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SELF-COMPLEMENTARY GRAPH DECOMPOSITIONS

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Abstract

A complementary decomposition of λK_n into a graph G is an edge-disjoint decomposition of λK_n into copies of G such that if each copy H of G is replaced by its complement in $V(H)$ then the result is an edge-disjoint decomposition of λK_n into copies of G^c; it is a self-complementary decomposition if $G = G^c$. The spectrum for the last self-complementary graph on at most 7 vertices is found.

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1. Introduction

A *G-design* (of λK_n) is an ordered triple (V, B, λ) where V is the vertex set of λK_n $(n = |V|)$ and B is a collection of graphs, each isomorphic to G, which form an edge-disjoint decomposition of λK_n; n is called the *order* and λ is called the *index* of the G-design. Let C_m denote a cycle of length m. Let H^c be the complement of H in $V(H)$.

In recent years, much attention has been focussed on G-designs and on G-designs with additional properties. For example, K_m-designs are just block designs, and C_m-designs have also been called balanced cycle designs and m-cycle systems. Perhaps the most natural question to ask about G-design is what is their *spectrum*, that is, for which values of n do they exist? In the case where $G = C_m$, the spectrum remains unknown, despite having been considered for at least 25 years (see [4] for example). More recently the

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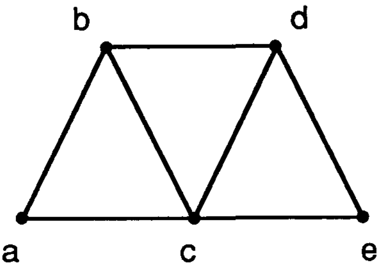
existence problem has been settled in the cases where G is a path [11], and where G is a star [10], and has nearly been settled when G is a graph with at most 5 vertices [1].

Related to this problem is the existence problem for G-designs that satisfy additional properties. For example, the spectra for C_m-designs that are resolvable, almost resolvable, or i-perfect have been successfully studied. Similar results exist when G is a star or a path. For a survey, see [7].

In this paper we consider the spectrum problem for self-complementary G-designs. A *complementary* G-design is a G-design (V, B, λ) with the additional property that replacing each copy H of G in B by its complement in $V(H)$ results in a G^c-design. For example if $\lambda = 1$ and $G = K_{1,3}$ then $G^c = K_3$ (together with an isolated vertex), so complementary $K_{1,3}$-designs are equivalent to nested Steiner triple systems; the spectrum for these has been found [8]. A *self-complementary* G-design is a complementary G-design in which $G \cong G^c$. For example, if $G = P_3$, a path of length 3, then $G = G^c$; the spectrum problem for self-complementary P_3-designs has been found when $\lambda = 1$ [2, 6]. The spectrum for self-complementary C_5-designs (also known as Steiner pentagon systems) has also been found [5]. Here we consider the remaining self-complementary graph with at most 7 vertices.

Let M be the graph with $V(M) = \{a, b, c, d, e\}$ and $E(M) = \{ab, bc, bd, cd, de\}$; throughout this paper we shall denote M by (a, b, c, d, e). Then $M \cong M^c$. The purpose of this paper is to find the spectrum for self-complementary M-designs, for all λ.



The graph $M = (a, b, c, d, e)$.

2. Preliminary results

We shall make use of quasigroups with various properties in constructing the self-complementary M-designs. The properties that are not defined here are well known, but can be found in [9]. Let $Z_x = \{0, 1, \ldots, x-1\}$, and let $(a, b, c, d, e) + i = (a+i, b+i, c+i, d+i, e+i)$.

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LEMMA 2.1. *For $n > 4$, $n \notin \{6, 10\}$, there exist 3 idempotent mutually orthogonal quasigroups of order n.*

LEMMA 2.2 [13]. *For all odd $n \geq 5$ there exists an idempotent self-orthogonal quasigroup of order n which is orthogonal to an idempotent commutative quasigroup.*

Let $h_i = \{2i, 2i+1\}$ and let $H = \{h_i \mid 0 \leq i \leq s-1\}$; the elements of H are called *holes*. A *self-orthogonal quasigroup with holes H* is a quasigroup $(Z_{2s}, \cdot)$ in which

(a) for $0 \leq i \leq s-1$, $2i \cdot 2i = 2i = (2i+1) \cdot (2i+1)$ and $(2i+1) \cdot 2i = 2i+1 = 2i \cdot (2i+1)$, and

(b) for all $(x, y) \in (Z_{2s} \times Z_{2s}) \backslash (\bigcup_{i=0}^{s-1} (h_i \times h_i))$ there exists a unique pair i and j such that $i \cdot j = x$ and $j \cdot i = y$.

