ON SIGNED BRANCHING MARKOV PROCESSES WITH AGE

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To Professor Kiyoshi Noshiro on the occasion of his 60th birthday

§ 1. **Introduction.** Many authors have considered branching Markov processes for the probabilistic treatment of semi-linear equations. Recently J.E. Moyal [11], [12] gave a formulation for a wide class of branching processes. A similar idea was used in A.V. Skorohod [18] and N. Ikeda-M. Nagasawa-S. Watanabe [4]-[7]. Applying their method, we shall consider in this paper the following problems (A) and (B).

(A): Let E be a compact Hausdorff space with the second axiom of countability and assume the following are given: (1) H_t: a strongly continuous semi-group on $\boldsymbol{C}(E)=\{f;$ continuous function on $E\}$, (2) $\mathscr{G}$: the infinitesimal operator of H_t, (3) $k(x)$, $q_n(x)$, $n=0,1,2,\cdots$, are continuous functions on E such that $k(x)=\sum_{n=0}^{\infty}q_n(x)$ and $\sum_{n=0}^{\infty}|q_n(x)|<\infty$. *How can we interprete probabilistically the following equation?*

$$\frac{\partial u(t,x)}{\partial t}=\mathscr{G}\,u(t,x)+k(x)F(x;u(t,x)),\qquad x\in E,\ t\geqq 0, \tag{1.1}$$

where

$$F(x;\xi)=\frac{1}{k(x)}\sum_{n=0}^{\infty}q_n(x)\xi^n,\qquad x\in E,\ \xi\in R^1. \tag{1.2}$$

(B): *How can we interprete probabilistcally the following equation?*

$$\frac{\partial u(t,x)}{\partial t}=\frac{1}{2}\Delta u(t,x)+G(u(t,x)),\qquad x\in R^d,\ t>0,^{1)} \tag{1.3}$$

where Δ denotes the Laplacian in x and $G(\xi)$ satisfies

$$G(0)=G(1)=0,\ G(\xi)>0 \text{ and } G'(0)>G'(\xi),\qquad 0<\xi<1. \tag{1.4}$$

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1) R^d denotes the d-dimensional Euclidian space.

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The equation (1. 3) for more general G was discussed by A. Kolmogoroff-I. Petrovsky-N. Piscounoff [9].

We first consider the problem (A). Among others, Ikeda-Nagasawa-Watanabe [4]-[7] have shown that (1. 1) can be interpreted probabilistically by means of branching Markov processes when the $q_n(x)$ are non-negative, $q_1(x)=0$ and

$$F(x;\xi)=\frac{1}{k(x)}\{\sum_{n\neq 1}q_n(x)\xi^n-\xi\},^{2)} \qquad x\in E,\ \xi\in R^1. \tag{1. 5}$$

Hence, problem (A) becomes a question of eliminating the restrictions concerning positivity of $q_n(x)$, $q_1(x)=0$ and the term $-\xi$ in the right hand side of (1. 5).

Let us next consider the following special case of (1. 1): (1) $E=R^d\cup\{\infty\}$ be the spaɔe obtained by the one-point compactification of R^d, (2) $q_0=q_1=0$ and the other q_n's are non-negative constants, (3) $\sum_{n=2}^{\infty}q_n=1$, (4) $\mathscr{G}=\frac{1}{2}\Delta$ and

$$F(\xi)=\sum_{n\neq 1}q_n\xi^n-\xi\,.$$

Then (1. 1) becomes a special case considered by Ikeda-Nagasawa-Watanabe [6], and is written as follows:

$$\frac{\partial u(t,x)}{\partial t}=\frac{1}{2}\Delta u(t,x)+F(u(t,x)), \qquad x\in R^d,\ t\geqq 0.^{3)}$$

where $F(0)=F(1)=0$. If we put $u(t,x)=1-v(t,x)$ and $G(\xi)=-F(1-\xi)$, then the above equation turns out to be the following equation

$$\frac{\partial v(t,x)}{\partial t}=\frac{1}{2}\Delta v(t,x)+G(v(t,x)), \qquad x\in R^d,\ t\geqq 0,$$

where G satisfies (1. 4). This means that problem (B) can be solved by means of a branching Markov process in the special case stated above.

Now, we shall sketch here the contents of §§ 2-8. In § 2, we shall give the notations which are used in the later discussions and give also the definitions of a branching Markov process with age and a signed branching

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[2] We can not regard here $q_1(x)=-1$ because $k(x)=\sum_{n\neq 1}q_n(x)$.

[3] The one-point compactification of R^d was used to apply the general theory and hence we omitted the point ∞ because we are interested in the equation whose variable domain is R^d.

Markov process with age after introducing extended state spaces $\hat{S}$ and $\tilde{S}$.

In §3, we shall consider a branching Markov process with age Y_t on $\hat{S}$ satisfying Condition 1 stated there. Then, for a given system $\{q_n(x); n = 0, 2, 3, \cdots\}$ of non-negative functions, $k(x) = \sum_{n \neq 1} q_n(x)$ and $F(x; \xi)$ defined by (1. 2) where $q_1(x) = 0$, we can discuss an integral equation which corresponds to the one called "S-equation" in [7]. Under certain conditions, the integral equation can be transformed into the equation of type (1. 1). In this case, $u(t, x) = T_t\widehat{f \cdot 2}(x, 0)$[4] is a solution of (1. 1) with $u(0, x) = f(x)$ if $u(t, x)$ is finite. This shows that we can eliminate the term $-\xi$ in the right hand side of (1. 4) by introducing of the notion of age. Moreover, for $G(\xi) = \xi^n$, the notion of branching Markov processes with age will serve to answer the question as to the existence of a non-trivial solution of (1. 3) which does not blow up in $[0, \infty)$. (See §6 and M. Nagasawa-T. Sirao [14].)

In §4, we shall consider a signed branching Markov process with age Z_t on $\tilde{S}$ satisfying Condition 2 which is essentially identical to Condition 1 except for the difference of branching (splitting) law caused by the difference of the state spaces $\hat{S}$ and $\tilde{S}$. After making the similar considerations as in §3, we can interpret (1. 1) probabilistically. That is to say $u(t, x) = U_t\widetilde{f \cdot 2}(x, 0, 0)$[5] is a solution of (1. 1) with $u(0, x) = f(x)$ if $u(t, x)$ is finite. This means that we can solve the problem (A) by means of signed branching Markov processes with age. (The existence of such (signed) branching Markov processes with age discussed in §§3-4 will be shown in §§7-8.)

In §5, we shall give a sufficient condition called Condition 3 in this paper which includes Condition 2 and makes a given Markov process Z_t become a signed branching Markov process with age on $\tilde{S}$. This part of the present paper, Ikeda-Nagasawa-Watanabe [7] and Nagasawa [13] overlap in some respects, because the proof of Theorem 5. 1 is essentially the same as one given in [7].

In §6, we shall consider a Markov process Z_t satisfying Condition 3 whose existence is shown in §§7-8. According to the discussions in §5, Z_t is a signed branching Markov process with age. Let f be a positive con-

4) T_t denotes the semi-group induced by Y_t, $\widehat{f \cdot 2}$ is a function of special type defined by (2. 1) and $(x, 0) \in \hat{S}$.

5) U_t denotes the semi-group induced by Z_t, $\widetilde{f \cdot 2}$ is a function of special type defined by (2. 2) and $(x, 0, 0) \in \tilde{S}$.

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tinuous function on R^d with $\|f\| = sup\{|f(x)|; x \in R^d\} < 1$ and $\alpha \in R^1$. When we consider $u(\alpha; t, x) = U_t \widetilde{\alpha f \cdot 2}(x, 0, 0)$, where U_t denotes the semi-group induced by Z_t and $x \in R^d$, $u(\alpha; t, x)$ can be expressed in the power series of α if $u(\alpha; t, x)$ is finite. But the solution of (1. 3) with initial value αf, in general, can not be expressed in the power series of α. Accordingly, if $G(\xi)$ is not an analytic function of ξ, then we can not obtain the solution of (1. 5) with initial value f directly by means of signed branching Markov processes with age as in the case of analytic G. However, if G is continuously differentiable on $[0,1]$ and satisfies the condition (1. 4), then we can express the solution $u(t, x)$ of (1. 3) with initial value f as the uniform limit of $u_n(t, x)$ in the wide sense where $u_n(t, x)$ is of the type considered in § 4, i.e. there exists a sequence of signed branching Markov processes with age $Z_t^{(n)}$ on $\tilde{\boldsymbol{S}}$ and corresponding semi-groups $U_t^{(n)}$ such that

$$u(t, x) = \lim_{n \to \infty} u_n(t, x), \qquad x \in R^d,$$

where

$$u_n(t, x) = U_t^{(n)} \widetilde{f \cdot 2}(x, 0, 0), \qquad x \in R^d, \ t \geqq 0.$$

In § 7, we shall construct a certain Markov process Y_t which will be used in § 8 in the constructions of branching Markov processes with age and signed branching Markov processes with age. We can regard this Markov process Y_t as corresponding to the creation of mass in the following sense. Let $k(x)$ be a bounded continuous function on E and consider the equation

$$\text{(1. 6)} \qquad \frac{\partial u(t, x)}{\partial t} = \mathscr{G} u(t, x) + k(x) u(t, x), \qquad x \in E, \ t \geqq 0,$$

where $\mathscr{G}$ is a infinitesimal operator of a semi-group H_t corresponding to a Markov process X_t on E. If $k(x)$ is non-positive, we can treat (1. 6) by killing X_t. So we may consider (1. 6) as the equation corresponding to the killing when $k(x)$ is non-positive. On the other hand, we may consider (1. 6) as the equation corresponding to the creation of mass when $k(x)$ is non-negative. In the theory of Markov processes, there are, as far as I know, two methods of interpreting (1. 6) when $k(x)$ is non-negative. One of them has been indicated by G.A. Hunt [3]. The other method is based on the theory of a branching Markov process, where (1. 6) appears as the mean number of particles. (cf. K.Ito - H.P.McKean [8] and Ikeda-Nagasawa-Watanabe [6] or [7].) Our method of describing the creation of mass uses age as an auxiliary variable. Let N be all the non-negative integers. We shall construct a strong Markov process $[X_t, N_t]$ on the state space $E \times N$ and consider the corresponding semi-group V_t. Then, for a given bounded continuous function f on E, $u(t, x) = V_t \widehat{f \cdot 2}(x, 0)$ is the solution of (1. 6) with the initial value f.

In § 8, we shall construct a Markov process Z_t satisfying Condition 3. Then, by the discussions in § 5, Z_t is a signed branching Markov process and the existence of the processes in §§ 3-4 is proved. We here note that the method of J.E. Moyal [10] will play an essential role in the construction of the processes dealt with in §§ 7-8.

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§ 2. **Notations and Definitions.** A branching process is one of the typical mathematical models used to describe the growth of the number of particles of a population in which each particle either produces new particles of the same character or dies out, and there is no interference among them. In order to describe the state of n particles, it seems to be natural to use the n-fold symmetric direct product space of the state space of each particle. Following [4], we here introduce some notations along this line.

Let E be a compact Hausdorff space satisfying the second axiom of countability. We denote the n-fold product space of E with itself by $E^{(n)}$ and say that $(x'_1, x'_2, \cdots, x'_n) \in E^{(n)}$ is equivalent to $(x_1, x_2, \cdots, x_n) \in E^{(n)}$ if and only if $(x'_1, x'_2, \cdots, x'_n)$ is obtainable from a permutation of $(x_1, x_2, \cdots, x_n)$. The E^n is defined as the quotient space of $E^{(n)}$ by the above equivalence relation. By the quotient topology, E^n is compact. A point $\boldsymbol{x}$ in E^n is also denoted by $[x_1, x_2, \cdots, x_n]$ as a collection of n-points $x_i \in E$ disregarding order. E^0 is considered as the set of the single point ∂, where ∂ denotes an extra point.

Let $N = \{0, 1, 2, \cdots\}$ and $N^{(n)}$ be the n-fold product space of N with itself. A point $(p'_1, p'_2, \cdots, p'_n)$ of $N^{(n)}$ is said to be equivalent to $(p_1, p_2, \cdots, p_n) \in N^{(n)}$ if $\sum_{i=1}^{n} p'_i = \sum_{i=1}^{n} p_i$.[6] The quotient space of $N^{(n)}$ by the

6) In the future discussions, $\sum_{i=1}^{n} p_i$ is essential in the role of $(p_1, p_2, \cdots, p_n)$ and hence we used here this equivalence relation.

above equivalence relation is denoted by N^n. A point $\boldsymbol{p}$ in N^n, $n \geqq 1$, is a collection of equivalent points in $N^{(n)}$ and is denoted by $[p_1, p_2, \cdots, p_n]$ if it contains $(p_1, p_2, \cdots, p_n) \in N^{(n)}$. $|\boldsymbol{p}|$ denotes $\sum_{i=1}^{n} p_i$ for $\boldsymbol{p} = [p_1, p_2, \cdots, p_n]$. Let $S = E \times N$ be the topological sum of $E \times \{p\}$, $p \in N$. Then S is a locally compact Hausdorff space satisfying the second axiom of countability. $S^{(n)}$ is defined as the n-fold product space of S with itself and $((x_1', p_1'), (x_2', p_2'), \cdots, (x_n', p_n'))$ is said to be equivalent to $((x_1, p_1), (x_2, p_2), \cdots, (x_n, p_n))$ if $[x_1', x_2', \cdots, x_n']$ is identical to $[x_1, x_2, \cdots, x_n]$ as a point of E^n and if $[p_1', p_2', \cdots, p_n']$ is identical to $[p_1, p_2, \cdots, p_n]$ as a point of N^n. S^n is defined as the quotient space by the above equivalence relation. Then S^n is locally compact with respect to the quotient topology. A point $\boldsymbol{z}$ in S^n is denoted by $[[x_1, x_2, \cdots, x_n], [p_1, p_2, \cdots, p_n]]$ or, for short, $[\boldsymbol{x}, \boldsymbol{p}]$ when $\boldsymbol{x} = [x_1, x_2, \cdots, x_n] \in E^n$ and $\boldsymbol{p} = [p_1, p_2, \cdots, p_n] \in N^n$.

