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On the Convergence of a Class of Nearly Alternating Series

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Abstract. Let C be the class of convex sequences of real numbers. The quadratic irrational numbers can be partitioned into two types as follows. If α is of the first type and $(c_k) \in C$, then $\sum(-1)^{\lfloor k\alpha \rfloor} c_k$ converges if and only if $c_k \log k \to 0$. If α is of the second type and $(c_k) \in C$, then $\sum(-1)^{\lfloor k\alpha \rfloor} c_k$ converges if and only if $\sum c_k/k$ converges. An example of a quadratic irrational of the first type is $\sqrt{2}$, and an example of the second type is $\sqrt{3}$. The analysis of this problem relies heavily on the representation of α as a simple continued fraction and on properties of the sequences of partial sums $S(n) = \sum_{k=1}^{n}(-1)^{\lfloor k\alpha \rfloor}$ and double partial sums $T(n) = \sum_{k=1}^{n} S(k)$.

1 Introduction

The goal of this paper is to provide necessary and sufficient conditions on the convex sequence $(c_k, k \geq 1)$ of real numbers for the convergence of the series

$$\sum(-1)^{\lfloor k\alpha \rfloor} c_k, \tag{1.1}$$

where α is a (real) quadratic irrational and $\lfloor x \rfloor$ denotes the largest integer not exceeding x. Series of type (1.1) for arbitrary irrational α are sometimes described as "almost alternating" or "nearly alternating" because the signs "balance out in the long run" in the sense that the ratio of the number of positive signs to the number of negative signs in the first n terms approaches unity. It will turn out that there are two classes of quadratic irrational numbers α, with the condition on the sequence (c_k) for convergence of (1.1) for the second class being more stringent than that for the first class. To which of the classes a given α belongs is determined by whether a certain functional of the periodic part of the continued fraction of α vanishes. The precise statement is Theorem 6.1.

It is instructive to look first at the same question for rational α. The analysis starts with a summation by parts:

$$\sum_{k=1}^{n}(-1)^{\lfloor k\alpha \rfloor} c_k = \sum_{k=1}^{n-1} S(k)\Delta c_k + S(n)c_n \tag{1.2}$$

where

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$$S(n) = \sum_{k=1}^{n} (-1)^{\lfloor k\alpha \rfloor} \quad \text{and} \quad \Delta c_k = c_k - c_{k+1}.$$

If $\alpha = p/q$ with $\gcd(p, q) = 1$, then the sequence $S(n)$ has period $2q$ if p is odd and is unbounded (with order of growth n) if p is even (the details are worked out in [10, Lemma 4]). It is then an easy exercise, using (1.2), to show that the rationals divide into two classes: if p is odd (respectively, even), then $S(n)$ is bounded (respectively, unbounded) and for a monotone sequence (c_k), (1.1) converges if and only if $c_k \to 0$ (respectively, $\sum c_k$ converges). We note that the classical alternating series theorem is the subcase p odd, $q = 1$.

For irrational α the sums $S(n)$ behave in a much less regular way, and we have to proceed to the second sums $T(n) = \sum_{k=1}^{n} S(k)$. This necessitates a second summation by parts:

$$(1.3) \qquad \sum_{k=1}^{n} (-1)^{\lfloor k\alpha \rfloor} c_k = \sum_{k=1}^{n-2} T(k) \Delta^2 c_k + T(n-1) \Delta c_{n-1} + S(n) c_n,$$

and the appearance of the second differences $\Delta^2 c_k$ in this formula suggests that the convexity of (c_k) will play the role that monotonicity played when α was rational. If we now assume that α is a quadratic irrational, we can exploit the periodicity of the continued fraction of α to show that $T(n)/n$ is either bounded or has order of growth $\log n$. We will then be able to show that for a convex sequence (c_k), (1.1) converges in the first case if and only if $c_k \log k \to 0$, and in the second case if and only if $\sum c_k/k$ converges. This result and the determination of which α belong to which of the two cases is the main result of this paper, as stated in Theorem 6.1.

The parallelism of the two situations is noteworthy. Periodicity of the base representation of α leads to enough regularity in $S(n)$ to get a nice theorem for rationals, and periodicity of the continued fraction representation of α leads to enough regularity in $T(n)$ to get a nice corresponding theorem for quadratic irrationals.

The convergence of (1.1) in the special case $\alpha = \sqrt{2}$, $c_k = 1/k$ was proposed as a problem to the American Mathematical Monthly by H. Ruderman [9] and solved by D. Borwein and others [1]. D. Borwein and W. Gawronski [2] then proved convergence for $\alpha = 1 - c + \sqrt{c^2 + 1}$ (c a positive integer) and $c_k = 1/k$, obtained good estimates for the sum, and investigated convergence under various summability methods. (Their convergence result is a special case of our Theorem 6.1. Example 7.1 elaborates on this.) P. Bundschuh [3] gave conditions on the sequence (c_k) for the convergence of (1.1) when the continued fraction of α has bounded partial quotients. Since he used bounds of $S(n)$ obtained from the theory of the discrepancy of sequences, he was able to give sufficient conditions only. More recently, series of the type (1.1) with $c_k = 1/k$ but with the signs chosen in a different way have been discussed by C. Feist and R. Naimi [4].

Section 2 of this paper is devoted to establishing notation and listing for reference the properties of continued fractions that are used in the sequel. In Sections 3 and 4 we develop the properties of the sequences $S(n)$ and $T(n)$. These results generalize those of [2] to all quadratic irrationals and are, we believe, of independent interest.

Section 5 contains some elementary lemmas on convex sequences in preparation for the theorem of Section 6. In Section 7 we provide some examples.

Since rational numbers have terminating continued fraction expansions, many of the statements that follow, and their proofs, would have to contain exceptions for rational α. In order to avoid this complication, for the rest of this paper α will always denote an irrational number.

2 Continued Fractions

In this section we collect the properties of continued fraction expansions that will be used in the remaining sections. Every irrational number α has an infinite continued fraction expansion $\alpha_0 + 1/(\alpha_1 + 1/(\alpha_2 + \dots))$, which is denoted by $[\alpha_0, \alpha_1, \alpha_2, \dots]$, and we write $\alpha = [\alpha_0, \alpha_1, \alpha_2, \dots]$. The integers α_i are the *partial quotients* and satisfy $\alpha_i \geq 1$ if $i \geq 1$. The sequence of partial quotients is periodic if and only if α is a quadratic irrational. (We shall follow the terminology of [7] in saying that a sequence (α_i) is *periodic* if there exists n such that $\alpha_{i+n} = \alpha_i$ for i sufficiently large, and *purely periodic* if $\alpha_{i+n} = \alpha_i$ for all i.) For $m \geq 0$, the m-th *convergent* is defined by $p_m/q_m = [\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_m]$ and $\gcd(p_m, q_m) = 1$. For the proofs of (2.1)–(2.6) below, see for example [6, 7]. A reference for (2.9) is [5].

$$p_{m+1} = \alpha_{m+1} p_m + p_{m-1}, \quad p_{-2} = 0, p_{-1} = 1, \tag{2.1}$$

$$q_{m+1} = \alpha_{m+1} q_m + q_{m-1}, \quad q_{-2} = 1, q_{-1} = 0.$$

$$p_{m+1} q_m - p_m q_{m+1} = (-1)^m, \quad m \geq -2. \tag{2.2}$$

$$\gcd(p_m, p_{m+1}) = \gcd(q_m, q_{m+1}) = 1, \quad m \geq -2. \tag{2.3}$$

$$\gcd(p_m, q_m) = 1, \quad m \geq -2. \tag{2.4}$$

$$\left| \alpha - \frac{p_m}{q_m} \right| < \frac{1}{q_m q_{m+1}}, \quad m \geq 0. \tag{2.5}$$

$$\frac{p_{2m}}{q_{2m}} < \alpha < \frac{p_{2m+1}}{q_{2m+1}}, \quad m \geq 0. \tag{2.6}$$

$$q_m \geq 2^{\lfloor m/2 \rfloor}, \quad m \geq 0. \tag{2.7}$$

For the proof of (2.7), use (2.1) to write $q_m \geq q_{m-1} + q_{m-2} \geq 2q_{m-2}$ and then use induction together with $q_0 = 1$ and $q_1 = \alpha_1 \geq 1$.

If $\alpha_i \leq K$ for all i, then

$$q_m \leq (K+1)^m, \quad m \geq 0. \tag{2.8}$$

For the proof of (2.8), use (2.1) to write $q_m \leq K q_{m-1} + q_{m-2} < (K+1) q_{m-1}$ and proceed by induction.

