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SUBDIRECTLY IRREDUCIBLE SEMIRINGS AND SEMIGROUPS WITHOUT NONZERO NILPOTENTS

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1. **Introduction.** It follows from [1, p. 377, Lemma 1] that a noncommutative subdirectly irreducible ring, with no nonzero nilpotent elements, cannot possess any proper zero-divisors. From [2, p. 193, Corollary 1] a subdirectly irreducible distributive lattice, with more than one element, is isomorphic to the chain with two elements. Hence we can say that a subdirectly irreducible distributive lattice with 0 possesses no proper zero-divisors.

In this paper we consider two generalizations of these results. Firstly, we show that there exists a commutative semiring with 0 and 1 having no nonzero nilpotents which is subdirectly irreducible and yet has proper zero-divisors. Secondly, it is proved that each subdirectly irreducible semigroup with 0 and no nonzero nilpotents cannot contain proper zero-divisors.

2. **Semirings.** A semiring is an algebra $(S, +, \cdot, 0)$ such that $(S, +)$ is a commutative semigroup, $(S, \cdot)$ is a semigroup, 0 is the zero, i.e. $x+0=x$ and $x\cdot 0=0=0\cdot x$ for every $x \in S$, and $\cdot$ distributes over $+$ from the left and the right. The rest of the terminology is used as in ring theory and universal algebra. In particular ω and ι respectively denote the smallest and largest congruences and a semiring is called simple if these are its only congruences.

THEOREM 2.1. *There exists a subdirectly irreducible cummutative semiring which has no nonzero nilpotents and yet contains proper divisors of zero.*

Proof. Let $S=\{0, a, b, c, 1\}$. Define addition as the supremum in the lattice of Figure 1 and multiplication as the infimum in the lattice of Figure 2.

The distributive law holds so that S is a commutative semiring with 1 as the identity element. It has no nonzero nilpotents and $a, b\neq 0$ while $a\cdot b=0$.

Besides ω and ι a routine computation shows that the only other congruences together with their associated partitions of S are: Θ with partition $\{0, a\}, \{b, c, 1\}$, Φ with partition $\{0, b\}, \{a, c, 1\}$, and $\Theta \wedge \Phi$ with partition $\{0\}, \{a\}, \{b\}, \{c, 1\}$. Thus S is subdirectly irreducible since $\Theta \wedge \Phi$ is the smallest congruence not equal to ω.

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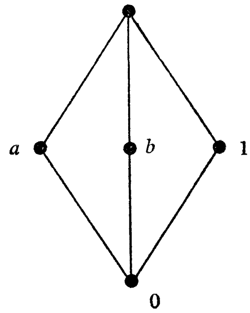

FIGURE 1

As a contrast we have the following positive result.

THEOREM 2.2. *A simple semiring with no nonzero nilpotents contains no proper divisors of zero.*

Proof. Let x be an arbitrary element of the semiring S. Define Φ_x by $y \equiv z(\Phi_x)$ if and only if $y+v=z+w$ for some v, $w \in J_x=\{s \in S : sx=0\}$. As S has no nonzero nilpotents $J_x=\{s \in S : xs=0\}$, Φ_x is a congruence and $J_x=\{s \in S : s \equiv 0(\Phi_x)\}$. Since S is simple, Φ_x is either ω or ι. In the first case $J_x=\{0\}$ so x is a non-divisor of zero. In the second case $J_x=S$ whence $x^2=0$ so $x=0$. Whence every nonzero element is a non-divisor of zero.

3. Semigroups period

THEOREM 3.1. *A subdirectly irreducible semigroup, with* 0 *and no nonzero nilpotents, contains no proper zero-divisors.*

Proof. Let S be any semigroup with 0 and no nonzero nilpotents. Though [1, p. 377, Lemma 1] is stated for rings it clearly applies to semigroups. Hence S possesses a set of ideals, $\{P_\alpha : \alpha \in I\}$, such that $xy \in P_\alpha$ implies $x \in P_\alpha$ or $y \in P_\alpha$ for

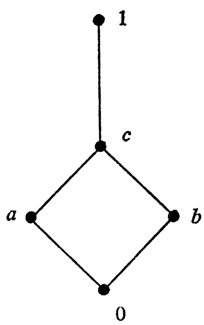

FIGURE 2

each $\alpha \in I$ and $\bigcap_{\alpha \in I} P_\alpha = \{0\}$. For each $\alpha \in I$ define Θ_α by $x \equiv y(\Theta_\alpha)$ iff $x, y \in P_\alpha$ or $x = y$. The following are easily verified: (i) each Θ_α is a congruence on S, (ii) for each α, the factor semigroup $S_\alpha = S/\Theta_\alpha$ is a semigroup with 0, and no proper zero divisors, (iii) $\wedge\, \Theta_\alpha = \omega$ in the lattice of congruences. Hence, S is a subdirect product of semigroups S_α with 0 and no proper zero-divisors. If S is subdirectly irreducible then S must be isomorphic to some S_α.

References

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