Let us consider the topological sum $\bigcup_{n=0}^{\infty} S^n$ where S^0 denotes $\{\partial\} \times N$, ∂ being an extra point. This topological sum is denoted by $\boldsymbol{S}$. Then $\boldsymbol{S}$ is a locally compact and non-compact Hausdorff space satisfying the second axiom of countability. If we consider the mapping g from $\boldsymbol{S}$ to $(\bigcup_{n=0}^{\infty} E^n) \times N$ defined by

$$g([\boldsymbol{x}, \boldsymbol{p}]) = [\boldsymbol{x}, |\boldsymbol{p}|],$$

then $\boldsymbol{S}$ is isomorphic to $(\bigcup_{n=0}^{\infty} E^n) \times N$, where $\bigcup_{n=0}^{\infty} E^n$ denotes the topological sum of E^n. $\hat{\boldsymbol{S}} = \boldsymbol{S} \cup \{\Delta\}$ is defined as the space obtained by the one-point compactification of $\boldsymbol{S}$. When A and B are subsets of E and E^n respectively, the sets $A \times \{p\}$ and $B \times \{\boldsymbol{p}\}$ are denoted by $[A, p]$ and $[B, \boldsymbol{p}]$ respectively.

Let J be the set $\{0, 1, 2, 3\}$ and $\tilde{\boldsymbol{S}}$ be the topological sum of $\hat{\boldsymbol{S}} \times \{j\}$, $j \in J$. A point in $\tilde{\boldsymbol{S}}$ is denoted by $[\boldsymbol{x}, \boldsymbol{p}, j]$, $[\boldsymbol{x}, \boldsymbol{p}] \in \hat{\boldsymbol{S}}$, but $\{\Delta\} \times J$ is considered as one point and is denoted simply by Δ. Then we may consider $\tilde{\boldsymbol{S}}$ is the space obtained by the one-point compactification of $\boldsymbol{S} \times J$. For a subset $[B, \boldsymbol{p}]$ of $\hat{\boldsymbol{S}}$, the set $[B, \boldsymbol{p}] \times \{j\}$ is denoted by $[B, \boldsymbol{p}, j]$.

Now let $\mathscr{X}$ be a compact or locally compact Hausdorff space. We shall introduce the following function spaces which are supposed to be real.

$\boldsymbol{C}(\mathscr{X})$ = the set of all bounded continuous functions on $\mathscr{X}$,

$\boldsymbol{B}(\mathscr{X})$ = the set of all bounded Borel measurable functions on $\mathscr{X}$,

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$$\boldsymbol{C}^*(\mathscr{X}) = \{f;\ f \in \boldsymbol{C}(\mathscr{X}) \text{ and } \|f\| = \sup_{x \in \mathscr{X}} |f(x)| < 1\},$$

$$\boldsymbol{B}^*(\mathscr{X}) = \{f;\ f \in \boldsymbol{B}(\mathscr{X}) \text{ and } \|f\| < 1\}.$$

When $\mathscr{X}$ is a locally compact Hausdorff space, let $X = \mathscr{X} \cup \{\infty\}$ be the space obtained by the one-point compactification of $\mathscr{X}$ and set

$$\boldsymbol{C}_0(\mathscr{X}) = \{f;\ f \in \boldsymbol{C}(X),\ \lim_{x \to \infty} f(x) = 0\},$$

$$\boldsymbol{C}_0^*(\mathscr{X}) = \{f;\ f \in \boldsymbol{C}_0(\mathscr{X}),\ \|f\| < 1\}.$$

$\boldsymbol{C}_0(\mathscr{X})$ and $\boldsymbol{C}_0^*(\mathscr{X})$ are denoted by $\boldsymbol{C}_0(X)$ and $\boldsymbol{C}_0^*(X)$ occasionally. The subclass of each function space introduced above formed of all non-negative elements is denoted by "$+$", e.g. $\boldsymbol{C}(\mathscr{X})^+$, $\boldsymbol{B}(\mathscr{X})^+, \cdots$, etc. We shall denote by "$-$" the closure with respect to the norm $\|\ \ \|$, so

$$\bar{\boldsymbol{C}}^*(\mathscr{X}) = \{f;\ f \in \boldsymbol{C}(\mathscr{X}) \text{ and } \|f\| \leqq 1\},$$

$$\bar{\boldsymbol{B}}^*(\mathscr{X}) = \{f;\ f \in \boldsymbol{B}(\mathscr{X}) \text{ and } \|f\| \leqq 1\},$$

and so on.

The set of all Borel subsets of $\mathscr{X}$ is denoted by $\mathscr{B}(\mathscr{X})$.

Now we shall define several operations on functions which will play an important role in the future discussions. First of all let us define a mapping from $\boldsymbol{B}(E)$ into the space of all measurable functions on $\hat{\boldsymbol{S}}$ by

$$\widehat{f\cdot\lambda}(\boldsymbol{z}) = \begin{cases} \lambda^p, & \text{if } \boldsymbol{z} = [\partial, p] \in S^0 \\ \lambda^{|\boldsymbol{p}|} \prod_{i=1}^n f(x_i), & \text{if } \boldsymbol{z} = [\boldsymbol{x}, \boldsymbol{p}] \in S^n \text{ and } \boldsymbol{x} = [x_1, x_2, \cdots, x_n], \\ 0, & \text{if } \boldsymbol{z} = \Delta, \end{cases} \tag{2.1}$$

where $\lambda \geqq 0$. If $f \in \boldsymbol{C}^*(E)$ and $0 \leqq \lambda < 1$, then $\widehat{f\cdot\lambda} \in \boldsymbol{C}_0^*(\hat{\boldsymbol{S}})$, while $\widehat{f\cdot\lambda}$ is unbounded for $\lambda > 1$.

Next we shall define a mapping $\sim$ from $\boldsymbol{B}(E)$ into the space of all measurable functions on $\tilde{\boldsymbol{S}}$ by

$$\widetilde{f\cdot\lambda}([\boldsymbol{x}, \boldsymbol{p}, j]) = (-1)^{\left[\frac{j}{2}\right]} \widehat{f\cdot\lambda}([\boldsymbol{x}, \boldsymbol{p}]), \qquad [\boldsymbol{x}, \boldsymbol{p}, j] \in \tilde{\boldsymbol{S}}, \tag{2.2}$$

where $\lambda \geqq 0$ and $[\ \ ]$ denotes Gauss' symbol.

For any function g on $\hat{\boldsymbol{S}}$, we define a function $g|_E$ on E by

$$g|_E(x) = g([x, 0]), \qquad x \in E. \tag{2.3}$$

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We define also $h|_E$ for any function on $\tilde{\boldsymbol{S}}$ by

(2. 4) $$h|_E(x) = h(x, 0, 0), \qquad x \in E.$$

Remark 1. Let $(\bigcup_{n=0}^{\infty} E^n) \cup \{\Delta\}$ be the space obtained by the one-point compactification of $\bigcup_{n=0}^{\infty} E^n$. Ikeda-Nagasawa-Watanabe [4]-[7], used a mapping $\widehat{\ }$ from $\boldsymbol{B}(E)$ into the space of all Borel measurable functions on $(\bigcup_{n=0}^{\infty} E^n) \cup \{\Delta\}$ defined by

(2. 5) $$\hat{f}(\boldsymbol{x}) = \begin{cases} 1 & , \text{ if } \boldsymbol{x} = \partial, \\ \prod_{i=1}^{n} f(x_i), & \text{ if } \boldsymbol{x} = [x_1, x_2, \cdots, x_n] \in E^n, \\ 0 & , \text{ if } \boldsymbol{x} = \Delta. \end{cases}$$

Then the linear hull of the set $\{\hat{f}; f \in \boldsymbol{C}^*(E)^+\}$ is dense in $\boldsymbol{C}_0(\bigcup_{n=1}^{\infty} E^n)$ (cf. Lemma 1. 4 in [7]). Accordingly, the linear hull of the set $\{\widehat{f \cdot \lambda}; f \in \boldsymbol{C}^*(E)^+, 0 \leqq \lambda < 1\}$ is dense in $\boldsymbol{C}_0(\hat{\boldsymbol{S}})$ because the linear hull of $\{\lambda; \lambda(p) = \lambda^p, 0 \leqq \lambda < 1, p \in N\}$ is also dense in $\boldsymbol{C}_0(N)$.

Comparing the two mappings $\widehat{\ }$ defined by (2. 1) and (2. 5), we have

$$\widehat{f \cdot 1}([\boldsymbol{x}, \boldsymbol{p}]) = \hat{f}([\boldsymbol{x}]).$$

So we need not distinguish between $\widehat{f \cdot 1}$ and $\hat{f}$ if there arises no danger of confusion.

Now we shall consider a Markov process $X = \{X_t, \zeta, \mathscr{B}_t, P_x; x \in \mathscr{X}\}$ on $\mathscr{X}$. Let $\mathscr{B}_\infty$ be the smallest σ-algebra which contains all elements of $\mathscr{B}_t$ for any $t \geqq 0$. A non-negative random variable τ is said to be a ($\mathscr{B}_t$-) Markov time if

$$\{w; \tau(w) \leqq t\} \in \mathscr{B}_t,$$

for any $t \geqq 0$. For each Markov time τ, we set

$$\mathscr{B}_\tau = \{A; A \in \mathscr{B}_\infty \text{ and } A \cap \{w; \tau(w) \leqq t\} \in \mathscr{B}_t \text{ for any } t \geqq 0\}.$$

Then it is easy to see that $\mathscr{B}_\tau$ is a σ-algebra. A measurable Markov process X is called a strong Markov process if for any Markov time τ and for any $t \geqq 0$, $x \in \mathscr{X}$, $f \in \boldsymbol{B}(\mathscr{X})$ and $A \in \mathscr{B}_\tau$

$$E_x[f(X_{t+\tau}); A \cap \{\tau < \zeta\}] = E_x[E_{X_\tau}[f(X_t)]; A \cap \{\tau < \zeta\}],$$

where E_x denotes the integral by P_x.

In this paper, with the exception of §§ 7-8, we shall assume that each sample function of a Markov process is right continuous in t and has its left limit at any $t>0$. We also use the same letter ζ for the terminal times of different Markov processes X and Y and the same letter $\mathscr{B}_t$ for the corresponding σ-algebras which make X_t or Y_t measurable if there arises no danger of confusion.

Let $Y=\{Y_t=[X_t, N_t], \zeta, \mathscr{B}_t, P_{[\boldsymbol{x},\boldsymbol{p}]}; [\boldsymbol{x},\boldsymbol{p}]\in\hat{\boldsymbol{S}}\}$ be a strong Markov process on $\hat{\boldsymbol{S}}$, where $[X_t(w), N_t(w)]=[\boldsymbol{x},\boldsymbol{p}]$ means $X_t(w)=\boldsymbol{x}$ and $N_t(w)=\boldsymbol{p}$. We shall define the functionals of Y by

$$\xi_t(w)=\begin{cases} n, & \text{if } [X_t(w),\ N_t(w)]\in S^n,\quad n\geqq 0\\ \infty, & \text{if } [X_t(w),\ N_t(w)]=\Delta,\end{cases}$$

(2. 6)
$$\begin{aligned}
&\tau(w)=\inf\{t>0;\ \xi_t(w)\neq\xi_0(w)\ \text{or}\ \sup_{s\leqq t}|N_s(w)|=\infty\}^{7)},\\
&\sigma(w)=\inf\{t<\tau(w);\ |N_t(w)|\neq|N_0(w)|\},\\
&\tau_0(w)=0,\ \ \tau_1(w)=\tau(w),\ \ \tau_{n+1}(w)=\tau_n(w)+\theta_{\tau_n}\tau(w),\ \ (n\geqq 1),\\
&\sigma_0(w)=0,\ \ \sigma_1(w)=\sigma(w)\ \text{and}\ \sigma_{n+1}(w)=\sigma_n(w)+\theta_{\sigma_n}\sigma(w),\ \ (n\geqq 1),
\end{aligned}$$

where θ denotes the shift operator (cf. E.B. Dynkin [1]).

Further let $Z=\{Z_t=[X_t, N_t, J_t], \zeta, \mathscr{B}_t, P_{[\boldsymbol{x},\boldsymbol{p},j]}; [\boldsymbol{x},\boldsymbol{p},j]\in\tilde{\boldsymbol{S}}\}$ be a strong Markov process on $\tilde{\boldsymbol{S}}$, where $[X_t(w), N_t(w), J_t(w)]=[\boldsymbol{x},\boldsymbol{p},j]$ means $X_t(w)=\boldsymbol{x}$, $N_t(w)=\boldsymbol{p}$ and $J_t(w)=j$. Then we define the functionals of Z by

(2. 7)
$$\begin{aligned}
&\eta(w)=\inf\{t>0;\ J_t(w)\neq J_0(w)\ \text{or}\ \sup_{s\leqq t}|N_s|(w)|=\infty\},\\
&\sigma(w)=\inf\{t<\eta(w);\ |N_t(w)|\neq|N_0(w)|\},\\
&\eta_0(w)=0,\ \ \eta_1(w)=\eta(w),\ \ \eta_{n+1}(w)=\eta_n(w)+\theta_{\eta_n}\eta(w),\ \ (n\geqq 1),\\
&\sigma_0(w)=0,\ \ \sigma_1(w)=\sigma(w)\ \text{and}\ \sigma_{n+1}(w)=\sigma_n(w)+\theta_{\sigma_n}\sigma(w),\ \ (n\geqq 1).^{8)}
\end{aligned}$$

Evidently τ_n, η_n and σ_n are Markov times.

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7) In this paper, we regard that inf $\phi=\infty$ where ϕ denotes the empty set.

8) Two functional σ's defined for Y and Z are denoted by the same letter because the definitions are identical except for the conditions $t<\tau$ and $t<\eta$, and this notation is convenient for the later use. Also, Y and Z are different Markov processes on the different state space $\hat{S}$ and $\tilde{S}$, and accordingly there arises no danger of confusion when we use the same letter σ.

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Now we shall give here the definitions of a branching Markov process with age on $\hat{S}$ and a signed branching Markov process with age on $\tilde{S}$.

DEFINITION 2. 1. A strong Markov process $Y = \{Y_t = [X_t, N_t], \zeta, \mathscr{B}_t, P_{[x,p]}; [x, p] \in \hat{S}\}$ is said to be *a branching Markov process with age*, if the semi-group $\{T_t; t \geqq 0\}$ on $\boldsymbol{B}(\hat{S})$ induced by Y satisfies

$$T_t \widehat{f \cdot \lambda} = \widehat{(T_t \widehat{f \cdot \lambda})|_E \cdot \lambda}, \qquad f \in \boldsymbol{C}^*(E), \tag{2. 8}$$

where $t \geqq 0$ and $0 \leqq \lambda < 1$.