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For a given α, every integer $n \geq 0$ has a unique representation

$$n = \sum_{i=0}^{m} b_i q_i,$$
$$(2.9) \qquad b_m \neq 0,\ 0 \leq b_i \leq \alpha_{i+1} \text{ for } i \geq 1,\ 0 \leq b_0 < \alpha_1;$$
$$b_i = \alpha_{i+1} \Longrightarrow b_{i-1} = 0 \text{ for } i \geq 1,$$

with coefficients b_i determined by the following division algorithm:

$$\begin{aligned} n &= b_m q_m + n_m, \quad 0 \leq n_m < q_m \\ n_m &= b_{m-1} q_{m-1} + n_{m-1}, \quad 0 \leq n_{m-1} < q_{m-1} \\ &\vdots \\ n_2 &= b_1 q_1 + n_1, \quad 0 \leq n_1 < q_1 \\ n_1 &= b_0 q_0. \end{aligned}$$

3 The Sequence $S(n)$

In this section we develop the properties of the sequence $S(n) = \sum_{k=1}^{n} (-1)^{\lfloor k\alpha \rfloor}$ that will be needed in Section 6. Theorem 3.3 follows from a standard result of discrepancy theory, but we include a proof to keep this paper self-contained. However, the lower bounds of discrepancy theory do not imply Theorem 3.14. (In general, $\max\{|S(k)|, 1 \leq k \leq n\}$ can grow arbitrarily slowly; see [8, §1].)

The proof of the first lemma is a simple exercise in modular arithmetic.

Lemma 3.1 *Let p/q be a reduced rational and let k run through the integers $1, 2, \ldots, 2q$.*

(a) *If p is odd, then $kp/q \bmod 2$ assumes the values $0, 1/q, \ldots (q-1)/q, 1, (q+1)/q, \ldots, (2q-1)/q$, each once.*
(b) *If p is even, then $kp/q \bmod 2$ assumes the values $0, 2/q, \ldots (q-1)/q, (q+1)/q, \ldots, (2q-2)/q$, each twice.*

Lemma 3.2 *If p_m/q_m is a convergent of the continued fraction of α, then*

$$\Big| \sum_{k=1}^{2q_m} (-1)^{\lfloor (t+k)\alpha \rfloor} \Big| \leq 6$$

for any real number t.

Proof From (2.5) we can write $\alpha - p_m/q_m = \theta/q_m q_{m+1}$, $|\theta| < 1$, from which we get $(t+k)\alpha = t\alpha + kp_m/q_m + k\theta/q_m q_{m+1}$ with $|k\theta/q_m q_{m+1}| \leq 2/q_{m+1} < 2/q_m$ for $1 \leq k \leq q_m$. Let M denote the multiset $\{(t\alpha + kp_m/q_m) \bmod 2,\ 1 \leq k \leq 2q_m\}$.

If p_m is odd, then from Lemma 3.1(a) M consists of $2q_m$ distinct values with equal spacing $1/q_m$, and each of the intervals $[0,1)$, $[1,2)$ contains q_m of these values. Adding $k\theta/q_mq_{m+1}$ to the k-th element of M moves all of these values to the right, or all of them to the left, according to the sign of θ, by amounts which are less than twice the spacing. So at most two of the original values leave $[0,1)$ and at most two leave $[1,2)$. It follows that $|\sum_{k=1}^{2q_m}(-1)^{\lfloor (t+k)\alpha\rfloor}| \le 4$ in this case.

If p_m is even, M contains q_m distinct values (each repeated twice) with equal spacing $2/q_m$. Counting repetitions, $q_m - 1$ of these values are in one of the intervals $[0,1)$, $[1,2)$ and $q_m + 1$ of them are in the other. Adding $k\theta/q_mq_{m+1}$ to the k-th element of M causes at most two values, counting repetitions, to move out of $[0,1)$ and at most two of them to move out of $[1,2)$. In this case $|\sum_{k=1}^{2q_m}(-1)^{\lfloor (t+k)\alpha\rfloor}| \le 6$. ■

Theorem 3.3 *If α is a quadratic irrational, then $S(n) = O(\log n)$.*

Proof It will suffice to show the result for n restricted to the even integers, since $|S(n+1) - S(n)| = 1$. Assuming n to be even, write $n/2 = \sum_{i=0}^{m} d_iq_i$ in the representation (2.9), so that $n = \sum_{i=0}^{m} d_i(2q_i)$. Partition the integers from 1 to n into d_i blocks of consecutive integers of length $2q_i$, $0 \le i \le m$. By Lemma 3.2, the sum of $(-1)^{\lfloor k\alpha\rfloor}$, where k runs over a block of length $2q_i$, has absolute value at most 6. Thus $|S_n| \le \sum_{i=0}^{m} 6d_i \le 6K(m+1)$, where K is an upper bound of $\{\alpha_{k+1}, k \ge 0\}$. Since $d_m \ne 0$, it follows from (2.7) that $n \ge 2q_m \ge 2^{m/2}$ and thus $\log n \ge (m\log 2)/2$. We then have $|S(n)| \le 6K(2\log n/\log 2 + 1)$. ■

Lemma 3.4 *For $m \ge 0$, $\lfloor kp_m/q_m\rfloor - \lfloor k\alpha\rfloor$ is equal to*

(a) 0 *if m is even and $k \in \{0,1,\dots,q_{m+1}\}$ or if m is odd and $k \in \{0,1,\dots,q_{m+1}\} \setminus \{q_m, 2q_m, \dots, \alpha_{m+1}q_m\}$;*
(b) 1 *if m is odd and $k \in \{q_m, 2q_m, \dots, \alpha_{m+1}q_m\}$.*

Proof For $0 \le k \le q_{m+1}$ it follows from (2.5) that $|k\alpha - kp_m/q_m| < 1/q_m$. Thus $\lfloor kp_m/q_m\rfloor$ and $\lfloor k\alpha\rfloor$ can differ by at most 1 and, using in addition (2.4), there is no integer strictly between $k\alpha$ and kp_m/q_m. If

$$k \in \{0,1,\dots,q_{m+1}\} \setminus \{q_m, 2q_m, \dots, \alpha_{m+1}q_m\},$$

then kp_m/q_m is not an integer and thus $\lfloor kp_m/q_m\rfloor = \lfloor k\alpha\rfloor$. If $k \in \{q_m, 2q_m, \dots, \alpha_{m+1}q_m\}$ and m is even, then by (2.6) $k\alpha - kp_m/q_m > 0$ and thus $\lfloor kp_m/q_m\rfloor = \lfloor k\alpha\rfloor$. If $k \in \{q_m, 2q_m, \dots, \alpha_{m+1}q_m\}$ and m is odd, then by (2.6) $kp_m/q_m - k\alpha > 0$ and thus $\lfloor k\alpha\rfloor = \lfloor kp_m/q_m\rfloor - 1$. ■

Lemma 3.5 *For n having the representation* (2.9),

$$S(n) = S(b_mq_m) + (-1)^{b_mp_m}S(n_m).$$

Proof Applying Lemma 3.4 and observing that $n < (b_m + 1)q_m$,

$$S(n) - S(b_m q_m) = \sum_{k=b_m q_m+1}^{n} (-1)^{\lfloor k\alpha \rfloor} = \sum_{k=1}^{n_m} (-1)^{\lfloor (k+b_m q_m)\alpha \rfloor}$$
$$= \sum_{k=1}^{n_m} (-1)^{\lfloor (k+b_m q_m) p_m/q_m \rfloor} = (-1)^{b_m p_m} \sum_{k=1}^{n_m} (-1)^{\lfloor k p_m/q_m \rfloor}$$
$$= (-1)^{b_m p_m} \sum_{k=1}^{n_m} (-1)^{\lfloor k\alpha \rfloor} = (-1)^{b_m p_m} S(n_m).$$

■

Lemma 3.6 *For* $1 \le b_m \le \alpha_{m+1}$,

$$S(b_m q_m) = S(q_m) \sum_{\nu=0}^{b_m-1} (-1)^{\nu p_m}.$$

Proof Applying Lemma 3.5 with $n = b_m q_m - 1 = (b_m - 1)q_m + (q_m - 1)$,

$$S(b_m q_m) = S(b_m q_m - 1) + (-1)^{\lfloor b_m q_m \alpha \rfloor}$$
$$= S((b_m - 1)q_m) + (-1)^{(b_m-1)p_m} S(q_m - 1) + (-1)^{\lfloor b_m q_m \alpha \rfloor}$$
$$= S((b_m - 1)q_m) + (-1)^{(b_m-1)p_m} (S(q_m) - (-1)^{\lfloor q_m \alpha \rfloor}) + (-1)^{\lfloor b_m q_m \alpha \rfloor}$$
$$= S((b_m - 1)q_m) + (-1)^{(b_m-1)p_m} S(q_m) - (-1)^{(b_m-1)p_m + \lfloor q_m \alpha \rfloor}$$
$$+ (-1)^{\lfloor b_m q_m \alpha \rfloor}.$$

From Lemma 3.4,

$$(b_m - 1)p_m + \lfloor q_m \alpha \rfloor - \lfloor b_m q_m \alpha \rfloor = (b_m - 1)p_m + \lfloor q_m p_m/q_m \rfloor - \lfloor b_m q_m p_m/q_m \rfloor$$
$$= (b_m - 1)p_m + p_m - b_m p_m = 0.$$

Thus

$$S(b_m q_m) = S((b_m - 1)q_m) + (-1)^{(b_m-1)p_m} S(q_m).$$

Replacing b_m by ν in the last equation and summing,

$$S(b_m q_m) = S(q_m) + \sum_{\nu=2}^{b_m} (-1)^{(\nu-1)p_m} S(q_m) = \sum_{\nu=0}^{b_m-1} (-1)^{\nu p_m} S(q_m).$$ ■

Lemma 3.7 *For n having the representation* (2.9),

$$S(n) = S(q_m) \sum_{\nu=0}^{b_m-1} (-1)^{\nu p_m} + S(n_m)(-1)^{b_m p_m}.$$

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Proof Put the formula of Lemma 3.6 into Lemma 3.5. ■

Lemma 3.8 $S(q_m) = S(q_m - 1) + (-1)^{m+p_m}$.