A self-orthogonal quasigroup with holes H, $(Z_{2s}, \cdot)$ is orthogonal to a commutative quasigroup with holes H, $(Z_{2s}, \circ)$ if for all $(x, y) \in (Z_{2s} \times Z_{2s}) \backslash (\bigcup_{i=0}^{s-1} (h_i \times h_i))$ there exists a unique pair i and j such that $i \cdot j = x$ and $i \circ j = y$.

LEMMA 2.3 [12]. *For all $n \equiv 2 \pmod 4$, $n \notin \{6, 30, 66, 174\}$ there exists a self-orthogonal quasigroup with holes H that is orthogonal to a commutative quasigroup with holes H.*

Clearly self-complementary M-designs bear some relation to K_5-designs (block designs with block size 5). We shall use the following result of Hanani.

LEMMA 2.4 [3]. *For all $n \equiv 1$ or $5 \pmod{20}$ there exists a K_5-design of K_n. There does not exist a K_5-design of $2K_{15}$.*

We will need some small M-designs.

LEMMA 2.5. *There exist self-complementary M-designs of K_n for $n \in \{5, 11, 31\}$.*

PROOF. $(Z_5, \{(0, 1, 2, 3, 4), (1, 4, 2, 0, 3)\}, 1)$ is a self-complementary M-design of K_5. $(Z_{11}, \{(0, 4, 1, 2, 7) + i \mid 0 \leq i \leq 10\}, 1)$ is a self-complementary M-design of K_{11} (reducing all sums modulo 11).

$(Z_{31}, \{(0, 1, 5, 17, 8)+i, (0, 2, 20, 12, 18)+i, (0, 5, 22, 25, 1)+i \mid 0 \leq i \leq 30\}, 1)$ is a self-complementary M-design of K_{31} (reducing all sums modulo 31).

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LEMMA 2.6. *There exist self-complementary M-designs of $2K_n$ for $n \in \{6, 10, 15, 16, 20, 30\}$.*

PROOF.

$$n = 6: \quad (Z_6, \{(0, 4, 1, 2, 3) + i \mid 0 \le i \le 5\}, 2)$$
$$n = 10: \quad (\{\infty\} \cup Z_9, \{(0,3,5,6,1) + i, (1, 2, \infty, 0, 4) + i \mid 0 \le i \le 8\}, 2)$$
$$n = 15: \quad (\{\infty\} \cup Z_{14}, \{(0, 1, 2, 4, 6) + i, (0,3,7,12,6) + i, (0,4,\infty,11,5) + i \mid 0 \le i \le 13\}, 2)$$
$$n = 16: \quad (Z_{16}, \{(0, 1, 2, 4, 7) + i, (0, 2, 7, 14, 6) + i, (0, 5, 11, 1, 8) + i \mid 0 \le i \le 15\}, 2)$$
$$n = 20: \quad (\{\infty\} \cup Z_{19}, \{(0, 1, 2, 4, 6) + i, (0, 3, 7, 11, 16) + i, (0,5,11,18,10) + i, (0, 9, \infty, 2, 12) + i \mid 0 \le i \le 18\}, 2)$$
$$n = 30: \quad (\{\infty\} \cup Z_{29}, \{(0, 1, 5, 11, 20) + i, (0, 3, 14, 7, 20) + i, (0, 1, 3, 6, 4) + i, (0, 5, 11, 21, 13) + i, (0, 7, 19, 27, 12) + i, (0, 12, \infty, 1, 16) + i \mid 0 \le i \le 28\}, 2)$$

Finally, we note the following necessary conditions.

LEMMA 2.7. *If there exists a self-complementary M-design of λK_n then*
(a) *if $\lambda \equiv 1, 3, 7$ or $9 \pmod{10}$ then $n \equiv 1$ or $5 \pmod{10}$, and if $\lambda = 1$ then $n \neq 15$,*
(b) *if $\lambda \equiv 2, 4, 6$ or $8 \pmod{10}$ then $n \equiv 0$ or $1 \pmod 5$,*
(c) *if $\lambda \equiv 5 \pmod{10}$ then $n \equiv 1 \pmod 2$, $n \neq 3$,*
(d) *if $\lambda \equiv 0 \pmod{10}$ then $n \notin \{2, 3, 4\}$.*

PROOF. If there exists a complementary M-design of λK_n then there exists a K_5-design of $2\lambda K_n$. Therefore, by Lemma 2.4, if $\lambda = 1$ then $n \neq 15$. The rest of the lemma follows from straightforward counting arguments.