DEFINITION 2. 2. A strong Markov process $Z = \{Z_t = [X_t, N_t, J_t], \zeta, \mathscr{B}_t, P_{[x,p,j]}; [x, p, j] \in \tilde{S}\}$ is said to be *a signed branching Markov process with age*, if the semi-group $\{U_t; t \geqq 0\}$ on $\boldsymbol{B}(\tilde{S})$ induced by Z satisfies

$$U_t \widetilde{f \cdot \lambda} = \widetilde{(U_t \widetilde{f \cdot \lambda})|_E \cdot \lambda}, \qquad f \in \boldsymbol{C}^*(E), \tag{2. 9}$$

where $t \geqq 0$ and $0 \leqq \lambda < 1$.

In both processes Y and Z, $|N_t|$ is considered as the total age of the particles and hence σ_n is called the nth jumping time of age N_t. τ_n and η_n are called the nth branching times of Y and Z respectively.

Remark 2. As was mentioned already, the linear hull of $\{\widehat{f \cdot \lambda}; f \in \boldsymbol{C}^*(E), 0 \leqq \lambda < 1\}$ is dense in $\boldsymbol{C}_0(\hat{S})$. Hence the process on $\hat{S}$ is uniquely determined by the values of $T_t \widehat{f \cdot \lambda}$ considered in (2. 8). But, unfortunately, the same unique insistence does not hold for the case of U_t (cf. Remark 2 in § 4).

Remark 3. When $Y_t = [X_t, N_t]$ is a branching Markov process with age, $[\partial, p]$, $p \in N$, and Δ are traps because

$$T_t \widehat{f \cdot \lambda}(\partial, p) = \widehat{(T_t \widehat{f \cdot \lambda})|_E \cdot \lambda}(\partial, p) = \lambda^p, \qquad p \in N,$$

and

$$T_t \widehat{f \cdot \lambda}(\Delta) = \widehat{(T_t \widehat{f \cdot \lambda})|_E \cdot \lambda}(\Delta) = 0,$$

for any $f \in \boldsymbol{C}^*(E)^+$ and $0 \leqq \lambda < 1$.[9)]

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9) cf. [6], Theorem 2.1.

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Remark 4. For any bounded continuous function f on E, $\alpha f \in \boldsymbol{C}^*(E)$ if $|\alpha| < 1/\|f\|$. So, if (2. 8) holds then we have for $f \in \boldsymbol{C}(E) - \boldsymbol{C}^*(E)$

$$T_t\widehat{\alpha f \cdot \lambda} = \widehat{(T_t\widehat{\alpha f \cdot \lambda})|_E} \cdot \lambda, \qquad |\alpha| < 1/\|f\|. \tag{2. 10}$$

On the other hand, both sides of the above equation can be expressed in the power series of α. So, if we put

$$T_t\widehat{\alpha f \cdot \lambda}([\boldsymbol{x}, \boldsymbol{p}]) = \sum_{n=0}^{\infty} E_{[\boldsymbol{x}, \boldsymbol{p}]}[\widehat{\alpha f \cdot \lambda}(Y_t)\,;\, Y_t \in S^n] = \sum_{n=0}^{\infty} a_n([\boldsymbol{x}, \boldsymbol{p}])\alpha^n$$

and

$$\widehat{(T_t\widehat{\alpha f \cdot \lambda})|_E} \cdot \lambda([\boldsymbol{x}, \boldsymbol{p}]) = \sum_{n=0}^{\infty} b_n([\boldsymbol{x}, \boldsymbol{p}])\alpha^n,$$

then (2. 10) shows that

$$a_n([\boldsymbol{x}, \boldsymbol{p}]) = b_n([\boldsymbol{x}, \boldsymbol{p}]), \qquad n = 0, 1, 2, \cdots. \tag{2. 11}$$

Since the finiteness of $T_t\widehat{f \cdot \lambda}$[10] and $\widehat{(T_t\widehat{f \cdot \lambda})|_E} \cdot \lambda$ implies that

$$\sum_{n=0}^{\infty} |a_n([x, p])| < \infty, \quad \sum_{n=0}^{\infty} |b_n([x, p])| < \infty$$

and

$$T_t\widehat{f \cdot \lambda}([\boldsymbol{x}, \boldsymbol{p}]) = \sum_{n=0}^{\infty} a_n([\boldsymbol{x}, \boldsymbol{p}]),$$

$$\widehat{(T_t\widehat{f \cdot \lambda})|_E} \cdot \lambda([\boldsymbol{x}, \boldsymbol{p}]) = \sum_{n=0}^{\infty} b_n([\boldsymbol{x}, \boldsymbol{p}]),$$

we have from (2. 11)

$$T_t\widehat{f \cdot \lambda} = \widehat{(T_t\widehat{f \cdot \lambda})|_E} \cdot \lambda, \qquad f \in \boldsymbol{C}(E), \ 0 \leqq \lambda < 1,$$

if both sides of the above equation are finite. By the same way, we may consider that if U_t satisfies (2. 9) and both sides of the following equation are finite then we have

---

[10] For any semi-group T_t on $\boldsymbol{B}(\mathscr{X})$ induced by a Markov process X_t on $\mathscr{X}$, we denote $E_x[f(X_t)]$ by $T_t f(x)$ in this paper even if f is unbounded but $E_x[|f(X_t)|]$ is finite. (E_x denotes the integral by the probability measure P_x of X_t.)

$$\widetilde{U_t f} \cdot \lambda = (\widetilde{U_t \widetilde{f} \cdot \lambda})|_E \cdot \lambda, \qquad f \in \boldsymbol{C}(E).$$

Similarly, we can see that

$$\widehat{T_t f} \cdot \lambda = (\widehat{T_t \widehat{f} \cdot \lambda})|_E \cdot \lambda, \qquad f \in \boldsymbol{C}(E),\ 0 \leqq \lambda, \tag{2. 12}$$

and

$$\widetilde{U_t f} \cdot \lambda = (\widetilde{U_t \widetilde{f} \cdot \lambda})|_E \cdot \lambda, \qquad f \in \boldsymbol{C}(E),\ 0 \leqq \lambda, \tag{2. 13}$$

if each member of (2. 12) and (2. 13) is finite.

§ 3. Branching Markov process with age. In this section, we restrict our attention to a branching Markov process with age $Y=\{Y_t=[X_t, N_t], \zeta, \mathscr{B}_t, P_{[x,p]}; [\boldsymbol{x}, \boldsymbol{p}] \in \hat{\boldsymbol{S}}\}$ satisfying the following condition (Condition 1), because it is sufficient to consider such a process for the probabilistic interpretation of equations of type (1. 1).

Let $\{q_n(x); n = 0, 2, 3, \cdots\}$ be a given system of bounded continuous and non-negative functions on E, let $k(x) = \sum_{n \neq 1} q_n(x)$ also be a non-negative bounded continuous function on E, and set

$$\pi([x, p]; [B, \boldsymbol{q}]) = \sum_{n \neq 1} \frac{q_n(x)}{k(x)} \delta_n([\boldsymbol{x}, \boldsymbol{p}], [B, \boldsymbol{q}]), \tag{3. 1}$$
$$[x, p] \in S,\ [B, \boldsymbol{q}] \in \mathscr{B}(\hat{\boldsymbol{S}}),$$

where δ_n is defined by

$$\delta_n([x, p], [B, \boldsymbol{q}]) = \begin{cases} 1, & \text{if } \boldsymbol{x} = [x, x, \cdots, x] \in B \cap E^n,\ |\boldsymbol{q}| = p,\ n \neq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Now we shall state the following

Condition 1. (i)

$$P_{[x,p]}(X_t \in A,\ N_t = p + q,\ t < \tau) = P_{[x,0]}(X_t \in A,\ N_t = q,\ t < \tau), \tag{3. 2}$$
$$[x, p] \in S,\ q \in N,\ A \in \mathscr{B}(E).$$

(ii) There exists a conservative Feller process[11] $X' = \{X'_t, \mathscr{B}'_t, P_x; x \in E\}$ on E such that

11) A right continuous strong Markov process on $\mathscr{X}$ is said to be a Feller process if the corresponding semi-group T_t maps $\boldsymbol{C}(\mathscr{X})$ into itself.

(3. 3)
$$P_{[x,0]}(X_{\tau-}\in A,\ \tau\in dt,\ \sigma_n\leqq t<\sigma_{n+1})$$
$$=E_x[e^{-2\int_0^t k(X'_s)ds}\frac{\left(\int_0^t k(X'_s)\,ds\right)^n}{n!}k(X'_t)I_A(X'_t)dt]$$

and

(3. 4)
$$P_{[x,0]}(X_{\sigma-}\in A,\ \sigma\in dt)=E_x[e^{-2\int_0^t k(X'_s)ds}k(X'_t)I_A(X'_t)dt],$$
$$[x,0]\in S,\ x\in E,\ A\in\mathscr{B}(E),$$

where E_x denotes the integral by P_x and I_A denotes the indicator function of A.

(iii) For any $\alpha>0$,

(3. 5)
$$E_{[x,p]}[e^{-\alpha\tau};[X_\tau,N_\tau]\in[B,\boldsymbol{q}]]$$
$$=E_{[x,p]}[e^{-\alpha\tau}\pi([X_{\tau-},\ N_{\tau-}];[B,\boldsymbol{q}])],$$
$$[x,p]\in S,\ [B,\boldsymbol{q}]\in\mathscr{B}(\hat{\boldsymbol{S}}),$$

and

(3. 6)
$$E_{[x,0]}[e^{-\alpha\sigma};[X_\sigma,N_\sigma]\in[A,q]]$$
$$=E_{[x,0]}[e^{-\alpha\sigma}\delta_{[X_{\sigma-},N_{\sigma-}+1]}([A,q])],$$
$$[x,0]\in S,\ A\in\mathscr{B}(E),\ q\in N,$$

where $E_{[x,p]}$ denotes the integral by $P_{[x,p]}$, π is given in (3. 1) and

$$\delta_{[x,p]}([A,q])=\begin{cases}1, & \text{if } x\in A,\ p=q,\\ 0, & \text{otherwise.}\end{cases}$$

For the process X' considered in (ii), we give the following

DEFINITION 3. 1. The process X' is called *the basic Markov process of* Y.

In the following, we consider the process $Y_t^0=[X_t^0,N_t^0]$ defined in the following way:

$$Y_t^0(w)=\begin{cases}Y_t(w), & \text{if } t<\tau(w)\\ \Delta\ , & \text{if } t\geqq\tau(w).\end{cases}$$

The probability measure for Y_t^0 is denoted by $P_{[x,p]}^0$ and the integral by

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$P^0_{[x,p]}$ is denoted by $E^0_{[x,p]}$. The semi-group on $\boldsymbol{B}(\hat{\boldsymbol{S}})$ induced by Y^0_t is denoted by T^0_t. Accordingly, we have for any $g \in \boldsymbol{B}(\hat{\boldsymbol{S}})$ with $g(\Delta)=0$

$$T^0_t g([x,p]) = E^0_{[x,p]}[g(Y^0_t)] = E_{[x,p]}[g(Y_t);\ t<\tau],$$
$$[x,p]\in \hat{S}.$$

DEFINITION 3. 2. When we restrict the starting points of Y^0_t on S, Y^0_t is called *the non-branching part of* Y.

Now we shall return to the discussion of Condition 1. (i) of the condition states that if we consider the process $[X^0_t, N^0_t - p]$ for the non-branching part $Y^0_t=[X^0_t, N^0_t]$ started from $[x,p]\in S$, then $[X^0_t, N^0_t - p]$ is stochastically equivalent to the one started from $[x,0]\in S$. (ii) of the condition states the relation between the first branching time τ and the nth jumping time σ_n of N_t. This condition holds if we consider a process such that (a) if we set, for $Y_t=[X_t,N_t]$ starting from $[x,0]\in S$,

$$X^0_t = \begin{cases} X_t, & \text{if } t<\sigma(w)\wedge\tau(w)^{12)} \\ \Delta, & \text{if } t\geqq\sigma(w)\wedge\tau(w), \end{cases}$$

then X^0_t is stochastically equivalent to the $\exp\left(-2\int_0^t k(X'_s)ds\right)$ sub-process of X' as a process on E, (b) each path of Y_t jumps from $[X_{\sigma(w)\wedge\tau(w)-}, 0]$ to either one of $[X_{\sigma(w)\wedge\tau(w)-}, 1]$ or some point in $\hat{S}-S$ at the time $\sigma\wedge\tau$ with probability 1/2. On the other hand (iii) states the branching law at the first branching time τ and the jumping law at the first jumping time σ of N_t. (We shall show in §8 that there exists a branching Markov process with age on $\hat{S}$ which satisfies Condition 1.) Moreover, if we combine (iii) with the stochastic equivalence of $[X^0_t, N^0_t - p]$ where $N^0_0=p$ and the non-branching part where $N^0_0=0$, then the strong Markov property of Y yields that for any $n\geqq 1$

$$P_{[x,p]}(N_{\sigma_n}\neq N_{\sigma_{n-1}}+1,\ \sigma_n<\tau)=0$$

or for any $n\geqq 1$ and $C\in\mathscr{B}_{\sigma_n}$,

$$P_{[x,p]}(C, N_{\sigma_n}=N_{\sigma_{n-1}}+1,\ \sigma_n<\tau) = P_{[x,p]}(C,\sigma_n<\tau),\quad [x,p]\in S, \tag{3. 7}$$

$$\begin{aligned} &E_{[x,p]}[e^{-\alpha\sigma};\ [X_\sigma,N_\sigma]\in[A,q]] \\ &= E_{[x,p]}[e^{-\alpha\sigma}\delta_{[X_{\sigma-},N_{\sigma-}+1]}([A,q])],\quad \alpha>0,\ [x,p]\in S,\ A\in\mathscr{B}(E), \end{aligned}\tag{3. 8}$$

12) $\sigma(w)\wedge\tau(w)$ denotes the minimum of $\sigma(w)$ and $\tau(w)$.

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and

$$(3.9)\qquad P_{[x,p]}([X_\tau, N_\tau] \in [B, \boldsymbol{p}+\boldsymbol{q}]) = P_{[x,0]}([X_\tau, N_\tau] \in [B, \boldsymbol{q}]),$$

$$[x,p] \in S,\ B \in \mathscr{B}(\bigcup_{n=0}^{\infty} E^n),$$

where $\boldsymbol{p}+\boldsymbol{q}$ denotes $[p+q_1, q_2, \cdots, q_n]$ for $\boldsymbol{q} = [q_1, q_2, \cdots, q_n]$.