Proof $S(q_m) - S(q_m - 1) = (-1)^{\lfloor q_m\alpha \rfloor}$. From Lemma 3.4, m even implies

$$(-1)^{\lfloor q_m\alpha\rfloor} = (-1)^{\lfloor q_m p_m/q_m\rfloor} = (-1)^{p_m} = (-1)^{m+p_m}$$

and m odd implies

$$(-1)^{\lfloor q_m\alpha\rfloor} = (-1)^{\lfloor q_m p_m/q_m\rfloor - 1} = (-1)^{p_m - 1} = (-1)^{m+p_m}. \quad ■$$

Lemma 3.9 *If p_m is even, then $S(q_m - 1) = 0$ and $S(q_m) = (-1)^m$.*

Proof By Lemma 3.4, $k \in \{1, \ldots, q_m - 1\}$ implies

$$\begin{aligned}(-1)^{\lfloor (q_m-k)\alpha\rfloor} &= (-1)^{\lfloor (q_m-k)p_m/q_m\rfloor} = (-1)^{p_m + \lfloor -kp_m/q_m\rfloor}\\ &= (-1)^{p_m-1-\lfloor kp_m/q_m\rfloor} = (-1)^{p_m-1}(-1)^{\lfloor k\alpha\rfloor},\end{aligned}$$

where we have used the fact that $\lfloor -x\rfloor = -\lfloor x\rfloor - 1$ for nonintegral x. Summing on k, $S(q_m - 1) = (-1)^{p_m-1}S(q_m - 1)$, which gives $S(q_m - 1) = 0$ for p_m even. Then Lemma 3.8 gives $S(q_m) = (-1)^m$ for p_m even. ■

Lemma 3.10 *For $m \geq 1$,*

$$S(q_{m+1}) = \beta_m S(q_m) + \gamma_m S(q_{m-1}), \tag{3.1}$$

where

$$\beta_m = \sum_{\nu=0}^{\alpha_{m+1}-1} (-1)^{\nu p_m}, \quad \gamma_m = (-1)^{\alpha_{m+1}p_m}. \tag{3.2}$$

Proof Replacing n by $q_{m+1} - 1 = \alpha_{m+1}q_m + (q_{m-1} - 1)$ in Lemma 3.7,

$$S(q_{m+1} - 1) = \beta_m S(q_m) + \gamma_m S(q_{m-1} - 1),$$

which by Lemma 3.8 implies

$$S(q_{m+1}) - (-1)^{m+1+p_{m+1}} = \beta_m S(q_m) + \gamma_m S(q_{m-1}) - (-1)^{\alpha_{m+1}p_m + m - 1 + p_{m-1}}.$$

Applying (2.1), we get (3.1). ■

Lemma 3.11 *If for some integer $m_1 \geq 0$ the sequence (α_{m+1} mod 2, $m \geq m_1$) is purely periodic with period π_α, then the sequence (p_m mod 2, $m \geq m_1$) is purely periodic with period π_p which is at most $3\pi_\alpha$.*

Proof Write the first formula of (2.1) in matrix notation as

$$(p_{m+1},\ p_m) = (p_m, p_{m-1}) \begin{pmatrix} \alpha_{m+1}. & 1 \\ 1 & 0 \end{pmatrix}.$$

Working modulo 2 and iterating this relation, we have for $k \geq 0$,

$$(p_{m_1+(k+1)\pi_\alpha}, p_{m_1+(k+1)\pi_\alpha-1}) = (p_{m_1+k\pi_\alpha}, p_{m_1+k\pi_\alpha-1})P,$$

where P mod 2 is independent of k by periodicity of $(\alpha_{m+1} \bmod 2, m \geq m_1)$ and is the product of π_α matrices, each of which is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. These two matrices generate the group

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

under matrix multiplication modulo 2, each of whose elements has order not exceeding 3. Let l be the order of P. Since P is in G, it follows that $l \leq 3$ and that

$$(p_{m_1+l\pi_\alpha},\ p_{m_1+l\pi_\alpha-1}) = (p_{m_1},\ p_{m_1-1})\, P^l = (p_{m_1},\ p_{m_1-1}).$$

The lemma is then proved by observing that (continuing to work modulo 2)

$$\begin{aligned}(p_{m_1+l\pi_\alpha+j},\ p_{m_1+l\pi_\alpha+j-1}) &= (p_{m_1+l\pi_\alpha},\ p_{m_1+l\pi_\alpha-1}) \prod_{i=1}^{j} \begin{pmatrix} \alpha_{m_1+l\pi_\alpha+i} & 1 \\ 1 & 0 \end{pmatrix} \\ &= (p_{m_1},\ p_{m_1-1}) \prod_{i=1}^{j} \begin{pmatrix} \alpha_{m_1+i} & 1 \\ 1 & 0 \end{pmatrix} \\ &= (p_{m_1+j},\ p_{m_1+j-1}),\end{aligned}$$

so that $l\pi_\alpha$ is a period of $(p_m \bmod 2,\ m \geq m_1)$. ■

Lemma 3.11 and the periodicity of the partial quotients of a quadratic irrational justify the following definition.

Definition 3.12 For a quadratic irrational α, let π denote the least common *even* multiple of the periods of $(\alpha_{m+1}, m \geq m_1)$ and $(p_m \bmod 2, m \geq m_1)$, where $m_1 \geq 0$ is such that $(\alpha_{m+1}, m \geq m_1)$ is purely periodic. (Note that π is independent of the choice of m_1.)

It will become clear in the proof of Lemma 3.15 why in this definition we require π to be even. Also, we want to have some flexibility in choosing m_1. In case of infinitely many p_m even, we will have to choose m_1 such that p_{m_1} is even in order for Lemma 3.16 to be correct.

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Lemma 3.13 *For a quadratic irrational α, let* $\max_m = \max\{S(n),\ 0 \le n < q_m\}$ *and* $\min_m = \min\{S(n),\ 0 \le n < q_m\}$*. If* $(\alpha_{m+1},\ m \ge m_1)$ *is purely periodic, then for* $m \ge m_1$*, either* $\max_{m+\pi} > \max_m$ *or* $\min_{m+\pi} < min_m$*.*

Proof If infinitely many p_m are even, then for $m \ge m_1$, there exists $j \in \{m, m+1, \dots, m+\pi-1\}$ such that p_j is even. For any n such that $q_j \le n < q_{j+1}$, Lemmas 3.7 and 3.9 imply that $S(n) = (-1)^j b_j + S(n_j)$, where $b_j > 0$ and $n_j \le q_j$ if $\alpha_{j+1} \ge 2$ and $n_j < q_{j-1}$ if $\alpha_{j+1} = 1$. If j is even, this implies that $\max_{j+1} \ge b_j + \max_{j-1} > \max_{j-1}$, and if j is odd, that $\min_{j+1} \le -b_j + \min_{j-1} < \min_{j-1}$. Since $\pi \ge 2$, the result of the lemma then follows in this case.

Now assume only finitely many p_m are even. Recalling that $p_{-2} = 0$, let $m_0 \ge -2$ be the largest value of m such that p_m is even. Then from (2.1) we conclude that α_m must be even for $m \ge m_0 + 3$ because $p_m,\ p_{m-1}$ and p_{m-2} are all odd, and that α_{m_0+2} must be odd because p_{m_0+2} and p_{m_0+1} are odd and p_{m_0} is even. Now q_{m_0} must be odd because p_{m_0} is even and $\gcd(q_{m_0},\ p_{m_0}) = 1$. If q_{m_0+1} is even, we have from (2.1) that q_{m_0+2} must be odd, and if q_{m_0+1} is odd, we have similarly that q_{m_0+2} must be even. Thus q_{m_0+1} and q_{m_0+2} have opposite parity. From the evenness of α_m for $m \ge m_0 + 3$ it then follows by induction that q_m and q_{m+1} have opposite parity for $m > m_0$.