3. The case $\lambda = 1$

THEOREM 3.1. *Let $n \equiv 5 \pmod{10}$. There exists a self-complementary M-design of K_n except if $n = 15$.*

PROOF. Let $n = 10s + 5 = 5(2s + 1)$. By Lemmas 2.5 and 2.7 we can assume that $2s + 1 \ge 5$. Let $(Z_{2s+1}, \cdot)$ be an idempotent self-orthogonal quasigroup that is orthogonal to the idempotent commutative quasigroup

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$(Z_{2s+1}, \circ)$ (these quasigroups exist by Lemma 2.2). Then define a self-complementary M-design $(Z_5 \times Z_{2s+1}, B, 1)$ as follows:

(a) for $0 \leq i \leq 2s$, let $\{((0, i), (1, i), (2, i), (3, i), (4, i)), ((1, i), (4, i), (2, i), (0, i), (3, i))\} \subseteq B$, and

(b) for $0 \leq i < j \leq 2s$, and for $0 \leq r \leq 4$, let $((2+r, i \cdot j), (r, i), (1+r, i \circ j), (r, j), (2+r, j \cdot i)) \in B$, (where the sums in the first coordinate are reduced modulo 5).

The fact that this defines an M-design easily follows from the fact that $(Z_{2s+1}, \circ)$ is a quasigroup and that $(Z_{2s+1}, \cdot)$ is an idempotent commutative quasigroup. The orthogonality of the quasigroups ensures that together the complements of each copy of M form an M-design.

To see this, notice that the complements of the copies of M in (a) produce the same set of copies of M. The complement of the graphs defined in (b) are

$$(*) \qquad \{(r, j), (2+r, i \cdot j), (1+r, i \circ j), (2+r, j \cdot i), (r, i)\}$$

for $0 \leq i < j \leq 2s$ and $0 \leq r \leq 4$. So, for example, the edge $\{(a, b), (a, c)\}$ is in the graph $(*)$ where $i \cdot j = b$ and $j \cdot i = c$; there is exactly one such choice for $i < j$ by the self-orthogonality of $(Z_{2s+1}, \cdot)$. Similarly the edge $\{(a, b), (a+1, c)\}$ is in the graph $(*)$ with $i \circ j = b$ and $i \cdot j = c$ (or $j \cdot i = c$); there is exactly one such choice for $i < j$ by the orthogonality of $(Z_{2s+1}, \circ)$ and $(Z_{2s+1}, \cdot)$. The remaining details are left to the reader.

THEOREM 3.2. *Let $n \equiv 11 \pmod{20}$. There exists a self-complementary M-decomposition of K_n except possibly if $n \in \{151, 331, 871\}$.*

PROOF. Let $n = 20s + 11 = 5(4s + 2) + 1$. Using Lemma 2.5 we can assume that $s \geq 2$. Let $(Z_{4s+2}, \cdot)$ be a self-orthogonal quasigroup with holes $\{\{2x, 2x + 1\} \mid 0 \leq x \leq 2s\}$ that is orthogonal to $(Z_{4s+2}, \circ)$, a commutative quasigroup with holes $\{\{2x, 2x+1\} \mid 0 \leq x \leq 2s\}$ (these quasigroups exist by Lemma 2.3). Define a self-complementary M-design $(\{\infty\} \cup (Z_5 \times Z_{4s+2}), B, 1)$ as follows:

(a) for $0 \leq x \leq 2s$, place a copy of the self-complementary M-design in Lemma 2.5 on the vertices $\{\infty\} \cup (Z_5 \times \{2x, 2x+1\})$ in B, and

(b) for $0 \leq i < j \leq 4s + 1$, $\{i, j\} \notin \{\{2x, 2x+1\} \mid 0 \leq x \leq 2s\}$ and for $0 \leq r \leq 4$ let $((2+r, i \cdot j), (r, i), (1+r, i \circ j), (r, j), (2+r, j \cdot i)) \in B$.

The fact that this defines a self-complementary M-design follows in the same way as the proof of Theorem 3.1.

LEMMA 3.3. *Let $n \equiv 1 \pmod{20}$. There exists a self-complementary M-design of K_n.*

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PROOF. By Lemma 2.4 there exists a K_5-design of K_n; replace each copy of K_5 with the self-complementary M-design of K_5 in Lemma 2.5.

LEMMA 3.4. *If there exist self-complementary M-designs of K_m and of K_{n+1} and if there exist* 3 *orthogonal quasigroups of order n then there exists a self-complementary M-design of K_{mn+1}.*

PROOF. Let $(Z_n, \cdot_1)$, $(Z_n, \cdot_2)$ and $(Z_n, \cdot_3)$ be 3 orthogonal quasigroups. Let $(Z_m, B_1, 1)$ be a self-complementary M-design of K_m and for each $i \in Z_m$ let $(\{\infty\} \cup (\{i\} \times Z_n), B(i), 1)$ be a self-complementary M-design of K_{n+1}. Then $(\{\infty\} \cup (Z_m \times Z_n), B, 1)$ is a self-complementary M-design of K_{mn+1}, where

$$B = \bigcup_{i \in Z_m} B_i \cup \{((a, i), (b, j), (c, i \cdot_1 j), (d, i \cdot_2 j), (e, i \cdot_3 j)) \mid i \in Z_n, \; j \in Z_n, \; (a, b, c, d, e) \in B_1\}.$$

COROLLARY 3.5. *There exist self-complementary M-designs of K_{151}, K_{331} and K_{871}.*

PROOF. Apply Lemma 3.4 with $(m, n) = (5, 30), (11, 30)$ and $(5, 174)$ respectively.