We have also from (ii) and the stochastic equivalence of $[X_t^0, N_t^0 - N_0^0]$ stated above

$$(3.10)\qquad \begin{aligned} &P_{[x,p]}(X_{\tau-} \in A,\ \tau \in dt,\ \sigma_n \leqq t < \sigma_{n+1}) \\ &= E_x[e^{-2\int_0^t k(X'_s)ds} \frac{\left(\int_0^t k(X'_s)\,ds\right)^n}{n!} k(X'_s) I_A(X'_t)dt] \end{aligned}$$

and

$$(3.11)\qquad \begin{aligned} &P_{[x,p]}(X_{\sigma_{n+1}-} \in A,\ \sigma_{n+1} \in dt) \\ &= E_x[e^{-2\int_0^t k(X'_s)ds} \frac{\left(\int_0^t k(X'_s)ds\right)^n}{n!} k(X'_t) I_A(X'_t)dt], \end{aligned}$$

$$n \geqq 0,\ [x,p] \in S,\ A \in \mathscr{B}(E).$$

Now we shall consider a family of measures $K([x,0]; \cdot, \cdot)$ on $\mathscr{B}([0,\infty)\times S)$ defined as follows: let Y_t^0 be the non-branching part of Y_t and set

$$(3.12)\qquad K([x,0]; dt,\ [A,p]) = P^0_{[x,0]}(\tau \in dt,\ Y^0_{\tau-} \in [A,p]),$$

$$[x,0] \in S,\ A \in \mathscr{B}(E),\ p \in N.$$

Evidently $K([x,0]; \cdot, \cdot)$ is a measure on $\mathscr{B}([0,\infty) \times S)$. Moreover, by (3.7) and (3.10), $K([x,0]; \cdot, \cdot)$ can be expressed in the following form:

$$(3.13)\qquad K([x,0]; dt,\ [A,p]) = E_x[e^{-2\int_0^t k(X'_s)ds} \frac{\left(\int_0^t k(X'_s)ds\right)^p}{p!} k(X'_t) I_A(X'_t)dt],$$

$$p \in N,\ A \in \mathscr{B}(E).$$

Further let T_t^0 be the semi-group on $\boldsymbol{B}(S)$ induced by Y_t^0 and F be a function defined by

$$(3.14)\qquad F(x;\xi) = \sum_{n\neq 1} \frac{q_n(x)}{k(x)} \xi^n, \qquad x \in E,\ \xi \in R^1,$$

where $q_n(x)$ and $k(x)$ are functions considered in Condition 1 (or in (3.1)).

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For a given system (T_t^0, K, F), consider the following equation:

$$u(t,x) = T_t^0\widehat{f\cdot\lambda}([x,0]) + \int_0^t\int_S K([x,0];ds,\ [dy,p])\lambda^p F(y;u(t-s,y)), \tag{3. 15}$$
$$f\in \boldsymbol{C}(E),\ 0\leqq\lambda,\ 0\leqq t\leqq T,\ x\in E,$$

where T is a positive constant.

Then we have

LEMMA 3. 1. *Let T_t be the semi-group on $\boldsymbol{B}(\hat{\boldsymbol{S}})$ induced by a branching Markov process with age Y_t on $\hat{\boldsymbol{S}}$ satisfying Condition* 1 *and let T_t^0 be the semi-group on $\boldsymbol{B}(\boldsymbol{S})$ induced by the non-branching part Y_t^0 of Y_t. Let also f be a bounded continuous function on E. If $u(t,x)=(T_t\widehat{f\cdot\lambda})|_E(x)$ is finite for any $x\in E$ and $0\leqq t\leqq T$, then $u(t,x)$ satisfies* (3. 15).

REMARK 1. With the exception of §6, "$(T_t\widehat{f\cdot\lambda})|_E(x)$ *is finite*" means in this paper that

$$E_{[x,0]}[|\widehat{f\cdot\lambda}|(Y_t)]<\infty.$$

(cf. Foot-note 10)). Let us set for any Borel measurable function g on $\hat{\boldsymbol{S}}$

$$g_n([\boldsymbol{x},\boldsymbol{p}]) = \begin{cases} g([\boldsymbol{x},\boldsymbol{p}]), & \text{if } |g([\boldsymbol{x},\boldsymbol{p}])|\leqq n, \\ 0, & \text{otherwise.}\end{cases}$$

If it holds that

$$E_{[\boldsymbol{x},\boldsymbol{p}]}(|g(Y_t)|)<\infty, \tag{3. 16}$$

then, by the strong Markov property of Y_t, we have for any Markov time σ

$$\begin{aligned} & E_{[\boldsymbol{x},\boldsymbol{p}]}[|g(Y_t)|;\sigma<t] \\ &= \lim_{n\to\infty} E_{[\boldsymbol{x},\boldsymbol{p}]}[|g_n(Y_t)|;\sigma<t] \\ &= \lim_{n\to\infty} E_{[\boldsymbol{x},\boldsymbol{p}]}[T_{t-\sigma}|g_n|(Y_\sigma);\sigma<t]^{13)} \\ &= E_{[\boldsymbol{x},\boldsymbol{p}]}[T_{t-\sigma}|g|(Y_\sigma);\sigma<t], \end{aligned}$$

Hence, if (3. 16) holds, then we have

$$E_{[\boldsymbol{x},\boldsymbol{p}]}[|g(Y_t)|;\sigma<t]<\infty$$

---

13) $T_{t-\sigma}g(Y_\sigma)$ or $E_{Y_\sigma}[g(Y_{t-\sigma})]$ denote $E_{Y_\sigma}[g(Y_{t-s})]$ at $s=\sigma$.

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and

$$E_{[x,p]}[g(Y_t);\sigma<t]=E_{[x,p]}[E_{[X_\sigma,N_\sigma]}[g(Y_{t-\sigma})];\sigma<t]. \tag{3.17}$$

Proof of Lemma 3.1. By the strong Markov property of Y_t and (3.17), it holds that

$$\begin{aligned}u(t,x)&=E_{[x,0]}[\widehat{f\cdot\lambda}(Y_t);t<\tau]+E_{[x,0]}[\widehat{f\cdot\lambda}(Y_t);\tau\leqq t]\\&=T_t^0\widehat{f\cdot\lambda}([x,0])+E_{[x,0]}[E_{[X_\tau,N_\tau]}\widehat{[f\cdot\lambda}(Y_{t-\tau})];\tau\leqq t].\end{aligned} \tag{3.18}$$

If we apply the branching property (2.12) to the second term of the right hand side of (3.18), then we can see

$$\begin{aligned}E_{[X_\tau,N_\tau]}[\widehat{f\cdot\lambda}(Y_{t-\tau})]&=T_{t-\tau}\widehat{f\cdot\lambda}([X_\tau,N_\tau])\\&=\widehat{(T_{t-\tau}\widehat{f\cdot\lambda})|_E\cdot\lambda}([X_\tau,N_\tau])\\&=\lambda^{|N_\tau|}\hat{u}(t-\tau,\cdot)(X_\tau)^{14)}.\end{aligned}$$

Combining (3.1), (3.5), (3.12) with the above equation, we have

$$\begin{aligned}&E_{[x,0]}[E_{[X_\tau,N_\tau]}[\widehat{f\cdot\lambda}(Y_{t-\tau})];\tau\leqq t]\\&=\int_0^t\int_{\hat{S}}P_{[x,0]}(\tau\in ds,\ X_\tau\in d\boldsymbol{y},\ N_\tau=\boldsymbol{p})\lambda^{|\boldsymbol{p}|}\hat{u}(t-s,\cdot)(\boldsymbol{y})\\&=\int_0^t\int_S K([x,0];ds,[dy,p])\lambda^p\sum_{n\neq1}\frac{q_n(y)}{k(x)}u(t-s,y)^n\\&=\int_0^t\int_S K([x,0];ds,\ [dy,p])\lambda^pF(y;u(t-s,y)).\end{aligned} \tag{3.19}$$

Now (3.18) and (3.19) prove the lemma. Q.E.D.

Next, we shall prove

LEMMA 3.2. *Let T_t^0 be the semi-group on $\boldsymbol{B}(S)$ induced by the non-branching part Y_t^0 of a branching Markov process with age Y_t satisfying Condition* 1 *and let H_t be the semi-group on $\boldsymbol{B}(E)$ induced by the basic Markov process X_t' of Y_t. Then we have*

$$T_t^0\widehat{f\cdot2}([x,0])=H_tf(x),\qquad f\in\boldsymbol{C}(E),\ x\in E. \tag{3.20}$$

Proof. Using (3.7) and (3.11), we can see that

14) cf. (2.5).

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$$P^0_{[x,0]}(X^0_t \in A,\ N^0_t = p) = P_{[x,0]}(X_t \in A,\ \sigma_p \leqq t < \sigma_{p+1} \wedge \tau)$$

$$= E_x[e^{-2\int_0^t k(X'_s)ds} \frac{\left(\int_0^t k(X'_s)ds\right)^p}{p!} I_A(X'_t)],$$

for any $A \in \mathscr{B}(E)$. Consequently, we have

$$T^0_t \widehat{f \cdot \lambda}([x,0]) = \sum_{p=0}^{\infty} E_{[x,0]}[\lambda^p f(X_t);\ N^0_t = p]$$

$$= \sum_{p=0}^{\infty} E_x[e^{-2\int_0^t k(X'_s)ds} \frac{\left(\lambda \int_0^t k(X'_s)ds\right)^p}{p!} f(X'_t)] \tag{3. 21}$$

$$= E_x[e^{-(2-\lambda)\int_0^t k(X'_s)ds} f(X'_t)], \qquad f \in \boldsymbol{C}(E).$$

If we put $\lambda = 2$, then (3. 20) follows from (3. 21) immediately. Q.E.D.

Now let H_t be the semi-group on $\boldsymbol{B}(E)$ and F be the function given in (3. 14). For a given system (H_t, k, F), consider the following equation:

$$u(t,x) = H_t f(x) + \int_0^t H_s(k(\cdot)F(\cdot\ ;\ u(t-s,\cdot)))(x)ds, \tag{3. 22}$$

$$f \in \boldsymbol{C}(E),\ x \in E,\ 0 \leqq t \leqq T,$$

where T is a positive constant.

Then we have

THEOREM 3. 1. *Let T_t be the semi-group on $\boldsymbol{B}(\hat{\boldsymbol{S}})$ induced by a branching Markov process with age Y_t on $\hat{\boldsymbol{S}}$ satisfying Condition* 1 *and let H_t be the semi-group on $\boldsymbol{B}(E)$ induced by the basic Markov process X'_t of Y_t. Further for, $f \in \boldsymbol{C}(E)$, set*

$$u(t,x) = (T_t \widehat{f \cdot 2})|_E(x), \qquad x \in E.$$

If $u(t,x)$ is finite for any $x \in E$ and $0 \leqq t \leqq T$, then it satisfies (3. 22).

Proof. It follows from (3. 13) that

$$\int_0^t \int_S K([x,0];\ ds,\ [dy,p]) 2^p F(y\ ;\ u(t-s,y))$$

$$= \int_0^t \sum_{p=0}^{\infty} 2^p E_x[e^{-2\int_0^s k(X'_v)dv} \frac{\left(\int_0^s k(X'_v)dv\right)^p}{p!} k(X'_s)F(X'_s;\ u(t-s,X'_s))]ds$$

$$= \int_0^t E_x[k(X'_s)F(X'_s;\ u(t-s,X'_s))]ds$$

$$= \int_0^t H_s(k(\cdot)F(\cdot\ ;\ u(t-s,\cdot)))(x)ds.$$

Then the theorem follows from Lemma 3. 1, Lemma 3. 2 and the above equation. Q.E.D.

Remark 2. For any $f \in C^*(E)$, there exists $\varepsilon > 0$ depending on $\|f\|$ such that $(\widehat{T_t f \cdot 2})|_E(x)$ is finite for $t \in [0, \varepsilon)$ (cf. Nagasawa [13], Proposition 5. 16 and also, for special cases, see Lemma 6. 1).

§ 4. **Signed branching Markov process with age.** We have considered the case where $q_n(x) \geqq 0$ and $q_1(x) = 0$ in the last section. In this section, we shall remove such restrictions.

Let $\{(q_n^+(x), q_n^-(x)); n = 0, 1, 2, \cdots\}$ be a system of pairs of non-negative bounded continuous functions on E such that

$$q_n^+(x) q_n^-(x) = 0, \qquad n = 0, 1, 2, \cdots.$$

Further let $k(x)$ defined by

$$k(x) = \sum_{n=0}^{\infty} \{q_n^+(x) + q_n^-(x)\}$$

be a non-negative bounded continuous function on E. Then we shall define the system $\{\pi(\cdot, \cdot)\}$ by

$$\begin{aligned}
&\pi([x, p, 0], [B, q, 1]) = \pi([x, p, 1], [B, q, 0]) \\
&= \pi([x, p, 2], [B, q, 3]) = \pi([x, p, 3], [B, q, 2]) \\
&= \sum_{n=0}^{\infty} \frac{q_n^+(x)}{k(x)} \delta_n([x, p], [B, q]), \\
&\pi([x, p, 0], [B, q, 3]) = \pi([x, p, 3], [B, q, 0]) \\
&= \pi([x, p, 1], [B, q, 2]) = \pi([x, p, 2], [B, q, 1]) \\
&= \sum_{n=0}^{\infty} \frac{q_n^-(x)}{k(x)} \delta_n([x, p], [B, q]), \\
&\pi([x, p, j], [B, q, j']) = 0 \quad \text{for the other pairs of } (j, j'), \\
&\qquad [x, p] \in S, \ j, j' \in J, \ [B, q] \in \mathscr{B}(\hat{S}),
\end{aligned} \tag{4. 1}$$

where δ_n is defined by

$$\delta_n([x, p], [B, q]) = \begin{cases} 1, & \text{if } \boldsymbol{x} = [x, x, \cdots, x] \in B \cap E^n, \ |q| = p, \ n \geqq 0, \\ 0, & \text{otherwise.} \end{cases}$$

For a given system $\{(q_n^+(x), q_n^-(x))\,;\, n=0,1,2,\cdots\}$ and $k(x)=\sum_{n=0}^{\infty}(q_n^+(x)+q_n^-(x))$, let us consider a signed branching Markov process with age $Z=\{Z_t=[X_t, N_t, J_t], \zeta, \mathscr{B}_t, P_{[x,p,j]}\,;\,[x,p,j]\in\tilde{S}\}$ on $\tilde{S}$ satisfying the following condition.

Condition 2. (i) For any fixed $j\in J$, the process $\{Y_t^{(j)}=[X_t,N_t], \zeta, \mathscr{B}_t, P_{[x,p,j]}\,;\,[x,p]\in\hat{S}\}$ is a strong Markov process on $\hat{S}$ and it satisfies (i) and (ii) of Condition 1 for given $k(x)$, but where σ and τ for $Y_t^{(j)}$ are replaced by σ and η for Z. Also $Y_t^{(j)}$, $j\in J$, are stochastically equivalent to each other.

