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SETS OF CONSTANT WIDTH, THE SPHERICAL INTERSECTION PROPERTY AND CIRCUMSCRIBED BALLS

G.T. SALLEE

It is well-known that if W is a set of constant width λ then the circumsphere and the insphere of W are concentric and their radii sum to λ. Here this fact is generalized to sets of constant relative width and it is shown that the result does not depend on W being of constant width, but rather on W satisfying the spherical intersection property; that is, $W = \cap\{w + \lambda B : w \in W\}$.

Let S be a compact, convex, centrally-symmetric set with non-empty interior in E^d, d-dimensional Euclidean space. Then S may be regarded as the unit ball of a Minkowski space in the usual way. A convex compact set K is said to be of constant width λ relative to S if $h(k,u) - h(K,-u) = 2\lambda h(S,u)$ for some constant $\lambda > 0$. Here, $h(K,u)$ represents the usual support function of K in direction u. (See [2] for terminology and notation.)

Let $R(K,S)$ denote the circumradius of K relative to S, that is, the smallest value β such that K is contained in some translate of βS. Similarly, $r(K;S)$, the relative inradius of K, is the largest value β such that K contains some translate of βS. It is

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standard fact that when $S = B^d$, the ordinary unit ball, and W has constant width λ, that the circumsphere and insphere of K are concentric and that $R(W,B^d) + r(W,B^d) = \lambda$. This fact has been generalized by Maehara [3]. He defines (X,Y) to be a pair of constant width if $h(X,u) - h(Y,-u) = \lambda$. In this case $R(X,B^d) + r(Y,B^d) = \lambda$.

It is another standard fact about sets of constant relative width λ that they satisfy the spherical intersection property with respect to their unit ball, that is, $X = \cap\{x + \lambda S: x \in X\}$. As Eggleston [1] also showed, the spherical intersection property does not imply constant width in general. The purpose of this note is to show that it is the spherical intersection property which implies the result about inspheres and circumspheres and, at the same time, provide a new proof of this result.

More precisely, we wish to establish:

THEOREM 1. *Suppose that* $C = \cap\{c + \lambda S: c \in C\}$ *and suppose that* $t + \mu S$ $(\mu \le \lambda)$ *is a homothet of* S *containing* C. *Then* $t + (\lambda - \mu)S$ *is a homothet of* S *contained in* C. *Moreover, if* $t + \tau S \subseteq C$, *then* $\tau + \mu \le \lambda$.

We will establish this result in terms of a more general result which generalizes Maehara's concept. For this, we say (X,Y) is a pair of constant S-width, λ, if $h(X,u) - h(Y,-u) - 2\lambda h(S,u)$. Finally, again following Maehara, let $\Omega(X,\tau S) = \cap\{x + \tau S: x \in X\}$. We say (X,Y) is a τS-pair if $\Omega(X,\tau S) = Y$ and $\Omega(Y,\tau S) = X$. Analogous to the connection between sets of constant width and the spherical intersection property, it is known [4] that if (X,Y) is a pair of constant S-width λ, then (X,Y) is a λS-pair, so our generalization reads

THEOREM 2. *Suppose* (X,Y) *forms a* λS*-pair, and that* $t + \mu S (\mu \le \lambda)$ *is a homothet of* S *containing* X. *Then* $t + (\lambda - \mu)S$ *is a homothet of* S *contained in* Y. *Moreover, if* $t + \tau S \subseteq Y$, *then* $t + (\lambda - \tau)S \supseteq X$. *Consequently,* $R(X;S) + r(Y;S) = \lambda$.

Proof. Without loss of generality, let $\lambda = 1$ and $t = \underline{0}$. From the definition of the Ω operator, it is clear that if $A \subseteq B$, then $\Omega(B,S) \subseteq \Omega(A,S)$. Hence, if $X \subseteq \mu S$, then $\Omega(\mu S,S) \subseteq \Omega(X,S) = Y$. But

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it is very easy to verify that $\Omega(\mu S,S) = (1-\mu)S$. Hence $(1-\mu)S \subseteq Y$, and so $r(Y;S) \geq 1 - R(X;S)$.

Now suppose that $\tau S \subseteq Y$. As above, $\Omega(Y,S) \subseteq \Omega(\tau S,S)$, and so $X \subseteq (1-\tau)S$. It then follows that $R(X;S) \leq 1 - r(Y;S)$. Combining this inequality with the one above, it follows that $R(X;S) + r(Y;S) = 1$.

It is clear that Theorem 1 is an immediate corollary since (C,C) forms a λS-pair.

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Department of Mathematics
University of California
Davis, California 95616
U.S.A.