If we apply Lemma 3.7 with both p_m and b_m odd, $1 \le b_m < \alpha_{m+1}$, we get $S(n) = S(q_m) - S(n_m)$, and if we apply it with p_m odd and b_m even, we get $S(n) = S(n_m)$. So if $\alpha_{m+1} \ge 2$, $\max_{m+1} = \max\{\max_m,\ S(q_m) - \min_m\}$ and $\min_{m+1} = \min\{\min_m,\ S(q_m) - \max_m\}$. In order to have both $\max_{m+1} = \max_m$ and $\min_{m+1} = \min_m$ when $\alpha_{m+1} \ge 2$, we would need $S(q_m) - \min_m \le \max_m$ and $S(q_m) - \max_m \ge \min_m$, which together imply that $S(q_m) = \max_m + \min_m$. If in addition $\max_{m+2} = \max_{m+1}$ and $\min_{m+2} = \min_{m+1}$ when $\alpha_{m+1} \ge 2$, we would have to have $S(q_{m+1}) = \max_{m+1} + \min_{m+1} = \max_m + \min_m = S(q_m)$. This is impossible if q_m and q_{m+1} have opposite parity, because the parity of $S(n)$ is the same as that of n. Thus for $m \ge m_0 + 3$, either $\max_{m+2} > \max_m$ or $\min_{m+2} < \min_m$. Since $\pi \ge 2$, the result of the lemma follows also in the case of only finitely many p_m even. ■

Theorem 3.14 *If α is a quadratic irrational, then there exists a constant $C > 1$ and a sequence of positive integers i_k such that $i_k \le C^k$ and $|S(i_k)| \ge k/2$.*

Proof In the notation of Lemma 3.13, $\max_{m_1+k\pi} - \min_{m_1+k\pi} \ge k$, from which we have $\max\{|S(n)|,\ 0 \le n \le q_{m_1+k\pi}\} \ge k/2$. Thus there exists a sequence of integers i_k such that $q_{m_1+(k-1)\pi} < i_k \le q_{m_1+k\pi}$ and $|S(i_k)| \ge k/2$. If K is an upper bound of the partial quotients α_m of α, we then have from (2.8) that $i_k \le (K+1)^{m_1+k\pi} \le C^k$, for some constant $C > 1$. ■

In preparation for the next lemma, we write (3.1) in the matrix form

$$(S(q_{m+1}),\ S(q_m)) = (S(q_m),\ S(q_{m-1}))\, B_m, \quad \text{where } B_m = \begin{pmatrix} \beta_m & 1 \\ \gamma_m & 0 \end{pmatrix}. \tag{3.3}$$

For a quadratic irrational α, we again let m_1 denote any nonnegative integer such that the sequence $(\alpha_{m+1}, m \ge m_1)$ is purely periodic. Applying (3.3) $k\pi + j$ times starting

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with $m = m_1$ and using the periodicity of B_m, we get

$$(3.4) \qquad (S(q_{k\pi+m_1+j}),\ S(q_{k\pi+m_1+j-1})) = (S(q_{m_1}), S(q_{m_1-1}))\, B^k \prod_{i=m_1}^{m_1+j-1} B_i,$$

where

$$(3.5) \qquad B = \prod_{i=m_1}^{m_1+\pi-1} B_i.$$

Lemma 3.15 *If α is a quadratic irrational with only finitely many p_m even, then B is the identity matrix, and hence $S(q_{m+\pi}) = S(q_m)$ for $m \geq m_1$.*

Proof For $m \geq m_1$, p_m is odd. Then (2.1) implies that α_{m+1} is even for $m \geq m_1 + 1$. By the periodicity of $(\alpha_{m+1},\ m \geq m_1)$ it must also be true that α_{m+1} is even for $m = m_1$. Then by (3.3) and (3.4), $B_m = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for all $m \geq m_1$ and $B = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)^\pi = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ since π is even. ∎

Lemma 3.15 says that the sequence $(S(q_m))$ is eventually periodic if only finitely many p_m are even. This result will not be true in general if infinitely many p_m are even, and the determination of the behavior of $(S(q_m))$ in that case requires a more detailed investigation, which we now begin.

If infinitely many p_m are even, we shall impose the additional requirement on m_1 that p_{m_1} be even. By (2.2), $\gcd(p_m,\ p_{m+1}) = 1$, so no two consecutive p_m's can be even. We can therefore partition the sequence $(p_m,\ m_1 \leq m \leq m_1 + \pi - 1)$ into one or more blocks of consecutive terms, each block consisting of an even integer followed by one or more odd integers. Suppose there are r such blocks starting at positions $m_1 < m_2 < \cdots < m_r$. Let j_k denote the length of the k-th block, so that $j_k = m_{k+1} - m_k (1 \leq j \leq r-1)$, $j_r = m_1 + \pi - m_r$, and $j_1 + \cdots + j_r = \pi$. Define

$$(3.6) \qquad \alpha'_{m_k+1} = \begin{cases} \alpha_{m_k+1} & j_k \text{ even,} \\ \alpha_{m_k+1} - 1 & j_k \text{ odd,} \end{cases}$$

and let

$$(3.7) \qquad A_i = \sum_{k=1}^{i} (-1)^{m_k} \alpha'_{m_k+1}, \quad 0 \leq i \leq r$$

with the usual convention that $A_0 = 0$. Notice that although A_i in general depends on our choice of m_1 (that is, where we choose to begin the period), A_r is independent of m_1 because the sum that defines it extends over all blocks within an entire (even) period.

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Lemma 3.16 *If α is a quadratic irrational with infinitely many p_m even, then $B = \left(\begin{smallmatrix} 1 & (-1)^{m_1} A_r \\ 0 & 1 \end{smallmatrix}\right)$ and hence $B^k = \left(\begin{smallmatrix} 1 & (-1)^{m_1} k A_r \\ 0 & 1 \end{smallmatrix}\right)$.*

Proof We first investigate the product of the matrices B_i over the k-th block. If $j_k = 2$, then p_{m_k} is even, and $p_{m_k+2} = p_{m_{k+1}}$ is even. This implies by (2.1) that α_{m_k+2} is even and hence by (3.2) that

$$\prod_{i=0}^{j_k-1} B_{m_k+i} = B_{m_k} B_{m_k+1} = \begin{pmatrix} \alpha_{m_k+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \alpha_{m_k+1} \\ 0 & 1 \end{pmatrix}, \quad j_k = 2. \tag{3.8}$$

If $j_k \geq 3$, then p_{m_k} is even, p_{m_k+i} is odd for $1 \leq i < j_k$, and $p_{m_k+j_k} = p_{m_{k+1}}$ is even, implying by (2.1) that α_{m_k+2} is odd, α_{m_k+i} is even for $3 \leq i < j_k$, and $\alpha_{m_k+j_k} = \alpha_{m_k+1}$ is odd. Thus

$$\prod_{i=0}^{j_k-1} B_{m_k+i} = \begin{pmatrix} \alpha_{m_k+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{j_k-3} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad j_k \geq 3. \tag{3.9}$$

For j_k odd, the right-hand side of (3.9) reduces to

$$\begin{pmatrix} \alpha_{m_k+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & \alpha_{m_k+1} - 1 \\ 0 & 1 \end{pmatrix},$$

and for j_k even it reduces to

$$\begin{pmatrix} \alpha_{m_k+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \alpha_{m_k+1} \\ 0 & 1 \end{pmatrix}.$$

So (3.8) and (3.9) can be combined into

$$\prod_{i=0}^{j_k-1} B_{m_k+i} = \begin{pmatrix} (-1)^{j_k} & \alpha'_{m_k+1} \\ 0 & 1 \end{pmatrix}. \tag{3.10}$$

Now we take the product of (3.10) over the r blocks to get

$$\begin{aligned} B &= \prod_{k=1}^{r} \begin{pmatrix} (-1)^{j_k} & \alpha'_{m_k+1} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} (-1)^{j_1+\dots+j_r} & \alpha'_{m_1+1} + (-1)^{j_1} \alpha'_{m_2+1} + \cdots + (-1)^{j_1+\cdots+j_{r-1}} \alpha'_{m_r+1} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} (-1)^{\pi} & (-1)^{m_1} \left[(-1)^{m_1} \alpha'_{m_1+1} + (-1)^{m_2} \alpha'_{m_2+1} + \cdots + (-1)^{m_r} \alpha'_{m_r+1} \right] \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & (-1)^{m_1} A_r \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

which proves the lemma. ■

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Lemma 3.17 *If α is a quadratic irrational and infinitely many p_m are even, then for $k \geq 0$, $0 \leq j < j_i$, $1 \leq i \leq r$ we have*

$$S(q_{k\pi+m_i+j}) = \begin{cases} (-1)^{m_i} & \text{if } j = 0, \\ kA_r + A_{i-1} + (-1)^{m_i}\alpha_{m_i+1} + S(q_{m_1-1}) & \text{if } j \text{ is odd}, \\ kA_r + A_{i-1} + (-1)^{m_i}\alpha_{m_i+1} + S(q_{m_1-1}) - (-1)^{m_i} & \text{if } j \text{ is even} \\ & \text{and } j > 0, \end{cases}$$

where A_i is given by (3.7).