(ii) It holds that for any $\alpha>0$

$$(4.2)\quad \begin{aligned} &E_{[x,p,j]}[e^{-\alpha\eta}\,;\,[X_\eta,N_\eta,J_\eta]\in[B,\boldsymbol{q},j']] \\ &= E_{[x,p,j]}[e^{-\alpha\eta}\pi([X_{\eta-},N_{\eta-},J_{\eta-}],\ [B,\boldsymbol{q},j'])]\,, \end{aligned}$$

and

$$(4.3)\quad \begin{aligned} &E_{[x,0,j]}[e^{-\alpha\sigma}\,;\,[X_\sigma,N_\sigma,J_\sigma]\in[A,q,j]] \\ &= E_{[x,0,j]}[e^{-\alpha\sigma}\delta_{[X_{\sigma-},N_{\sigma-}+1]}([A,q])]\,, \\ &j,j'\in J,\ [x,p]\in S,\ [B,\boldsymbol{q}]\in\mathscr{B}(\hat{S}),\ A\in\mathscr{B}(E)\,, \end{aligned}$$

where $E_{[x,p,j]}$ denotes the integral by $P_{[x,p,j]}$, π is given in (4.1) and $\delta_{[x,p]}(\cdot)$ denotes the δ-measure assigned to $[x,p]$.

The existence of a signed branching Markov process with age satisfying condition 2 will be shown in §8.

(i) of the condition states that two processes satisfying Condition 1 and 2 have the same character until their first branching, while (ii) gives the new branching law attached to the new space $\tilde{S}$.

Similarly as in the case of a branching Markov process with age, we shall give the following

Definition 4.1. The process X' considered in (i) of Condition 2 (or (ii) of Condition 1) is called *the basic Markov process of* Z.

Let us set

$$Z_t^0(w)=\begin{cases} Z_t(w), & \text{if } t<\eta(w) \\ \Delta\ , & \text{if } t\geqq\eta(w)\,. \end{cases}$$

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Then the probability measure for Z_t^0 is denoted by $P^0_{[x,p,j]}$ and the integral by $P^0_{[x,p,j]}$ is denoted by $E^0_{[x,p,j]}$. The semi-group induced by Z_t^0 is denoted by U_t^0. Then we have

$$U_t^0 h([x,p,j]) = E^0_{[x,p,j]}[h(Z_t)] = E_{[x,p,j]}[h(Z_t); t < \eta], \quad [x,p,j] \in \tilde{S}, \ h \in B(\tilde{S}). \tag{4.4}$$

Definition 4. 2. When we restrict the starting point of Z_t^0 on $S \times J$, Z_t^0 is called *the non-branching part of* Z.

Now we shall define K and F as follows:

$$K([x,p,j]; ds, [A,q,j]) = P_{[x,p,j]}(\eta \in ds, Z_{\eta-} \in [A,q,j]), \quad [x,p] \in S, \ j \in J, \tag{4.5}$$

and

$$F(x;\xi) = \sum_{n=0}^{\infty} \frac{\{q_n^+(x) - q_n^-(x)\}}{k(x)} \xi^n, \qquad x \in E, \ \xi \in R^1. \tag{4.6}$$

Then $K([x,p,j]; \cdot, \cdot)$ is a measure on $\mathscr{B}([0,\infty) \times (S \times J))$ and it follows from (i) of Condition 2 and (3. 13) that

$$K([x,0,j]; ds, [A,p,j]) = E_x[e^{-2\int_0^t k(X'_s)ds} \frac{\left(\int_0^t k(X'_s)ds\right)^p}{p!} k(X'_t) I_A(X'_t)dt], \quad A \in \mathscr{B}(E), \tag{4.7}$$

where E_x denotes the integral by the probability measure P_x of the basic Markov process X'. Also, we can see that (3. 7)-(3. 11) hold if we replace τ by η. Then we have

Lemma 4. 1. *Let U_t^0 be the semi-group on $B(S \times J)$ induced by the non-branching part Z_t^0 of a signed branching Markov process with age Z_t satisfying Condition 2 and let H_t be the semi-group on $B(E)$ induced by the basic Markov process X'_t of Z_t. Then we have*

$$U_t^0 \widetilde{f \cdot 2}([x,0,0]) = H_t f(x), \qquad f \in C(E), \ x \in E. \tag{4.8}$$

Now, for a given system (U_t^0, K, F), consider the following equation:

$$u(t,x) = U_t^0 \widetilde{f \cdot \lambda}([x,0,0]) + \int_0^t \int_S K([x,0,0]; ds, [dy,p,0]) \lambda^p F(y; u(t-s,y)), \quad f \in C(E), \ 0 \leq \lambda, \ 0 \leq t \leq T, \ x \in E. \tag{4.9}$$

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where T is a positive constant.

Then we have

LEMMA 4. 2. *Let U_t be the semi-group on $\boldsymbol{B}(\tilde{\boldsymbol{S}})$ induced by a signed branching Markov process with age Z_t on $\tilde{\boldsymbol{S}}$ satisfying Condition 2 and let U_t^0 be the semi-group on $\boldsymbol{B}(S\times J)$ induced by the non-branching part Z_t^0 of Z_t. Let also f be a bounded continuous function on E. If $u(t,x)=(U_t\widetilde{f\cdot\lambda})|_E(x)$ is finite*[15] *for any $x\in E$ and $0\leqq t\leqq T$, then $u(t,x)$ satisfies* (4. 9).

Proof. Let us assume that $u(t,x)$ is finite for $x\in E$ and $0\leqq t\leqq T$. By the strong Markov property of Z_t, it holds that

$$(4.10)\qquad (U_t\widetilde{f\cdot\lambda})|_E(x)=E_{[x,0,0]}[\widetilde{f\cdot\lambda}(Z_t);\,t<\eta]+E_{[x,0,0]}[U_{t-\eta}\widetilde{f\cdot\lambda}(Z_\eta);\,\eta\leqq t]\,.$$

On the other hand, we have from the signed branching property (2. 13) of Z_t

$$U_t\widetilde{f\cdot\lambda}([\boldsymbol{x},\boldsymbol{p},j])=(-1)^{\left[\frac{j}{2}\right]}\lambda^{|\boldsymbol{p}|}\hat{u}(t,\cdot)([\boldsymbol{x}])\,,$$

and hence, by (4. 2), (4. 5) and (4. 6), we have

$$\begin{aligned}
&E_{[x,0,0]}[U_{t-\eta}\widetilde{f\cdot\lambda}(Z_\eta);\,\eta\leqq t]\\
&=\int_0^t\int_{\tilde{S}}P_{[x,0,0]}(\eta\in ds,\ Z_\eta\in[d\boldsymbol{y},\boldsymbol{p},j])U_{t-s}\widetilde{f\cdot\lambda}([\boldsymbol{y},\boldsymbol{p},j])^{16)}\\
&=\int_0^t\int_S K([x,0,0];\,ds,\ [dy,p,0])\lambda^p\left\{\sum_{n=0}^{\infty}\frac{q_n^+(y)}{k(y)}u(t-s,y)^n-\sum_{n=0}^{\infty}\frac{q_n^-(y)}{k(y)}u(t-s,y)^n\right\}\\
&=\int_0^t\int_S K([x,0,0];\,ds,\ [dy,p,0])\lambda^pF(y;\,u(t-s,y))\,.
\end{aligned}$$

Thus the lemma is obtained from (4. 10) and the above equation. Q.E.D.

Now let H_t be the semi-group on $\boldsymbol{B}(E)$ and F be the function given in (4. 6). For a given system (H_t,k,F), we consider the following equation

$$(4.11)\qquad u(t,x)=H_tf(x)+\int_0^t H_s(k(\cdot)F(\cdot\,;\,u(t-s,\ \cdot)))(x)ds\,,$$
$$f\in\boldsymbol{C}(E),\ x\in E,\ 0\leqq t\leqq T\,,$$

15) "$(U_t\widetilde{f\cdot\lambda})|_E(x)$ is finite" means that $E_{[x,0,0]}[|\widetilde{f\cdot\lambda}|(Z_t)]<\infty$.

16) cf. Remark 1 in § 3.

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where T is a positive constant. Then we have

THEOREM 4. 1. *Let U_t be the semi-group on $\boldsymbol{B}(\tilde{\boldsymbol{S}})$ induced by a signed branching Markov process with age Z_t on $\tilde{\boldsymbol{S}}$ satisfying Condition 2 and let H_t be the semi-group on $\boldsymbol{B}(E)$ induced by the basic Markov process X_t' of Z_t. Further, for $f \in \boldsymbol{C}(E)$, set*

$$u(t,x) = (U_t\widetilde{f\cdot 2})|_E(x)\,, \qquad x \in E\,.$$

If $u(t,x)$ is finite for any $x \in E$ and $0 \leqq t \leqq T$, then it satisfies (4. 11).

Proof. By the same method as in the proof of Theorem 3. 1, we have from (4. 7)

$$\int_0^t\int_S K([x,0,0];\,ds,\,[dy,p,0])2^pF(y\,;u(t-s,y))$$
$$=\int_0^t\int_S H_s(k(\cdot)F(\cdot\,;u(t-s,\,\cdot)))\,(x)ds\,.$$

Then the theorem follows from Lemma 4. 1 and Lemma 4. 2. Q.E.D.

Now let $E=R^d \cup \{\infty\}$ be the space obtained by the one-point compactification of R^d and consider the standard Brownian motion X' on R^d. Considering the point ∞ is the trap of X', the process X' can be regarded as the process on E. Then we can consider a signed branching Markov process with age Z_t on $\tilde{\boldsymbol{S}}$ corresponding to the basic Markov process X'. But when we take a starting point of Z_t in R^d a branching law at ∞ is not needed because almost all sample paths do not reach ∞. Hence it is sufficient in the present case that $q_n^+(x)$, $q_n^-(x)$ and $k(x)$ are bounded and continuous in R^d, and we may consider $q_n^+(\infty)=q_n^-(\infty)=k(\infty)=0$.

Remark 1. For the case stated above, Theorem 4. 1 holds for $f\in \boldsymbol{C}(R^d)$ with $f(\infty)=0$. The proof is given as follows: let $\rho_n(x)$, $n \geqq 1$, be bounded continuous functions such that

$$\begin{cases} \rho_n(x)=1\quad, & \text{if } \|x\|<n \\ 0\leqq \rho_n(x)\leqq 1\,, & \text{if } n\leqq \|x\|\leqq n+1\,, \\ \rho_n(x)=0\quad, & \text{if } \|x\|>n+1 \text{ or } x=\infty\,. \end{cases}$$

Then $\rho_n f \in \boldsymbol{C}(E)$. On the other hand, if $U_t\widetilde{f\cdot\lambda}$ is finite[17] we have by Lebesgue's convergence theorem

17) cf. Foot-note 10).

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$$\widetilde{U_t f \cdot \lambda}([\boldsymbol{x}, \boldsymbol{p}, j]) = \lim_{n\to\infty} U_t \widetilde{\rho_n f \cdot \lambda}([\boldsymbol{x}, \boldsymbol{p}, j]), \qquad [\boldsymbol{x}, \boldsymbol{p}, j] \in \tilde{\boldsymbol{S}}.$$

Hence it follows from (2. 13)

$$\widetilde{U_t f \cdot \lambda} = \widetilde{(U_t \widetilde{f \cdot \lambda})|_E} \cdot \lambda,$$

provided each side is finite, because it holds for $\rho_n f$. Then we can see, as in the proof of Lemma 4. 2, that Lemma 4. 2 holds for $f \in \boldsymbol{C}(R^d)$ with $f(\infty) = 0$. Evidently Lemma 4. 1 holds for our f and accordingly we can see that Theorem 4. 1 holds for our f.

COROLLARY 4. 1. *Let Z_t be a signed branching Markov process with age satisfying Condition 2 whose basic Markov process is a standard Brownian motion on R^d and let U_t be the semi-group on $\boldsymbol{B}(\tilde{\boldsymbol{S}})$ induced by Z_t. Let us assume that $k(x)F(x; \xi)$ satisfies Lipschitz's condition:*

$$|k(x_1)F(x_1; \xi_1) - k(x_2)F(x_2; \xi_2)| \leqq K\{\| x_1 - x_2 \| + |\xi_1 - \xi_2|\},$$
$$x_1, x_2 \in R^d, \quad \xi_1, \xi_2 \in R^1,$$

where K is a positive constant and $\| x_1 - x_2 \|$ denotes the Euclidian distance between x_1 and x_2. If, for $f \in \boldsymbol{C}(E)$, $u(t, x) = (U_t \widetilde{f \cdot 2})|_E(x)$ is bounded for any $x \in R^d$ and $0 \leqq t < T$, then $u(t, x)$ is the bounded solution of parabolic equation

$$\frac{\partial u(t, x)}{\partial t} = \frac{1}{2} \Delta u(t, x) + k(x)F(x; u(t, x)), \qquad x \in R^d, \ 0 < t < T,$$

with the initial condition $u(0+, x) = f(x)$.

Now we shall give a simple remark on a signed branching Markov process with age.

Remark 2. Let Z_t be a signed branching Markov process with age on $\tilde{\boldsymbol{S}}$ satisfying Condition 2 and let U_t be the semi-group induced by Z_t. If, for instance, we replace π in (4. 2) by π_1 defined by

$$\pi_1([x, p, 0], [B, \boldsymbol{q}, 1]) = \pi_1([x, p, 1], [B, \boldsymbol{q}, 0])$$
$$= \pi_1([x, p, 2], [B, \boldsymbol{q}, 3]) = \pi_1([x, p, 3], [B, \boldsymbol{q}, 2])$$
$$= \sum_{n=0}^{\infty} \frac{q_n^+(x)}{k(x)} \delta_n([x, p], [B, \boldsymbol{q}]) \ (= \pi([x, p, 0], [B, \boldsymbol{q}, 1])),$$
$$\pi_1([x, p, 0], [B, \boldsymbol{q}, 2]) = \pi_1([x, p, 2], [B, \boldsymbol{q}, 0])$$

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$$= \pi_1([x, p, 1],\ [B, \boldsymbol{q}, 3]) = \pi_1([x, p, 3],\ [B, \boldsymbol{q}, 1])$$

$$= \sum_{n=0}^{\infty} \frac{q_n^-(x)}{k(x)} \delta_n([x, p],\ [B, \boldsymbol{q}])\ (= \pi([x, p, 0],\ [B, \boldsymbol{q}, 3])\,,$$

$$\pi_1([x, p, j],\ [B, \boldsymbol{q}, j']) = 0 \quad \text{for the other pairs of } (j, j')\,,$$

$$[x, p] \in S,\ \ j, j' \in J,\ \ [B, \boldsymbol{q}] \in \mathscr{B}(\hat{S})\,,$$

then we have a new process Z'_t and the corresponding semi-group U'_t. Evidently Z'_t is not stochastically equivalent to Z_t, but it holds that $U_t\widetilde{f\cdot\lambda} = U'_t\widetilde{f\cdot\lambda}$ for any $f \in \boldsymbol{C}(E)$ provided each side exist. Therefore $U_t\widetilde{f\cdot\lambda} = U'_t\widetilde{f\cdot\lambda}$ does not imply the stochastic equivalence of the processes Z_t and Z'_t.

