ to be $U(n+1, 1)$ we consider the function $y_n \in C$ defined by $y_n(x)=0$ for $x\le n$ and $y_n(x)=x-n$ for $x>n$. Then obviously $b\cdot y_n \in U(n, a)$ for any constant b but on the other hand for every $x>n$ and any $k\ge 1$ we have $F^k y_n(x)>0$. Thus for any k we can choose b_k in such a way that

$$b_k F^k y_n(n+1)\cdot e^{-2(n+1)} \ge 1$$

showing that the sets $F^k(U(n, a))$ do not shrink into the set $U(n+1, 1)$ as would follow from the Lemma and the contradiction thus obtained completes the proof of our theorem.

REMARK. If X is a metrizable topological space and $F:X \to X$ a continuous selfmapping then the sufficient and necessary conditions for F to be a topological contraction have been found by Ph. Meyers ([3]). It is an open problem to establish a similar characterization for generalized contractions dropping at the same time the hypothesis of metrizability of the space X. The question is:

Given a completely regular topological space X, how to characterize those continuous selfmappings $F:X \to X$ for which there exists a family $\{\rho_i \mid i \in I\}$ of pseudometrics ρ_i on X inducing the topology of X and a constant $q \in (0, 1)$ such that

$$\rho(_iFx, Fy) \leq q\rho_i(x, y)$$

for all $x, y \in X$ and all $i \in I$?

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