Proof Recalling (Lemma 3.9) that $S(q_m) = (-1)^m$ for p_m even, applying (3.3) $k\pi + m_i + j - m_1$ times, and using Lemma 3.16,

$$\begin{aligned}(3.11)\quad &\big(S(q_{k\pi+m_i+j}), S(q_{k\pi+m_i+j-1})\big) \\ &\qquad = \big((-1)^{m_1}, S(q_{m_1-1})\big)\begin{pmatrix}1 & (-1)^{m_1}kA_r \\ 0 & 1\end{pmatrix}\prod_{\nu=m_1}^{m_i-1}B_\nu \prod_{\nu=m_i}^{m_i+j-1}B_\nu.\end{aligned}$$

By (3.10) and (3.7),

$$\prod_{\nu=m_1}^{m_i-1}B_\nu = \prod_{k=1}^{i-1}\begin{pmatrix}(-1)^{j_k} & \alpha'_{m_k+1} \\ 0 & 1\end{pmatrix} = \begin{pmatrix}(-1)^{j_1+\cdots+j_{i-1}} & (-1)^{m_1}A_{i-1} \\ 0 & 1\end{pmatrix},$$

from which we get

$$\begin{aligned}&((-1)^{m_1}, S(q_{m_1-1}))\begin{pmatrix}1 & (-1)^{m_1}kA_r \\ 0 & 1\end{pmatrix}\prod_{\nu=m_1}^{m_i-1}B_\nu \\ &\qquad = ((-1)^{m_1}, S(q_{m_1-1}))\begin{pmatrix}(-1)^{j_1+\cdots+j_{i-1}} & (-1)^{m_1}A_{i-1} + (-1)^{m_1}kA_r \\ 0 & 1\end{pmatrix} \\ &\qquad = ((-1)^{m_1+j_1+\cdots+j_{i-1}}, A_{i-1} + kA_r + S(q_{m_1-1})) \\ &\qquad = ((-1)^{m_i}, A_{i-1} + kA_r + S(q_{m_1-1})).\end{aligned}$$

Putting this result into (3.11), we now have

$$(3.12)\quad (S(q_{k\pi+m_i+j}), S(q_{k\pi+m_i+j-1})) = ((-1)^{m_i}, A_{i-1} + kA_r + S(q_{m_1-1}))\prod_{\nu=m_i}^{m_i+j-1}B_\nu.$$

For $j = 0$, $\prod_{\nu=m_i}^{m_i+j-1}B_\nu$ is an empty product, which evaluates to the identity matrix. In this case we get $S(q_{k\pi+m_i+j}) = (-1)^{m_i}$ from (3.12) by equating first components. This proves the first formula of the lemma.

For $j = 1$, by (3.2) and (3.3),

$$\prod_{\nu=m_i}^{m_i+j-1} B_\nu = B_{m_i} = \begin{pmatrix} \alpha_{m_i+1} & 1 \\ 1 & 0 \end{pmatrix},$$

while for $j \geq 2$,

$$\prod_{\nu=m_i}^{m_i+j-1} B_\nu = B_{m_i} = \begin{pmatrix} \alpha_{m_i+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{j-2},$$

which reduces to

$$\begin{pmatrix} \alpha_{m_i+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \alpha_{m_i+1} - 1 & \alpha_{m_i+1} \\ 1 & 1 \end{pmatrix}$$

if j is even and to

$$\begin{pmatrix} \alpha_{m_i+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \alpha_{m_i+1} & \alpha_{m_i+1} - 1 \\ 1 & 1 \end{pmatrix}$$

if j is odd. So for $j \geq 1$ we get from (3.12) by equating first components that $(q_{k\pi+m_i+j}) = (-1)^{m_i}(\alpha_{m_{i+1}} - 1) + A_{i-1} + kA_r + S(q_{m_1-1})$ if j is even, $j \geq 2$, and $S(q_{k\pi+m_i+j}) = (-1)^{m_i}\alpha_{m_{i+1}} + A_{i-1} + kA_r + S(q_{m_1-1})$ if j is odd. ■

If we replace $k\pi + m_i + j$ by m in Lemma 3.17, we can rephrase the lemma in this way:

If infinitely many p_m are even, then

$$S(q_m) = \begin{cases} (-1)^m & \text{if } p_m \text{ is even,} \\ (A_r/\pi)m + c_m & \text{if } p_m \text{ is odd,} \end{cases} \tag{3.13}$$

where $c_{m+\pi} = c_m$ for $m \geq m_1$. In view of Lemmas 3.9 and 3.15, (3.13) remains true in the case of only finitely many p_m even if we replace A_r/π by 0. This motivates the following definition.

Definition 3.18 For a quadratic irrational α, we define $A = A(\alpha)$ by

(i) $A = 0$ if only finitely many p_m are even, and

(ii) $A = A_r/\pi = \dfrac{1}{\pi} \displaystyle\sum_{\substack{m=m_1 \\ p_m \text{ even}}}^{m_1+\pi-1} (-1)^m \alpha'_{m+1}$ if infinitely many p_m are even,

where α'_{m+1} is given by (3.6) and π and m_1 are given by Definition 3.12 with the additional stipulation that p_{m_1} be even.

It should be noted that A can be equal to 0 even in case (ii) of Definition 3.18, as Example 7.2 will show.

From the remarks immediately preceding Definition 3.18, we then have the following theorem.

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Theorem 3.19 *If α is a quadratic irrational, then*

$$S(q_m) = \begin{cases} (-1)^m & \text{if } p_m \text{ is even,} \\ Am + c_m & \text{if } p_m \text{ is odd,} \end{cases}$$

where $c_{m+\pi} = c_m$ for m sufficiently large and A is given by Definition 3.18.

Definition 3.18 allows us to classify all quadratic irrationals according to the following simple scheme:

Class I: $A = 0$, in which case $(S(q_m))$ is a bounded sequence.

Class II: $A \neq 0$, in which case $(S(q_m))$ is unbounded.

It is this classification that determines the convergence behavior of (1.1), as we shall see in Section 6.

4 The Sequence $T(n)$

We now return to the double sums $T(n)$ defined in the Introduction.

Lemma 4.1 *For $n \geq 0$,*

$$T(n) = T(b_m q_m) + (-1)^{b_m p_m} T(n_m) + n_m D_m S(q_m),$$

where m, b_m and n_m are defined by the representation (2.9) *and*

$$D_m = \sum_{\nu=0}^{b_m-1} (-1)^{\nu p_m}. \tag{4.1}$$

Proof $T(n) - T(b_m q_m) = \sum_{k=b_m q_m+1}^{b_m q_m + n_m} S(k) = \sum_{k=1}^{n_m} S(b_m q_m + k)$. By Lemma 3.5, the last sum is equal to $\sum_{k=1}^{n_m} (S(b_m q_m) + (-1)^{b_m p_m} S(k)) = n_m S(b_m q_m) + (-1)^{b_m p_m} T(n_m)$, and by Lemma 3.6, $n_m S(b_m q_m) = n_m D_m S(q_m)$. ■

Lemma 4.2 *For $m \geq 0$, and $b_m \in \{1, 2, \ldots, \alpha_{m+1}\}$*

$$T(b_m q_m) = D_m T(q_m) + q_m C_m S(q_m)$$

where D_m is given by (4.1) *and*

$$C_m = (-1)^{(b_m-1)p_m} \sum_{\nu=0}^{b_m-1} \nu(-1)^{\nu p_m}. \tag{4.2}$$

Proof We begin by writing

$$T(b_m q_m) = T(b_m q_m - 1) + S(b_m q_m) = T((b_m - 1)q_m + q_m - 1) + S(b_m q_m). \tag{4.3}$$

By Lemma 4.1

$$T((b_m-1)q_m+q_m-1)=T((b_m-1)q_m)+(-1)^{(b_m-1)p_m}T(q_m-1)\\ +(q_m-1)S(q_m)\sum_{\nu=0}^{b_m-2}(-1)^{\nu p_m} \tag{4.4}$$

and by Lemma 3.6,

$$S(b_mq_m)=S(q_m)\sum_{\nu=0}^{b_m-1}(-1)^{\nu p_m}. \tag{4.5}$$

Putting (4.4) and (4.5) into (4.3),

$$T(b_mq_m)=T((b_m-1)q_m)+(-1)^{(b_m-1)p_m}(T(q_m)-S(q_m))\\ +(q_m-1)S(q_m)\sum_{\nu=0}^{b_m-2}(-1)^{\nu p_m}+S(q_m)\sum_{\nu=0}^{b_m-1}(-1)^{\nu p_m},$$

which simplifies to

$$T(b_mq_m)=T((b_m-1)q_m)+(-1)^{(b_m-1)p_m}T(q_m)+q_mS(q_m)\sum_{\nu=0}^{b_m-2}(-1)^{\nu p_m}. \tag{4.6}$$

Now replace b_m by μ in (4.6) and sum μ from 1 to b_m :

$$\begin{aligned}T(b_mq_m)&=T(0)+T(q_m)\sum_{\mu=1}^{b_m}(-1)^{(\mu-1)p_m}+q_mS(q_m)\sum_{\mu=1}^{b_m}\sum_{\nu=0}^{\mu-2}(-1)^{\nu p_m}\\ &=T(q_m)\sum_{\nu=0}^{b_m-1}(-1)^{\nu p_m}+q_mS(q_m)\sum_{\mu=1}^{b_m}\sum_{\nu=0}^{\mu-2}(-1)^{\nu p_m}.\end{aligned}$$

To complete the proof, we reverse the order of summation in the double sum:

$$\begin{aligned}\sum_{\mu=1}^{b_m}\sum_{\nu=0}^{\mu-2}(-1)^{\nu p_m}&=\sum_{\mu=2}^{b_m}\sum_{\nu=0}^{\mu-2}(-1)^{\nu p_m}=\sum_{\nu=0}^{b_m-2}\sum_{\mu=\nu+2}^{b_m}(-1)^{\nu p_m}\\ &=\sum_{\nu=0}^{b_m-2}(b_m-\nu-1)(-1)^{\nu p_m}=\sum_{\nu=0}^{b_m-1}\nu(-1)^{(b_m-1-\nu)p_m}\\ &=(-1)^{(b_m-1)p_m}\sum_{\nu=0}^{b_m-1}\nu(-1)^{\nu p_m}=C_m.\end{aligned}$$