§5. **A sufficient condition.** Let $Z = \{Z_t = [X_t, N_t, J_t],\ \zeta,\ \mathscr{B}_t,\ P_{[x,\boldsymbol{p},j]};\ [x, \boldsymbol{p}, j] \in \tilde{\boldsymbol{S}}\}$ be a strong Markov process on $\tilde{\boldsymbol{S}}$ which is not assumed *a priori* to be a signed branching Markov process with age. In this section, we shall give a sufficient condition which makes the process Z_t on $\tilde{\boldsymbol{S}}$ a signed branching Markov process with age on $\tilde{\boldsymbol{S}}$.

Now let us define U_t^0 and Ψ by

$$U_t^0 h([x, \boldsymbol{p}, j]) = E_{[x,\boldsymbol{p},j]}[h(Z_t);\ t < \eta]\,, \tag{5. 1}$$

$$\Psi([x, \boldsymbol{p}, j];\ ds,\ [B, \boldsymbol{p}', j']) = P_{[x,\boldsymbol{p},j]}(\eta \in ds, Z_\eta \in [B, \boldsymbol{p}', j'])\,, \tag{5. 2}$$

$$[x, \boldsymbol{p}, j] \in \tilde{\boldsymbol{S}},\ \ [B, \boldsymbol{p}', j'] \in \mathscr{B}(\tilde{\boldsymbol{S}})\,,$$

where h is a Borel measurable function on $\tilde{\boldsymbol{S}}$, $\eta(w) = \inf\{t > 0;\ J_t(w) \neq J_0(w)$ or $\sup_{s\leq t} |N_s(w)| = \infty\}$ and $E_{[x,\boldsymbol{p},j]}$ denotes the integral by $P_{[x,\boldsymbol{p},j]}$. $\Psi([x, \boldsymbol{p}, j];\ \cdot, \cdot)$ is a measure on $\mathscr{B}([0, \infty) \times \tilde{\boldsymbol{S}})$. Then we consider the following

Condition 3. (i) Condition 2 holds.

(ii)

$$U_t^0\widetilde{f\cdot\lambda}([x, \boldsymbol{p}, j]) = \widetilde{(U_t^0 f\cdot\lambda)|_E}\cdot\lambda([x, \boldsymbol{p}, j])\,, \qquad f \in \boldsymbol{C}^*(E),\ [x, \boldsymbol{p}, j] \in \tilde{\boldsymbol{S}}\,. \tag{5. 3}$$

(iii) For $f \in \boldsymbol{B}^*([0, \infty) \times E)$, set

$$\widetilde{f^{(t)}\cdot\lambda} = \widetilde{f(t, \cdot)\cdot\lambda}\,,$$

where $f(t, \cdot)$ denotes the function on E for fixed $t \geqq 0$. Then it holds that for any $t \geqq 0$, $m \geqq n-1$

$$(5.4) \qquad \int_0^t \int_{S^m \times \{j'\}} \Psi([\boldsymbol{x}, \boldsymbol{p}, j]; ds, [d\boldsymbol{y}, \boldsymbol{q}, j']) \widetilde{f^{(s)} \cdot \lambda}([\boldsymbol{y}, \boldsymbol{q}, j'])$$

$$= \sum_{i=1}^{\infty} \int_0^t \int_{S^{m-n+1} \times \{j'\}} \Psi([x_i, p_i, j]; ds, [d\boldsymbol{y}, \boldsymbol{p}', j']) \widetilde{f^{(s)} \cdot \lambda}([\boldsymbol{y}, \boldsymbol{p}', j'])$$

$$\cdot \prod_{l \neq i} U_s^0 \widetilde{f^{(s)} \cdot \lambda}([x_l, p_l, 0]), \qquad m \geqq n-1,$$

where $\boldsymbol{x} = [x_1, x_2, \cdots, x_n]$ and $\boldsymbol{p} = [p_1, p_2, \cdots, p_n]$.

(iv) Δ and $[\partial, p, j]$, $p \in N$ and $j \in J$, are traps.

(v) Let

$$\eta_\infty = \lim_{n \to \infty} \eta_n, \qquad e_\Delta = \inf \{t > 0; Z_t = \Delta\},$$

where η_n is given in (2. 7). Then it holds that

$$P_{[\boldsymbol{x}, \boldsymbol{p}, j]}(\eta_\infty = e_\Delta, \ \eta_\infty < \zeta) = P_{[\boldsymbol{x}, \boldsymbol{p}, j]}(\eta_\infty < \zeta), \qquad [\boldsymbol{x}, \boldsymbol{p}, j] \in \tilde{\boldsymbol{S}}.$$

(ii) requires the independence of the motions of n-particles starting from $[x_1, p_1, j]$, $[x_2, p_2, j]$, $\cdots$, $[x_n, p_n, j]$ and ending at the minimum of their first branching times $\inf\{t > 0; J_t(w) \neq J_0(w)$ or $\sup_{s \leqq t} |N_s(w)| = \infty\}$, while (5. 4) means that only one of them branches at the first branching time η and the others do not. ((ii) and (iii) of Condition 3 correspond to the property B III in [7].) The existence of a strong Markov process satisfying Condition 3 will be proved in § 8.

Now our purpose is to prove the following

THEOREM 5. 1. *If a strong Markov process* $Z = \{Z_t = [X_t, N_t, J_t], \zeta, \mathscr{B}_t, P_{[\boldsymbol{x}, \boldsymbol{p}, j]}; [\boldsymbol{x}, \boldsymbol{p}, j] \in \tilde{\boldsymbol{S}}\}$ *on* $\tilde{\boldsymbol{S}}$ *satisfies Condition* 3, *then* Z *is a signed branching Markov process with age.*

First we shall prepare some lemmas. Let U_t be the semi-group on $\boldsymbol{B}(\tilde{\boldsymbol{S}})$ induced by Z_t and set

$$(5.5) \qquad U_t^{(r)} h([\boldsymbol{x}, \boldsymbol{p}, j]) = E_{[\boldsymbol{x}, \boldsymbol{p}, j]}[h(Z_t); \eta_r \leqq t < \eta_{r+1}], \ r \geqq 0, \ [\boldsymbol{x}, \boldsymbol{p}, j] \in \tilde{\boldsymbol{S}},$$

for a Borel measurable function h on $\tilde{\boldsymbol{S}}$ provided that $E_{[\boldsymbol{x}, \boldsymbol{p}, j]}(h(Z_t))$ exists. In the following lemmas, it is always assumed that Condition 3 holds, $f \in \boldsymbol{C}^*(E)$ and $0 \leqq \lambda < 1$.

LEMMA 5. 1. *For any $r \geqq 0$, we have*

$$U_t^{(r)}\widetilde{f\cdot\lambda}([x, p, j]) = (-1)^{\left[\frac{j}{2}\right]}\lambda^{|p|}U_t^{(r)}\widetilde{f\cdot\lambda}([x, 0, 0]), \qquad [x, p, j] \in \tilde{S}. \tag{5. 6}$$

Proof. (5. 6) holds for $r = 0$ and hence it suffices to prove (5. 6) for $r+1$ under the assumption that (5. 6) holds for r.

Now it follows from (5. 4) and (ii) of Condition 2 that for any $[x, p] \in \hat{S}$ and $[B, q] \in \mathscr{B}(\hat{S})$

$$\begin{aligned}
&\Psi([x, p, 0];\ dt,\ [B, q, 1]) = \Psi([x, p, 1];\ dt,\ [B, q, 0]) \\
&= \Psi([x, p, 2];\ dt,\ [B, q, 3]) = \Psi([x, p, 3];\ dt,\ [B, q, 2]), \\
&\Psi(]x, p, 0];\ dt,\ [B, q, 3]) = \Psi([x, p, 3];\ dt,\ [B, q, 0]) \\
&= \Psi([x, p, 1];\ dt,\ [B, q, 2]) = \Psi([x, p, 2];\ dt,\ [B, q, 1]), \\
&\Psi([x, p, j];\ dt,\ [B, q, j']) = 0, \qquad \text{for other pairs of } (j, j'), \\
&\Psi([x, p, j];\ dt,\ [B, q, j']) = \Psi([x, 0, j];\ dt,\ [B, q-p, j']), \quad j, j' \in J,
\end{aligned} \tag{5. 7}$$

where $q-p$ denotes p' with $|p'| = |q| - |p|$. Then we can see from (5. 7) and the strong Markov property of Z_t that

$$\begin{aligned}
&U_t^{(r+1)}\widetilde{f\cdot\lambda}([x, p, j]) \\
&= \int_0^t\int_{\tilde{S}}\Psi([x, p, j];\ ds,\ [dy, q, j'])U_{t-s}^{(r)}\widetilde{f\cdot\lambda}([y, q, j']) \\
&= (-1)^{\left[\frac{j}{2}\right]}\int_0^t\int_{\tilde{S}}\Psi([x, 0, 0];\ ds,\ [dy, q, j'])U_{t-s}^{(r)}\widetilde{f\cdot\lambda}([y, p+q, j]),
\end{aligned}$$

where $|p+q| = |p| + |q|$. By the assumption of induction, we have from the above equation

$$\begin{aligned}
&U_t^{(r+1)}\widetilde{f\cdot\lambda}([x, p, j]) \\
&= (-1)^{\left[\frac{j}{2}\right]}\lambda^{|p|}\int_0^t\int_{\tilde{S}}\Psi([x, 0, 0];\ ds,\ [dy, q, j'])U_{t-s}^{(r)}\widetilde{f\cdot\lambda}([y, q, j']) \\
&= (-1)^{\left[\frac{j}{2}\right]}\lambda^{|p|}U_t^{(r+1)}\widetilde{f\cdot\lambda}([x, 0, 0]),
\end{aligned}$$

as was to be proved. Q.E.D.

LEMMA 5. 2. *For any $r \geqq 0$ and $[x, p] = [[x_1, x_2, \cdots, x_n], [p_1, p_2, \cdots, p_n]]$,*

$$(5.8)\qquad \sum_{r_1+r_2+\cdots+r_n=r}\int_0^t\sum_{i=1}^n\int_{\tilde S}\Psi([x_i,p_i,j];ds,[dy,q,j'])U_{t-s}^{(r_i)}\widetilde{f\cdot\lambda}([y,q,j'])\prod_{l\neq i}U_s^0U_{t-s}^{(r_l)}\widetilde{f\cdot\lambda}([x_l,p_l,0])$$

$$=(-1)^{\left[\frac{j}{2}\right]}\sum_{r_1+r_2+\cdots+r_n=r+1}\prod_{i=1}^n U_t^{(r_i)}\widetilde{f\cdot\lambda}([x_i,p_i,0]).$$

Proof. According to (i) of Condition 2 and Lemma 5.1, it holds that

$$U_s^0U_{t-s}^{(r)}\widetilde{f\cdot\lambda}([x,p,0])=\lambda^pU_s^0U_{t-s}^{(r)}\widetilde{f\cdot\lambda}([x,0,0]),\qquad [x,p]\in S.$$

Hence, by (5.7) and Lemma 5.1, it suffices to prove (5.8) for the case $\boldsymbol{p}=\boldsymbol{0}$ and $j=0$.

Now let us put $g^{(r_i)}(s)=U_s^0U_{t-s}^{(r_i)}\widetilde{f\cdot\lambda}([x_i,0,0])$. Then $g^{(0)}(s)$ is independent of s by the semi-group property of U_t^0. Further we can see from the strong Markov property of Z_t that for $r\geqq 1$

$$U_{t-s}^{(r)}\widetilde{f\cdot\lambda}([x,p,j])=\int_0^{t-s}\int_{\tilde S}\Psi([x,p,j];dv,[dy,q,j'])U_{t-s-v}^{(r-1)}\widetilde{f\cdot\lambda}([y,q,j']),$$

and hence we have

$$g^{(r_i)}(s)=\int_s^t\int_{\tilde S}\Psi([x_i,0,0];dv,[dy,q,j'])U_{t-v}^{(r_i-1)}\widetilde{f\cdot\lambda}([y,q,j']).$$

Then the left hand side of (5.8), where $\boldsymbol{p}=\boldsymbol{0}$ and $j=0$, is equal to

$$\sum_{r_1+r_2+\cdots+r_n=r}\int_0^t\sum_{i=1}^n d_s(-g^{(r_i+1)}(s))\prod_{l\neq i}g^{(r_l)}(s).$$

Writing r_i+1 as r_i and noting $d_sg^{(0)}(s)=0$ and $g^{(r_i)}(t)=0$ for $r_i\geqq 1$, the above expression is equal to

$$\sum_{r_1+r_2+\cdots+r_n=r+1}\int_0^t\sum_{i=1}^n d_s(-g^{(r_i)}(s))\prod_{l\neq i}g^{(r_l)}(s)$$

$$=\sum_{r_1+r_2+\cdots+r_n=r+1}\prod_{i=1}^n g^{(r_i)}(0)=\sum_{r_1+r_2+\cdots+r_n=r+1}\prod_{i=1}^n U_t^{(r_i)}\widetilde{f\cdot\lambda}([x_i,0,0]),$$

as was to be proved. Q.E.D.