THEOREM 3.6. *The spectrum for self-complementary M-designs of index* 1 *is $n \equiv 1$ or 5 (mod 10), $n \neq 15$.*

PROOF. This follows from Lemmas 2.7, 3.3 and 3.4, Corollary 3.5 and Theorems 3.1 and 3.2.

4. The cases $\lambda > 1$

THEOREM 4.1. *Let $n \equiv 0$ or 1 (mod 5). There exists a self-complementary M-design of $2K_n$.*

PROOF. Of course if there exists a self-complementary M-design of K_n then there also exists one of $2K_n$. For $n \in \{6, 10, 15, 16, 20, 30\}$ self-complementary M-designs of $2K_n$ are constructed in Lemma 2.6. For $n = 50$, modify the construction in Theorem 3.2 with $s = 2$ by using a copy of the self-complementary M-design of $2K_{10}$ in (a) and taking two copies of each of the blocks in (b); this produces a self-complementary M-design of $2K_{50}$ on the vertex set $Z_5 \times Z_{10}$.

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In any other case, $n = 5s$ or $5s+1$ where s is an integer for which there exist 3 idempotent mutually orthogonal quasigroups of order s, say $(Z_s, \cdot_1)$, $(Z_s, \cdot_2)$ and $(Z_s, \cdot_3)$ (see Lemma 2.1). If $n = 5s$ then a self-complementary M-design $(Z_5 \times Z_s, B, 2)$ can be formed as follows:

(a) for $0 \le x \le s-1$, B contains a copy of a self-complementary M-design of $2K_5$ on the vertex set $Z_5 \times \{x\}$, and

(b) for $0 \le i \le s-1$, $0 \le j \le s-1$, $i \ne j$ and $0 \le r \le 4$ let $((2+r, i \cdot_1 j), (r, i), (1+r, i \cdot_2 j), (r, j), (2+r, i \cdot_3 j)) \in B$.

If $n = 5s+1$ then a self-complementary M-design $(\{\infty\} \cup (Z_5 \times Z_s), B, 2)$ can be produced by using a self-complementary M design of $2K_6$ on the vertex set $\{\infty\} \cup (Z_5 \times \{x\})$ in part (a) above.

The fact that these constructions produce self-complementary M-designs follows in the same way as the proof of Theorem 3.1.

Theorem 4.2. *Let* $n \equiv 1 \pmod 2$. *For all* $n \ne 3$ *there exists a self-complementary* M*-design of* $5K_n$.

Proof. Let $n = 2s+1$. By Lemma 2.2 there exists an idempotent self-orthogonal quasigroup $(Z_{2s+1}, \cdot)$ that is orthogonal to an idempotent commutative quasigroup $(Z_{2s+1}, \circ)$. Then $(Z_{2s+1}, \{(i \cdot j, i, i \circ j, j, j \cdot i) \mid 0 \le i < j \le 2s\}, 5)$ is a self-complementary M-design of $5K_n$.

Theorem 4.3. *For all* $n \ge 5$ *there exists a self-complementary* M*-design of* $10K_n$.

Proof. For $n = 6$ or 10, such a design can be produced by taking 5 copies of the designs in Lemma 2.6. For any other n, by Lemma 2.1 there exist 3 idempotent mutually orthogonal quasigroups of order n, say $(Z_n, \cdot_1)$, $(Z_n, \cdot_2)$ and $(Z_n, \cdot_3)$. Then $(Z_n, \{(i \cdot_1 j, i, i \cdot_2 j, j, i \cdot_3 j) \mid \{i, j\} \subseteq Z_n, i \ne j\}, 10)$ is a self-complementary M-design of $10K_n$.

5. Conclusions

The results in the previous sections can be combined to give the following theorem.

Theorem 5.1. *The necessary conditions in Lemma* 2.7 *for the existence of a self-complementary* M*-design of* λK_n *are sufficient.*

Proof. This follows immediately from Lemma 2.7 and Theorems 3.6, 4.1, 4.2 and 4.3 by combining self-complementary M-designs with $\lambda \in$

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$\{1, 2, 5, 10\}$ to obtain self-complementary M-designs for other values of λ, except when $\lambda = 3$ and $n = 15$. Recently a self-complementary M-design of $3K_{15}$ was constructed by Elizabeth J. Billington (private communication).

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