■

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Lemma 4.3 *For $n \geq 0$ having the representation* (2.9),

$$T(n) = D_m T(q_m) + (-1)^{b_m p_m} T(n_m) + q_m C_m S(q_m) + n_m D_m S(q_m).$$

Proof Put Lemma 4.2 into Lemma 4.1. ■

Lemma 4.4 *For $n \geq 0$,*

$$T(q_{m+1}) = \beta_m T(q_m) + \gamma_m T(q_{m-1}) + \left(\delta_m q_m + \beta_m q_{m-1}\right) S(q_m),$$

where β_m, γ_m, are given by (3.2) *and where*

$$\delta_m = (-1)^{(\alpha_{m+1}-1)p_m} \sum_{\nu=0}^{\alpha_{m+1}-1} \nu(-1)^{\nu p_m}.$$

Proof Write $T(q_{m+1}) = T(q_{m+1}-1) + S(q_{m+1}) = T(\alpha_{m+1}q_m + q_{m-1} - 1) + S(q_{m+1})$ and apply Lemma 4.3 with $n = \alpha_{m+1}q_m + q_{m-1} - 1,\ b_m = \alpha_{m+1},\ n_m = q_{m-1} - 1$. Next, rewrite $T(q_{m-1} - 1)$ as $T(q_{m-1}) - S(q_{m-1})$ and use the definition of β_m, γ_m together with (3.1). ■

Lemma 4.5 *For $m \geq 0$, $q_m S(q_m - 1) = (1 - (-1)^{p_m})\, T(q_m - 1)$.*

Proof Take $k \in \{0, 1, \dots, q_m - 1\}$. Using Lemma 3.4,

$$\begin{aligned} S(q_m - 1) - S(k) &= \sum_{j=k+1}^{q_m-1} (-1)^{\lfloor j\alpha \rfloor} = \sum_{j=1}^{q_m-k-1} (-1)^{\lfloor (q_m-j)p_m/q_m \rfloor} \\ &= \sum_{j=1}^{q_m-k-1} (-1)^{p_m-1-\lfloor jp_m/q_m \rfloor} = -(-1)^{p_m} S(q_m - 1 - k), \end{aligned}$$

where we have used the fact that $\lfloor -x \rfloor = -\lfloor x \rfloor - 1$ for nonintegral x. Summing over k, $q_m S(q_m - 1) - T(q_m - 1) = -(-1)^{p_m} T(q_m - 1)$. ■

Lemma 4.6 *For $m \geq 1$,*

$$2T(q_m) = \begin{cases} (q_m + 2)S(q_m) + (-1)^m q_m & \text{if } p_m \text{ is odd,} \\ q_m S(q_{m-1}) + (-1)^{m-1} q_{m-1} + 2(-1)^m & \text{if } p_m \text{ is even.} \end{cases}$$

Proof If p_m is odd, $q_m S(q_m - 1) = 2T(q_m - 1)$ from Lemma 4.5. Applying Lemma 3.8 we then have $q_m(S(q_m) + (-1)^m) = 2(T(q_m) - S(q_m))$, which is equivalent to the first formula of the lemma. If p_m is even, we write the formula of Lemma 4.4 in the form

$$2\beta_m T(q_m) = 2T(q_{m+1}) - 2\gamma_m T(q_{m-1}) - 2\left(\delta_m q_m + \beta_m q_{m-1}\right) S(q_m).$$

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From (3.3) we compute $\beta_m = \alpha_{m+1},\ \gamma_m = 1,\ \delta_m = \alpha_{m+1}(\alpha_{m+1} - 1)/2$ and we use Lemma 3.9 to replace $S(q_m)$ by $(-1)^m$:

$$2\alpha_{m+1}T(q_m) = 2T(q_{m+1}) - 2T(q_{m-1}) \\ - (-1)^m \left(2\alpha_{m+1}q_{m-1} + \alpha_{m+1}(\alpha_{m+1} - 1)q_m\right).$$

The evenness of p_m implies by (2.3) that p_{m-1} and p_{m+1} are both odd, so we can apply the first formula of this lemma to the first two terms of the right side of the last equation:

$$\begin{aligned} 2\alpha_{m+1}T(q_m) = (q_{m+1} + 2)S(q_{m+1}) + (-1)^{m+1}q_{m+1} \\ - (q_{m-1} + 2)S(q_{m-1}) - (-1)^{m-1}q_{m-1} \\ - (-1)^m \left(2\alpha_{m+1}q_{m-1} + (\alpha_{m+1}(\alpha_{m+1} - 1)q_m\right). \end{aligned}$$

We then replace q_{m+1} by $\alpha_{m+1}q_m + q_{m-1}$ according to (2.1), and using Lemma 3.10, replace $S(q_{m+1})$ by $\alpha_{m+1}S(q_m) + S(q_{m-1})$, which by Lemma 3.9 is equal to $\alpha_{m+1}(-1)^m + S(q_{m-1})$, to get

$$\begin{aligned} 2\alpha_{m+1}T(q_m) = \left(\alpha_{m+1}q_m + q_{m-1} + 2\right)\left(\alpha_{m+1}(-1)^m + S(q_{m-1})\right) \\ - (-1)^m(\alpha_{m+1}q_m + q_{m-1}) - (q_{m-1} + 2)S(q_{m-1}) \\ + (-1)^m q_{m-1} - (-1)^m \left(2\alpha_{m+1}q_{m-1} + (\alpha_{m+1}(\alpha_{m+1} - 1)q_m\right), \end{aligned}$$

which simplifies algebraically to

$$2\alpha_{m+1}T(q_m) = \alpha_{m+1}q_m S(q_{m-1}) + \alpha_{m+1}(-1)^m q_{m-1} + 2\alpha_{m+1}(-1)^m.$$

Dividing by α_{m+1} then produces the second formula of the lemma. ■

In preparation for Lemma 4.7, we use Lemma 4.3 and the representation (2.9) to write

$$T(n) - nq_m^{-1}T(q_m) = (-1)^{b_m p_m}\left(T(n_m) - n_m q_{m-1}^{-1}T(q_{m-1})\right) + R(n) \tag{4.7}$$

where

$$\begin{aligned} R(n) = \left(D_m - nq_m^{-1}\right)T(q_m) + (-1)^{b_m p_m} n_m q_{m-1}^{-1} T(q_{m-1}) \\ + \left(q_m C_m + n_m D_m\right)S(q_m) \end{aligned} \tag{4.8}$$

and where $D_m,\ C_m$ are given by (4.1) and (4.2).

From (4.7) we then have

$$\left|T(n) - nq_m^{-1}T(q_m)\right| \leq \left|T(n_m) - n_m q_{m-1}^{-1}T(q_{m-1})\right| + |R(n)|. \tag{4.9}$$

In Theorem 4.8 we shall apply (4.9) recursively to bound $\left|T(n) - nq_m^{-1}T(q_m)\right|$, but we first need to get the following bound for $R(n)$.

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Lemma 4.7 *If α is a quadratic irrational, there exists a constant K_1 such that $|R(n)| \leq K_1 q_m$ for all $m \geq 0$.*

Proof From Theorem 3.19 and Lemma 4.6 we have

$$T(q_m) = \frac{1}{2}Amq_m + d_m \tag{4.10}$$

where $|d_m| \leq K_2 q_m$ for $m \geq 0$ and for some constant K_2. From (4.8) and (4.10),

$$\begin{aligned} R(n) &= \left(D_m - nq_m^{-1}\right)\frac{1}{2}Amq_m + (-1)^{b_m p_m} n_m q_{m-1}^{-1}\frac{1}{2}Amq_{m-1} \\ &\quad + \left(q_m C_m + n_m D_m\right) S(q_m) + e_m \\ &= \frac{1}{2}Am\left(D_m q_m - n + n_m(-1)^{b_m p_m}\right) + \left(q_m C_m + n_m D_m\right) S(q_m) + e_m, \end{aligned}$$

where $|e_m| \leq K_3 q_m$ for $m \geq 0$ and some constant K_3.

If p_m is even, then $D_m q_m - n + n_m(-1)^{b_m p_m} = b_m q_m - n + n_m = 0$ and $S(q_m) = (-1)^m$. It follows that $|R(n)| \leq K_4 q_m$ for $m \geq 0$ and for some constant K_4.

If p_m is odd, we have from Theorem 3.19 that

$$R(n) = \frac{1}{2}Am\left(D_m q_m - n + n_m(-1)^{b_m p_m} + 2q_m C_m + 2n_m D_m\right) + f_m,$$

where $|f_m| \leq K_5 q_m$ for $m \geq 0$ and for some constant K_5. For p_m odd and b_m odd,

$$\begin{aligned} D_m q_m - n + n_m(-1)^{b_m p_m} + 2q_m C_m + 2n_m D_m &= q_m - n - n_m + q_m(b_m - 1)) + 2n_m \\ &= -n + q_m b_m + n_m = 0, \end{aligned}$$

For p_m odd and b_m even,

$$D_m q_m - n + n_m(-1)^{b_m p_m} + 2q_m C_m + 2n_m D_m = -n + n_m + b_m q_m = 0.$$

Thus for p_m odd, $R(n) = f_m$ and so $|R(n)| \leq K_5 q_m$ for $m \geq 0$. ∎

Theorem 4.8 *Let α be a quadratic irrational, and let A be given by Definition* 3.18.