LEMMA 5.3. *For any* $r\geqq 0$, $[\boldsymbol{x},\boldsymbol{p}]=[[x_1,x_2,\cdots,x_n],[p_1,p_2,\cdots,p_n]]$ *and* $j\in J$, *we have*

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$$U_t^{(r)}\widetilde{f\cdot\lambda}([\boldsymbol{x},\boldsymbol{p},j]) = (-1)^{\left[\frac{j}{2}\right]}\sum_{r_1+r_2+\cdots+r_n=r}\prod_{i=1}^{n}U_t^{(r_i)}\widetilde{f\cdot\lambda}([x_i,p_i,0]), \tag{5. 9}$$

and

$$\begin{aligned}&\sum_{m=n-1}^{\infty}\int_0^t\int_{S^m\times J}\Psi([\boldsymbol{x},\boldsymbol{p},j];\,ds,\ [d\boldsymbol{y},\boldsymbol{q},j'])\\ &\qquad\cdot\{(-1)^{\left[\frac{j'}{2}\right]}\sum_{r_1+r_2+\cdots+r_m=r}\prod_{i=1}^{m}U_{t-s}^{(r_i)}\widetilde{f\cdot\lambda}([y_i,q_i,0])\}\\ &=\sum_{r_1+r_2+\cdots+r_n=r}\int_0^t\sum_{i=1}^{n}\int_{\tilde{S}}\Psi([x_i,p_i,j];\,ds,\ [d\boldsymbol{y},\boldsymbol{p}',j'])U_{t-s}^{(r_i)}\widetilde{f\cdot\lambda}([\boldsymbol{y},\boldsymbol{p}',j'])\\ &\qquad\prod_{l\neq i}U_s^0U_{t-s}^{(r_l)}\widetilde{f\cdot\lambda}([x_l,p_l,0]),\end{aligned} \tag{5. 10}$$

where $\boldsymbol{y}=[y_1,y_2,\cdots,y_m]$ *and* $\boldsymbol{q}=[q_1,q_2,\cdots,q_m]$.

Proof. For $r=0$, (5. 9) follows from Lemma 5. 1, and (5. 10) follows from Lemma 5. 1 and (iii) of Condition 3. Hence we shall prove the validity of (5. 9) and (5. 10) for $r+1$ under the assumption that (5. 9) and (5. 10) hold up to r. Further, by Lemma 5. 1, we may assume $\boldsymbol{p}=\boldsymbol{0}$ and $j=0$.

By Lemma 5. 2 and the assumption of induction, we have

$$\begin{aligned}&\sum_{r_1+r_2+\cdots+r_n=r+1}\prod_{i=1}^{n}U_t^{(r_i)}\widetilde{f\cdot\lambda}([x_i,0,0])\\ &=\sum_{m=n-1}^{\infty}\int_0^t\int_{S^m\times J}\Psi([\boldsymbol{x},\boldsymbol{0},0];\,ds,\ [d\boldsymbol{y},\boldsymbol{p},j])\\ &\qquad\cdot\{(-1)^{\left[\frac{j}{2}\right]}\sum_{r_1+r_2+\cdots+r_m=r}\prod_{i=1}^{m}U_{t-s}^{(r_i)}\widetilde{f\cdot\lambda}([y_i,p_i,0])\}\\ &=\int_0^t\int_{\tilde{S}}\Psi([\boldsymbol{x},\boldsymbol{0},0];\,ds,\ [d\boldsymbol{y},\boldsymbol{p},j])U_{t-s}^{(r)}\widetilde{f\cdot\lambda}([\boldsymbol{y},\boldsymbol{p},j])\\ &=U_t^{(r+1)}\widetilde{f\cdot\lambda}([\boldsymbol{x},\boldsymbol{0},0]).\end{aligned}$$

So (5. 9) holds for $r+1$.

Now we note that for any (m,m)-matrix $A=(a_{i,j})_{i,j=1}^m$

$$\begin{aligned}\sum_{\pi}\prod_{i=1}^{m}a_{\pi(i),i}&=\prod_{i=1}^{m}(\sum_{k=1}^{m}a_{k,i})-\sum_{(k_1,k_2,\cdots,k_{m-1})}\prod_{i=1}^{m}(\sum_{j=1}^{m-1}a_{k_j,i})\\ &+\sum_{(k_1,k_2,\cdots,k_{m-2})}\prod_{i=1}^{m}(\sum_{j=1}^{m-2}a_{k_j,i})-\cdots+(-1)^{m-1}\sum_{(k)}\prod_{i=1}^{m}a_{k,i},\end{aligned} \tag{5. 11}$$

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holds, where $\sum_\pi$ denotes the summation over all permutations $(\pi(1), \pi(2), \cdots, \pi(m))$ of $(1,2,\cdots,m)$ and $\sum_{(k_1,k_2,\cdots,k_r)}$, $r \leqq m-1$, denotes the summation over all $(k_1,k_2,\cdots,k_r)$ such that $1 \leqq k_i \leqq m$ and all k_i are different.[18]

Let $h_i \in \boldsymbol{B}^*([0,\infty)\times E)$, $i=1,2,\cdots,m$. Considering $\widetilde{h_j^{(s)}\cdot\lambda}([y_i,p_i,0])$ in the place of $a_{j,i}$ in (5. 11), we have for $[\boldsymbol{x}]=[x_1,x_2,\cdots,x_n]$

(5. 12)
$$\begin{aligned}&\int_0^t\int_{S^m\times\{j\}}\Psi([\boldsymbol{x},0,0];ds,[d\boldsymbol{y},\boldsymbol{p},j])\{(-1)^{\left[\frac{j}{2}\right]}\sum_\pi\prod_{i=1}^m\widetilde{h_{\pi(i)}^{(s)}\cdot\lambda}([y_i,p_i,0])\}\\&=\sum_{\nu=0}^{m-1}(-1)^\nu\sum_{(k_1,k_2,\cdots,k_{m-\nu})}\int_0^t\int_{S^m\times\{j\}}\Psi[\boldsymbol{x},0,0]);ds,[d\boldsymbol{y},\boldsymbol{p},j])\\&\qquad\cdot(-1)^{\left[\frac{j}{2}\right]}\prod_{i=1}^m\{\sum_{r=1}^{m-\nu}\widetilde{h_{k_r}^{(s)}\cdot\lambda}([y_i,p_i,0])\}.\end{aligned}$$

According to Lemma 5. 1, and (5. 4), the right hand side of the above equation is equal to

$$\begin{aligned}&\sum_{\nu=0}^{m-1}(-1)^\nu\sum_{(k_1,k_2,\cdots,k_{m-\nu})}\int_0^t\int_{S^m\times\{j\}}\Psi([\boldsymbol{x},0,0];ds,[d\boldsymbol{y},\boldsymbol{p},j])\\&\qquad\qquad\cdot\widetilde{(\sum_{r=1}^{m-\nu}h_{k_r}^{(s)})\cdot\lambda}[\boldsymbol{y},\boldsymbol{p},j]\\&=\int_0^t\sum_{i=1}^n\int_{S^{m-n+1}\times\{j\}}\Psi([x_i,0,0];ds,[d\boldsymbol{y},\boldsymbol{p},j])(-1)^{\left[\frac{j}{2}\right]}\{\sum_{\nu=0}^{m-1}(-1)^\nu\sum_{(k_1,k_2,\cdots,k_{m-\nu})}\\&\qquad\widetilde{(\sum_{r=1}^{m-\nu}h_{k_r}^{(s)})\cdot\lambda}([\boldsymbol{y},\boldsymbol{p},0])\prod_{l\neq i}U_s^0(\widetilde{\sum_{r=1}^{m-\nu}h_{k_r}^{(s)}\cdot\lambda})([x_l,0,0])\}.\end{aligned}$$

Now noting that for $\boldsymbol{y}=[y_1,y_2,\cdots,y_{m-n+1}]$, $\boldsymbol{p}=[p_1,p_2,\cdots,p_{m-n+1}]$

$$\widetilde{(\sum_{r=1}^{m-\nu}h_{k_r}^{(s)})\cdot\lambda}([\boldsymbol{y},\boldsymbol{p},0])=\prod_{\mu=1}^{m-n+1}\sum_{r=1}^{m-\nu}\widetilde{h_{k_r}^{(s)}\cdot\lambda}([y_\mu,p_\mu,0]),$$

and applying again (5. 11) to the integrand { }, the above expression is equal to

(5. 13)
$$\begin{aligned}&\int_0^t\sum_{i=1}^n\int_{S^{m-n+1}\times\{j\}}\Psi([x_i,0,0];ds,[d\boldsymbol{y},\boldsymbol{p},j])\\&\cdot(-1)^{\left[\frac{j}{2}\right]}\{\sum_\pi\prod_{\mu=1}^{m-n+1}\widetilde{h_{\pi(\mu)}^{(s)}\cdot\lambda}([y_\mu,p_\mu,0])\cdot\prod_{l\neq i}U_s^0\widetilde{h_{\pi(\mu_l)}^{(s)}\cdot\lambda}([x_l,0,0]),\end{aligned}$$

where $\{\mu_l;1\leqq l\leqq n,\ l\neq i\}=\{m-n+2,\ m-n+3,\cdots,m\}$ and π is a permutation on $(1,2,\cdots,m)$. If we use the following notations:

18) cf. Ryser [15], Th. 4. 1 (p. 26)

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$\sum_{<k_1,k_2,\cdots,k_{m-n+1}>}$: the sum over all choices $(k_1,k_2,\cdots,k_{m-n+1})$ from $(1,2,\cdots,m)$,

$\sum_{\pi}^{(k)}$: the sum over all permutations π on $(k_1,k_2,\cdots,k_{m-n+1})$,

$\sum_{\hat{\pi}}^{(\hat{k})}$: the sum over all permutations $\hat{\pi}$ on $(\hat{k}_1,\hat{k}_2,\cdots,\hat{k}_{n-1})$ which is the remainder of $(1,2,\cdots,m)$ excluding $(k_1,k_2,\cdots,k_{m-n+1})$,

then (5. 13) is equal to

$$\int_0^t\sum_{i=1}^n\sum_{<k_1,k_2,\cdots,k_{m-n+1}>}\sum_{\hat{\pi}}^{(\hat{k})}\sum_{\pi}^{(k)}\int_{S^{m-n+1}\times\{j\}}\Psi([x_i,0,0];ds,[dy,p,j])$$
$$\cdot(-1)^{\left[\frac{j}{2}\right]}\prod_{\mu=1}^{m-n+1}\widetilde{h_{k_{\pi(\mu)}}^{(s)}\cdot\lambda}([y_\mu,p_\mu,0])\cdot\prod_{l\neq i}U_s^0\widetilde{h_{k_{\hat{\pi}(\mu_l)}}^{(s)}\cdot\lambda}([x_l,0,0]).$$

Now putting $h_k^{(s)}=(U_{t-s}^{(k)}\widetilde{f\cdot\lambda})|_E$, we can see from (5. 9) for $r_i\leqq r+1$

$$\int_0^t\int_{S^m\times\{j\}}\Psi([x,0,0];ds,[dy,p,j])$$
$$\cdot\{(-1)^{\left[\frac{j}{2}\right]}\sum_{r_1+r_2+\cdots+r_m=r+1}\prod_{i=1}^m U_{t-s}^{(r_i)}\widetilde{f\cdot\lambda}([y_i,p_i,0])\}$$
$$=\int_0^t\sum_{i=1}^n\sum_{r_i=0}^{r+1}\sum_{\hat{r}_1+\hat{r}_2+\cdots+\hat{r}_{n-1}=r+1-r_i}\int_{S^{m-n+1}\times\{j\}}\Psi([x_i,0,0];ds,[dy,p,j])$$
$$\cdot\{(-1)^{\left[\frac{j}{2}\right]}\sum_{r_1+r_2+\cdots+r_{m-n+1}=r_i}\prod_{\mu=1}^{m-n+1}U_{t-s}^{(r_\mu)}\widetilde{f\cdot\lambda}([y_\mu,p_\mu,0])\}$$
$$\cdot\prod_{l\neq i}U_s^0U_{t-s}^{(\hat{r}_l)}\widetilde{f\cdot\lambda}([x_l,0,0])$$
$$=\sum_{r_1+r_2+\cdots+r_n=r+1}\int_0^t\sum_{i=1}^n\int_{S^{m-n+1}\times\{j\}}\Psi([x_i,0,0];ds,[dy,p,j])$$
$$\cdot U_{t-s}^{(r_i)}\widetilde{f\cdot\lambda}([y,p,j])\cdot\prod_{l\neq i}U_s^0U_{t-s}^{(r_l)}\widetilde{f\cdot\lambda}([x_l,0,0]).$$

Summing up both sides of the above equation over all $m\geqq n-1$, we have (5. 10) for $r+1$. Q.E.D.

LEMMA 5. 4.

$$U_t\widetilde{f\cdot\lambda}([x,p,j])=\sum_{r=0}^{\infty}U_t^{(r)}\widetilde{f\cdot\lambda}([x,p,j]),\qquad [x,p,j]\in\tilde{S}.$$

Proof. By (iv) and (v) of Condition 3, we have

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$$U_t\widetilde{f\cdot\lambda}[\boldsymbol{x},\boldsymbol{p},j]) = E_{[\boldsymbol{x},\boldsymbol{p},j]}[\widetilde{f\cdot\lambda}(Z_t);\, t<\eta_\infty] + E_{[\boldsymbol{x},\boldsymbol{p},j]}[\widetilde{f\cdot\lambda}(Z_t);\, t\geqq\eta_\infty]$$

$$=\sum_{r=0}^{\infty} E_{[\boldsymbol{x},\boldsymbol{p},j]}[\widetilde{f\cdot\lambda}(Z_t);\, \eta_r\leqq t<\eta_{r+1}] + E_{[\boldsymbol{x},\boldsymbol{p},j]}[U_{t-\eta_\infty}\widetilde{f\cdot\lambda}(\Delta);\, t\geqq\eta_\infty]$$

$$=\sum_{r=0}^{\infty} U_t^{(r)}\widetilde{f\cdot\lambda}([\boldsymbol{x},\boldsymbol{p},j]).$$ Q.E.D.

We are now in a position to prove Theorem 5. 1.

Proof of Theorem 5. 1. It suffices to prove (2. 9). By Lemma 5. 4 and 9), we have

$$U_t\widetilde{f\cdot\lambda}([\boldsymbol{x},\boldsymbol{p},j]) = \sum_{r=0}^{\infty} U_t^{(r)}\widetilde{f\cdot\lambda}([\boldsymbol{x},\boldsymbol{p},j])$$

$$=(-1)^{\left[\frac{j}{2}\right]}\lambda^{|\boldsymbol{p}|}\sum_{r=0}^{\infty}\sum_{r_1+r_2+\cdots+r_n=r}\prod_{i=1}^{n}U_t^{(r_i)}\widetilde{f\cdot\lambda}([x_i,0,0])$$

$$=(-1)^{\left[\frac{j}{2}\right]}\lambda^{|\boldsymbol{p}|}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\prod_{i=1}^{n}U_t^{(r_i)}\widetilde{f\cdot\lambda}([x_i,0,0])$$

$$=(-1)^{\left[\frac{j}{2}\right]}\lambda^{|\boldsymbol{p}|}\prod_{i=1}^{n}\sum_{r_i=0}^{\infty}U_t^{(r_i)}\widetilde{f\cdot\lambda}([x_i,0,0])$$

$$=(-1)^{\left[\frac{j}{2}\right]}\lambda^{|\boldsymbol{p}|}\prod_{i=1}^{n}(U_t\widetilde{f\cdot\lambda})|_E(x_i)$$

$$=\widetilde{(U_t\widetilde{f\cdot\lambda})|_E\cdot\lambda}([\boldsymbol{x},\boldsymbol{p},j]),\qquad f\in \boldsymbol{C}^*(E),\ 0\leqq\lambda<1,$$

'here $\boldsymbol{x}=[x_1,x_2,\cdots,x_n]$ and $\boldsymbol{p}=[p_1,p_2,\cdots,p_n]$. Q.E.D.