(a) *If $A = 0$, then $T(n) = O(n)$.*
(b) *If $A \neq 0$, there exists a positive constant K_6 such that for $n \geq 2$, $T(n) \geq K_6 n \log n$ if $A > 0$ and $T(n) \leq -K_6 n \log n$ if $A < 0$.*

Proof From (4.9) and Lemma 4.7,

$$\left|T(n) - nq_m^{-1}T(q_m)\right| \leq \left|T(n_m) - n_m q_{m-1}^{-1}T(q_{m-1})\right| + K_1 q_m. \tag{4.11}$$

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We then apply (4.11) recursively, first with n replaced by n_m and n_m replaced by $n_{m-1} = n_m - b_{m-1}q_{m-1}$, then with n_m replaced by n_{m-1} and n_{m-1} replaced by $n_{m-2} = n_{m-1} - b_{m-2}q_{m-2}$, etc. and add the results to get

$$\left|T(n) - nq_m^{-1}T(q_m)\right| \leq K_1 \sum_{i=0}^{m} q_i. \tag{4.12}$$

From (2.1) we have $q_i \geq q_{i-1} + q_{i-2}$. If we sum this inequality on i from 0 to m and subtract $\sum_{i=0}^{m-1} q_i$ from both sides, we get $q_m \geq \sum_{i=0}^{m-2} q_i + 2q_{-1} + q_{-2} \geq \sum_{i=0}^{m-2} q_i$. Then $3q_m \geq 2q_m + q_{m-1} \geq \sum_{i=0}^{m} q_i$. If then follows from (4.12) that

$$T(n) - nq_m^{-1}T(q_m) = O(q_m),$$

which, in view of (4.10) and the fact that $q_m \leq n$, implies that

$$T(n) = \frac{1}{2}Amn + O(n). \tag{4.13}$$

If $A = 0$, (4.13) becomes part (a) of the theorem. Now assume $A \neq 0$. We have from (2.8) that $q_m \leq (K+1)^m$. Since $n < q_{m+1}$, this implies that $\log n < (m+1)\log(K+1)$ and hence that $m \geq K_6 \log n$ for some positive K_6. Putting the last inequality into (4.13) then proves part (b) of the theorem. ∎

5 Convex Sequences

In this section we collect the properties of convex sequences $(c_k,\ k \geq 1)$ that will be needed to prove Theorem 6.1. We shall use the notation of Section 1, $\Delta c_k = c_k - c_{k+1}$ and $\Delta^2 c_k = \Delta(\Delta c_k) = c_k - 2c_{k+1} + c_{k+2}$, and we shall say that (c_k) is *decreasing* if $\Delta c_k \geq 0$ and *convex* if $\Delta^2 c_k \geq 0$. We begin by listing two familiar properties.

(5.1) Let (c_k) be decreasing. If $\sum c_k$ converges, then $kc_k \to 0$.

(5.2) Let (c_k) be convex. If $\lim c_k$ is finite, then (c_k) is decreasing.

Lemma 5.1 *Let (c_k) be convex. If $c_k \to 0$, then*

(a) $c_k \geq 0$ *for all k;*
(b) $k\Delta c_k \to 0$*;*
(c) $\sum k\Delta^2 c_k < \infty$.

Proof (a) By (5.2), (c_k) is decreasing, so $c_k \geq \lim c_k = 0$.

(b) From the convexity of (c_k), Δc_k is decreasing. Further,

$$\sum_{k=1}^{n} \Delta c_k = c_1 - c_{n+1} \to c_1.$$

So by (5.1), $k\Delta c_k \to 0$.

(c) It is a simple induction to show that $\sum_{k=1}^{n-1} k\Delta^2 c_k = c_1 - n\Delta c_n - c_{n+1}$, which has limit c_1 by (b). ∎

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Lemma 5.2 *Let (c_k) be convex. If $\sum c_k/k$ converges, then*

(a) *(c_k) is decreasing and $c_k \geq 0$ for all k;*
(b) *$c_k \log k \to 0$ and $\sum(\Delta c_k)\log k < \infty$;*
(c) *$(\Delta c_k)k\log k \to 0$ and $\sum(\Delta^2 c_k)k\log k < \infty$.*

Proof (a) By convexity, (c_k) converges to ∞, $-\infty$, or a finite number c. If $c_k \to \infty$, then $c_k \geq 1$ eventually and $\sum c_k/k$ diverges by comparison with $\sum 1/k$. If $c_k \to -\infty$ or $c_k \to c \neq 0$ then $\sum c_k/k$ diverges similarly. Thus $c_k \to 0$. From (5.2) we conclude that (c_k) is decreasing, and from Lemma 5.1 (a) that $c_k \geq 0$ for all k.

(b) Let $h_n = \sum_{k=1}^{n} 1/k$ and use summation by parts to write

$$\sum_{k=1}^{n} c_k/k = \sum_{k=1}^{n-1} h_k \Delta c_k + h_n c_n. \tag{5.3}$$

From (a) we know that $h_n c_n \geq 0$ and that $\Delta c_n \geq 0$, so from (5.3) and the convergence of $\sum c_k/k$ we get the convergence of $\sum h_k \Delta c_k$. Using $h_k \sim \log k$ we then have $\sum(\Delta c_k)\log k < \infty$. Applying (5.3) again, we get that $\lim h_n c_n = l$ exists, from which it follows that $c_n \log n \to l$. If $l > 0$, $\sum c_k/k$ would diverge by comparison with $\sum 1/(k\log k)$. So $l = 0$ and $c_n \log n \to 0$.

(c) Let $H_n = \sum_{k=1}^{n} h_k$ and perform a second summation by parts to write

$$\sum_{k=1}^{n} c_k/k = \sum_{k=1}^{n-2} H_k \Delta^2 c_k + H_{n-1}\Delta c_{n-1} + h_n c_n. \tag{5.4}$$

By (a), $H_{n-1}\Delta c_{n-1} \geq 0$ and $h_n c_n \geq 0$. The convergence of $\sum c_k/k$ then implies that of $\sum H_k \Delta^2 c_k$ and, in view of $H_n \sim n\log n$, that of $\sum(\Delta^2 c_k)k\log k$. In the proof of part (b), we saw that $h_n c_n \to 0$, which together with (5.4) and the convergence of $\sum c_k/k$, implies $\lim H_{n-1}\Delta c_{n-1} = l$ exists. Thus $\lim(\Delta c_n)n\log n = l$. If $l \neq 0$, we would have $\sum(\Delta c_k)\log k = \infty$, contradicting (b). Thus $(\Delta c_n)n\log n \to 0$. ∎

Example 5.3 Let $c_k = 1/(\log k \log\log k)$, $k \geq 3$. Then (c_k) is convex and $c_k \log k \to 0$ but $\sum c_k/k = \infty$. So for convex sequences, the convergence of $\sum c_k/k$ is stronger than the condition $c_k \log k \to 0$.

6 The Convergence Theorem

We now present the main theorem of this paper.

Theorem 6.1 *Let α be a quadratic irrational, let A be defined by Definition* 3.18*, let $(c_k, k \geq 1)$ be a convex sequence, and let S denote the series $\sum(-1)^{\lfloor k\alpha \rfloor} c_k$.*

(a) *If $A = 0$, then S converges if and only if $c_k \log k \to 0$.*
(b) *If $A \neq 0$, then S converges if and only if $\sum c_k/k$ converges.*

We shall prove this theorem in a sequence of lemmas. For the remainder of this section, α will be a quadratic irrational and (c_k) will be convex.

Lemma 6.2 *If $A = 0$ and $c_k \log k \to 0$, then S converges.*

Proof By Theorem 3.3, $c_k \log k \to 0$ implies $c_k S(k) \to 0$. By Theorem 4.8(a) and Lemma 5.1(b), $T(k-1)\Delta c_{k-1} \to 0$. By Theorem 4.8(a) and Lemma 5.1(c), $\sum T(k)\Delta^2 c_k$ converges. It then follows from (1.3) that S converges. ■

Lemma 6.3 *If $A = 0$ and S converges, then $c_k \log k \to 0$.*

Proof The convergence of S obviously implies $c_k \to 0$, which by (5.2) implies that (c_k) is decreasing and by Lemma 5.1(a) that $c_k \geq 0$ for all k. By Theorem 4.8(a), $T(n) = O(n)$. Then by Lemma 5.1(b) and (c) we have $T(n-1)\Delta c_{n-1} \to 0$ and $\sum T(k)\Delta^2 c_k < \infty$. The convergence of S and (1.3) imply that $S(n)c_n \to 0$. From Theorem 3.14, there exists a sequence of positive integers $(i_k,\ k \geq 1)$ such that $i_k \leq C^k$ and $|S(i_k)| \geq k/2$. For $C^k \leq n < C^{k+1}$ we have $0 \leq c_n \log n \leq c_{i_k} \log C^{k+1} = c_{i_k}(k+1)\log C \leq c_{i_k}(2|S(i_k)|+1)\log C \to 0$ as $k \to \infty$, and hence $c_n \log n \to 0$ as $n \to \infty$. ■

Lemmas 6.2 and 6.3 together prove part (a) of Theorem 6.1.

Lemma 6.4 *Independently of A, the convergence of $\sum c_k/k$ implies the convergence of S.*

Proof Theorem 3.3 and Lemma 5.2(b) imply $c_k S(k) \to 0$. Also, Theorem 3.3 implies $T(k) = O(k \log k)$ which, together Lemma 5.2(c), implies $T(k-1)\Delta c_{k-1} \to 0$ and $\sum T(k)\Delta^2 c_k$ converges. The convergence of S than follows from (1.3). ■

Lemma 6.5 *If $A \neq 0$, the convergence of S implies the convergence of $\sum c_k/k$.*

Proof The proof for $A < 0$ is obtained by reversing the inequality signs in the proof for $A > 0$, so we shall give only the proof for $A > 0$. As in Lemma 6.3, the convergence of S implies that $c_k \to 0$, (c_k) is decreasing, and $c_k \geq 0$ for all k. If p_m is even, (1.3) and Lemma 3.9 imply that

(6.1)
$$\sum_{k=1}^{n}(-1)^{\lfloor k\alpha \rfloor} c_k = \sum_{k=1}^{n-2} T(k)\Delta^2 c_k + T(n-1)\Delta c_n, \quad \text{if } n = q_m - 1 \text{ and } p_m \text{ is even.}$$

The condition $T(n) \geq Kn \log n$ for n sufficiently large and for a positive constant K, from Theorem 4.8(b), together with the convexity of (c_k), imply that for n sufficiently large, $\sum_{k=1}^{n-2} T(k)\Delta^2 c_k$ is increasing and $T(n-1)\Delta c_{n-1} \geq 0$. The case $A \neq 0$ can occur only if infinitely many p_m are even, which means that (6.1) holds for infinitely many n. It then follows from the convergence of S that the partial sums of $\sum T(k)\Delta^2 c_k$ are bounded on an infinite subsequence, and thus bounded because $\sum_{k=1}^{n-2} T(k)\Delta^2 c_k$ increases. Hence $\sum T(k)\Delta^2 c_k$ converges. Using (1.3) again, we get the finiteness of the limit $L = \lim(T(n-1)\Delta c_{n-1} + S(n)c_n)$. The condition $T(n) \geq Kn\log n$ implies that $S(n) \geq (K/2)\log n$ for infinitely many n. Thus $K(n-1)\log(n-1)\Delta c_{n-1} + (K/2)\log nc_n \leq L+1$ for infinitely many n. This implies that $H_{n-1}\Delta c_{n-1} + h_n c_n \leq 2(L+2)/K$ for infinitely many n, where h_n and H_n were defined in the proof of Lemma 5.2. Also, the convergence of $\sum T(k)\Delta^2 c_k$ and the condition $T(n) \geq Kn\log n$ for n sufficiently large implies the convergence of $\sum H_k\Delta^2 c_k$. We thus see from (5.4) that there is an infinite sequence of integers n on which $\sum_{k=1}^{n} c_k/k$ is bounded. The nonnegativity of (c_k) then implies the convergence of $\sum c_k/k$. ∎

Part (b) of Theorem 6.1 then follows from Lemmas 6.4 and 6.5.

7 Examples

We conclude by giving four examples of the determination of π and the computation of A. Recall (Lemma 3.11 and Definition 3.12) that π_α is the period of (α_{m+1} mod $2, m \geq m_1$), π_p is the period of (p_m mod 2, $m \geq m_1$) and π is the least even multiple of π_p and the period of (α_{m+1} mod $2, m \geq m_1$). Also, in the case of infinitely many p_m even, m_1 is chosen so that p_{m_1} is even. (In the examples below we always choose the least such m_1.) Thus we have to carry out the tables below to include $\operatorname{lcm}(3\pi_\alpha, 2)$ periods of $(\alpha_{m+1}, m \geq m_1)$ to be sure that we see the entire period of p_m mod 2. For the computation of A, the tables have to be carried out to $m = m_1 + \pi - 1$.

Example 7.1 $\alpha = 1 - c + \sqrt{c^2+1} = [1, 2c, 2c, 2c, \dots] = [1, \overline{2c}]$ for c a positive integer. From (2.1), p_m satisfies the recursion $p_{m+1} = 2cp_m + p_{m-1}$. Noting that $p_{-1} = 1$ and $p_0 = 1$, it follows by induction that p_m is odd for $m \geq 0$. Thus $A = 0$ and case (a) of Theorem 6.1 applies. (Since the sequence $(1/k)$ is convex, the convergence result of Borwein and Gawronski [2], noted in the Introduction, is a special case of Theorem 6.1.) This example, of course, contains the special case $\alpha = \sqrt{2}$ mentioned in the abstract.

Example 7.2 $\alpha = (1+\sqrt{5})/2 = [\overline{1}]$. The recursion is now $p_{m+1} = p_m + p_{m-1}$. Since $\pi_\alpha = 1$ and $m_1 = 1$, we carry the table out to 6 periods of (α_{m+1} mod 2),

m	−1	0	1	2	3	4	5	6
α_{m+1}	1	1	1	1	1	1	1	1
p_m mod 2	1	1	0	1	1	0	1	1

from which we see that in fact $\pi_p = 3$. Thus $\pi = 6$. For $1 \leq m \leq 6$, there are two values of m with p_m even, so there are two blocks of length 3 each, $r = 2$, $m_2 = 4$, $j_1 = m_2 - m_1 = 3$, $j_2 = m_1 + \pi - m_2 = 3$. Therefore

$$A = \frac{1}{6}\big((-1)^1\alpha_2' + (-1)^4\alpha_5'\big) = \frac{1}{6}\big((-1)^1(\alpha_2 - 1) + (-1)^4(\alpha_5 - 1)\big) = 0.$$

The golden ratio is an example with infinitely many p_m even and $A = 0$.

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Example 7.3 $\alpha = \sqrt{3} = [1, \overline{1, 2}]$. Here, $\pi_\alpha = 2$ and from the table

m	-1	0	1	2	3	4	5
α_{m+1}	1	1	2	1	2	1	2
p_m mod 2	1	1	0	1	1	1	0

we obtain $m_1 = 1,\ \pi_p = 4$, and $\pi = 4$. For $1 \le m \le 4$ there is only one p_m even and thus there is only one block of length 4, so $r = 1,\ j_1 = 4$ and

$$A = \frac{1}{4}(-1)^1 \alpha_2' = \frac{1}{4}(-1)^1 \alpha_2 = -\frac{1}{2}.$$

So $\sqrt{3}$ is an example with infinitely many p_m even and $A \neq 0$.

Example 7.4 $\alpha = (-1 + \sqrt{442})/9 = [2, \overline{4, 2, 4}]$. In this example, $\pi_\alpha = 1$, but the period of $(\alpha_m,\ m \ge m_1)$ is 3. From the below table, $m_1 = 0,\ \pi_p = 2$ and thus $\pi = 6$.

m	-1	0	1	2	3	4	5	6
α_{m+1}	2	4	2	4	4	2	4	4
$p_m$mod 2	1	0	1	0	1	0	1	0

For $0 \le m \le 6$, there are three p_m even, so there are three blocks, length 2 each. Thus $m_2 = 2,\ m_3 = 4$ and $j_i = 2$ for $i = 1, 2, 3$. Finally,

$$A = \frac{1}{6}\big((-1)^0 \alpha_1' + (-1)^2 \alpha_3' + (-1)^4 \alpha_5'\big) = \frac{1}{6}(\alpha_1 + \alpha_3 + \alpha_5) = \frac{5}{3}.$$

So $(-1 + \sqrt{442})/9$ is also an example with infinitely many p_m even and $A \neq 0$.

Applying Theorem 6.1 to Examples 7.1 and 7.3 with (c_k) the sequence of Example 5.3, we get the interesting concrete result that $\sum (-1)^{\lfloor k\sqrt{2} \rfloor}/(\log k \log\log k)$ converges and $\sum (-1)^{\lfloor k\sqrt{3} \rfloor}/(\log k \log\log k)$ diverges.

References

[1] D. Borwein, *Solution to problem no. 6105.* Amer. Math. Monthly **85**(1978), no. 3, 207.

[2] D. Borwein, and W. Gawronski, *On certain sequences of plus and minus ones.* Canad. J. Math. **30**(1978), no. 1, 170–179.

[3] P. Bundschuh, *Konvergenz unendlicher Reihen und Gleichverteilung* mod *1.* Arch. Math. **29**(1977), no. 5, 518–523.

[4] C. Feist and R. Naimi, *Almost alternating harmonic series.*, College Math. Jour. **35**(2004), no. 3, 183–191.

[5] A. S. Fraenkel, *System of enumeration.* Amer. Math. Monthly, **92**(1985), no. 2, 105–114.

[6] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers.* Oxford, Oxford University Press, 1960.

[7] I. Niven and H. Zuckerman, *An Introduction to the Theory of Numbers.* John Wiley and Sons, New York, 1966.

[8] K. O'Bryant, B. Reznick, and M. Serbinowska, *Almost alternating sums.* Amer. Math. Monthly **113**(2006), no. 8, 673–688.

[9] H. D. Ruderman, *Problem no. 6105.* Amer. Math. Monthly, **83**(1970), 573.
[10] M. Serbinowska, *A case of an almost alternating series*. Unpublished manuscript (2003), available from the author on request.

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