§ 6. Semi-linear equation. In this section, we shall consider an pplication of Corollary 4. 1 to a probabilistic interpretation of the following ›mi-linear equation:

$$\text{}. 1)\qquad \frac{\partial u(t,x)}{\partial t}=\frac{1}{2}\Delta u(t,x)+k(x)F(x;\,u(t,x))^{19)},$$

'here $k(x)$ is a non-negative bounded continuous function on R^d and $F(x;\xi)$ ıtisfies the following conditions: there exists a positive constant K such that

$$\text{}. 2)\qquad |k(x)F(x;\xi)-k(x')F(x';\xi')|\leqq K\{\|x-x'\|+|\xi-\xi'|\},\quad x,x'\in R^d,\ \xi,\xi'\in[0,1],$$

19) Semi-linear equations of this type are discussed in Kolmogoroff-Petrovsky-Piscounoff 3].

where $\| x - x' \|$ denotes the Euclidian distance between x and x', and it also holds that

(6. 3) $$F(x; 0) = F(x; 1) = 0 \text{ and } 0 \leqq F(x; \xi) \text{ for } 0 < \xi < 1.$$

Throughout this section, we shall consider a strong Markov process Z_t satisfying Condition 3 (and hence, by Theorem 5. 1, a signed branching Markov process with age) whose basic Markov process is a standard Brownian motion on $E = R^d \cup \{\infty\}$ which is obtained by the one-point compactification of R^d.[20]

We first consider the special case satisfying Condition (Q):

(a) Let $q_n^+(x)$ and $q_n^-(x)$ be functions given *a priori* in Condition 3 (through the part (i)). Then $q_0^+(x) = q_0^-(x) = 0$ and there exists an integer $M > 0$ such that $q_n^+(x) = q_n^-(x) = 0$ for $n > M$.

(b) Set

$$F(x; \xi) = \sum_{n=1}^{M} \frac{\{q_n^+(x) - q_n^-(x)\}}{k(x)} \xi^n,$$

where $k(x) = \sum_{n=1}^{M} (q_n^+(x) + q_n^-(x))$. Then

$$0 \lessdot F(x; \xi), \qquad x \in R^d, \ \xi \in (0,1),$$

and also there exists a positive constant K such that

$$|k(x)F(x; \xi) - k(x')F(x'; \xi')| \leqq K\{\| x - x' \| + |\xi - \xi'|\}, \ x, x' \in R^d, \ \xi, \xi' \in [0,1].$$

(c) $$\sum_{n=1}^{M} \{q_n^+(x) - q_n^-(x)\} = 0, \qquad x \in R^d.$$

Condition 3 is called "*Condition* 3 *with* (Q)" when q_n^+ and q_n^- satisfy Condition (Q).

LEMMA 6. 1. *Let $Z = \{Z_t = [X_t, N_t, J_t],\ \zeta,\ \mathscr{B}_t, P_{[\boldsymbol{x}, \boldsymbol{p}, j]};\ [\boldsymbol{x}, \boldsymbol{p}, j] \in \tilde{\boldsymbol{S}}\}$ be a signed branching Markov process with age on $\tilde{\boldsymbol{S}}$ satisfying Condition 3 with (Q) and let U_t be the semi-group on $\boldsymbol{B}(\tilde{\boldsymbol{S}})$ induced by Z[21]. Then there exists a positive number δ_0 such that $U_t\widetilde{f \cdot 2}([\boldsymbol{x}, \boldsymbol{p}, j])$ exists for any $0 \leqq t < \delta_0$, any $f \in \boldsymbol{C}^*(R^d)$ with $f(\infty) = 0$[22] and $[\boldsymbol{x}, \boldsymbol{p}, j] \in \tilde{\boldsymbol{S}}$.*

---

20) cf. §4.

21) In Lemma 6.1, (b) and (c) in Condition (Q) are not necessary.

22) "$U_t\widetilde{f \cdot 2}$ exists" means that $E_{[\boldsymbol{x}, \boldsymbol{p}, j]}[|\widetilde{f \cdot 2}|(Z_t)] < \infty$. Also the condition "$f(\infty) = 0$" does not have any influence in the sequel, because ∞ is a trap of X_t^i and almost all sample functions of X_t^i with $X_0^i(w) \neq \infty$ do not reach ∞ in any finite time interval.

Proof. First of all, we shall prove the existence of $\delta_0 > 0$ such that $U_t|\widetilde{1\cdot 2}|([x,0,0]) = E_{[x,0,0]}[|\widetilde{1\cdot 2}|(Z_t)]$ is finite for any $t < \delta_0$ and $x \in R^d$, where $1(x) \equiv 1$.

By the same method as in the proof of Lemma 5.4, we have

$$U_t|\widetilde{1\cdot 2}|([\boldsymbol{x}, \boldsymbol{p}, j]) = \sum_{r=0}^{\infty} U_t^{(r)}|\widetilde{1\cdot 2}|([\boldsymbol{x}, \boldsymbol{p}, j]), \qquad [\boldsymbol{x}, \boldsymbol{p}, j] \in \tilde{\boldsymbol{S}}, \tag{6.4}$$

where

$$U_t^{(r)}|\widetilde{1\cdot 2}|([\boldsymbol{x}, \boldsymbol{p}, j]) = E_{[\boldsymbol{x},\boldsymbol{p},j]}[|\widetilde{1\cdot 2}|(Z_t); \eta_r \leqq t < \eta_{r+1}], \quad r \geqq 0.$$

Then, by (ii) of Condition 3, Lemma 5.1 and Lemma 4.1, we have

$$U_t^0|\widetilde{1\cdot 2}|([\boldsymbol{x}, \boldsymbol{p}, j]) = 2^{|\boldsymbol{p}|}U_t^0|\widetilde{1\cdot 2}|([\boldsymbol{x}, \boldsymbol{0}, 0]) = 2^{|\boldsymbol{p}|}\prod_{i=1}^{n} E_{x_i}[1(X'_t)] = 2^{|\boldsymbol{p}|}, \tag{6.5}$$
$$\boldsymbol{x} = [x_1, x_2, \cdots, x_n], \; x_i \in R^d, \; j \in J,$$

where E_x denotes the integral by the probability measure of a standard Brownian motion X'_t. Accordingly, it follows from the strong Markov property of Z_t that

$$\begin{aligned} U_t^{(1)}|\widetilde{1\cdot 2}|([\boldsymbol{x}, \boldsymbol{0}, 0]) &= E_{[\boldsymbol{x},\boldsymbol{0},0]}[U_{t-\eta}^0|\widetilde{1\cdot 2}|(Z_\eta); \eta \leqq t] \\ &= \sum_{|\boldsymbol{p}|=0}^{\infty} 2^{|\boldsymbol{p}|}\int_0^t P_{[\boldsymbol{x},\boldsymbol{0},0]}(N_\eta = \boldsymbol{p}, \; \eta \in ds). \end{aligned} \tag{6.6}$$

On the other hand, if we apply (5.4) to $f(t, \cdot) = 1$, then we have

$$\begin{aligned} &P_{[\boldsymbol{x},\boldsymbol{0},0]}(N_\eta = \boldsymbol{p}, \; \eta \in ds) \\ &= \sum_{i=1}^{n} \sum_{\substack{\Sigma p_l + |\boldsymbol{p}_i| = |\boldsymbol{p}| \\ l \neq i}} \sum_{m=1}^{M+n-1}\sum_{j=0}^{3} \Psi([x_i, 0, 0]; ds, \; [E^m, \boldsymbol{p}_i, j]) \\ &\qquad \cdot \prod_{l \neq i} P_{[x_l,0,0]}(N_s = p_l, \; s < \eta). \end{aligned} \tag{6.7}$$

Since Z_t satisfies Condition 2, (4.7) holds and hence we have

$$\begin{aligned} &\sum_{m=1}^{M+n-1}\sum_{j=0}^{3} \Psi([x_i, 0, 0]; ds, \; [E^m, \boldsymbol{p}_i, j]) \\ &= P_{[x_i,0,0]}(X_{\eta-} \in R^d, \; \eta \in ds, \; N_{\eta-} = |\boldsymbol{p}_i|, \; \sigma_{|\boldsymbol{p}_i|} \leqq s < \sigma_{|\boldsymbol{p}_i|+1}) \\ &= E_{x_i}\left[e^{-2\int_0^s k(X'_v)dv} \frac{\left(\int_0^s k(X'_v)dv\right)^{|\boldsymbol{p}_i|}}{|\boldsymbol{p}_i|!} k(X'_v)ds\right], \end{aligned} \tag{6.8}$$

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and by (3. 11)

$$P_{[x_l,0,0]}(N_s = p_l,\ s<\eta) = E_x[e^{-2\int_0^s k(X'_v)dv} \frac{\left(\int_0^s k(X'_v)dv\right)^{p_l}}{p_l!}]. \tag{6. 9}$$

Now let us consider an nd-dimensional standard Brownian motion $(X_s^{(1)}, X_s^{(2)}, \cdots, X_s^{(n)})$ and denote by $E_{(x_1,x_2,\cdots,x_n)}$ the integral with respect to the probability measure $P_{(x_1,x_2,\cdots,x_n)}$ corresponding to $(X_s^{(1)}, X_s^{(2)}, \cdots, X_s^{(n)})$.[23] Also we set

$$\check{g}(x_1, x_2, \cdots, x_n) = \sum_{i=1}^n g(x_i), \qquad g \in \boldsymbol{B}(R^d).$$

Then it is obtained from (6. 7), (6. 8) and (6. 9) that

$$P_{[\boldsymbol{x},0,0]}(N_\eta = \boldsymbol{p},\ \eta \in ds)$$

$$= \sum_{\substack{i=1}}^n \sum_{\substack{\sum p_l + |\boldsymbol{p}_i| = |\boldsymbol{p}| \\ l \neq i}} E_{x_i}[e^{-2\int_0^s k(X'_v)dv} \frac{\left(\int_0^s k(X'_v)dv\right)^{|\boldsymbol{p}_i|}}{|\boldsymbol{p}_i|!} k(X'_s)ds]$$

$$\cdot \prod_{l \neq i} E_{x_l}[e^{-2\int k(X'_v)dv} \frac{\left(\int_0^s k(X'_v)dv\right)^{p_l}}{p_l!}]$$

$$= E_{(x_1,x_2,\cdots,x_n)}[e^{-2\int_0^s \check{k}(X_v^{(1)}, X_v^{(2)}, \cdots, X_v^{(n)})dv} \tag{6. 10}$$

$$\cdot \{ \sum_{\sum_i p_i = |\boldsymbol{p}|} \prod_{i=1}^n \frac{\left(\int_0^s k(X_v^{(i)})dv\right)^{p_i}}{p_i!} \} \check{k}(X_s^{(1)}, X_s^{(2)}, \cdots, X_s^{(n)})ds]$$

$$= E_{(x_1,x_2,\cdots,x_n)}[e^{-2\int_0^s \check{k}(X_v^{(1)}, X_v^{(2)}, \cdots, X_v^{(n)})dv} \frac{1}{|\boldsymbol{p}|!} \left\{\int_0^s \check{k}(X_v^{(1)}, X_v^{(2)}, \cdots, X_v^{(n)})dv\right\}^{|\boldsymbol{p}|}$$

$$\cdot \check{k}(X_s^{(1)}, X_s^{(2)}, \cdots, X_s^{(n)})ds].$$

Applying the above result to the right hand side of (6. 6), we have

$$U_t^{(1)}\widetilde{|1\cdot 2|}([\boldsymbol{x},0,0]) = \int_0^t E_{[x_1,x_2,\cdots,x_n]}[\check{k}(X_s^{(1)}, X_s^{(2)}, \cdots, X_s^{(n)})]ds$$

$$= \sum_{i=1}^n \int_0^t E_{x_i}[k(X'_s)]ds$$

$$\leq n \, \|k\| \, t.$$

---

[23] $X_s^{(i)}$ are mutually independent and equivalent standard d-dimensional Brownian motions and $x_i \in R^d$, $i=1,2,\cdots,n$.

Now we shall assume that for any $x=[x_1, x_2, \cdots, x_n]$ and $r \geqq 1$

$$U_t^{(r)}|\widetilde{1\cdot 2}|([x,0,0]) \leqq n(n+M)\cdots(n+(r-1)M)\frac{(\|k\|\,t)^r}{r!}. \tag{6. 11}$$

Since we have by the same method as in the proof of Lemma 5. 1

$$U_t^{(r)}|\widetilde{1\cdot 2}|([y,p,j]) = 2^{|p|}U_t^{(r)}|\widetilde{1\cdot 2}|([y,0,0]), \tag{6. 12}$$

it follows from (6. 11) and the strong Markov property of Z_t that for $x=[x_1, x_2, \cdots, x_n]$

$$\begin{aligned}
&U_t^{(r+1)}|\widetilde{1\cdot 2}|([x,0,0])\\
&=\int_0^t\int_{\tilde{S}}\Psi([x,0,0];\,ds,\ [dy,p,j])2^{|p|}U_{t-s}^{(r)}|\widetilde{1\cdot 2}|([y,0,0])\\
&\leqq\int_0^t\int_{\tilde{S}}\Psi([x,0,0];\,ds,\ [dy,p,j])2^{|p|}\,(n+M)\,(n+2M)\,\cdots\,(n+rM)\,\frac{(\|k\|\,(t-s))^r}{r!},
\end{aligned}$$

because, by the assumption that $q_n^+(x)=q_n^-(x)=0$ for $n>M$, $\Psi([x,0,0];ds,$ $[E^m,p,j])=0$ for $m>n+M$. Applying (6. 10), the right hand side of the above inequality equals

$$\begin{aligned}
&(n+M)(n+2M)\cdots(n+rM)\int_0^t E_{(x_1,x_2,\cdots,x_n)}[\check{k}(X_s^{(1)}, X_s^{(2)}, \cdots, X_s^{(n)})]\frac{(\|k\|\,(t-s))^r}{r!}ds\\
&\quad\leqq n(n+M)\cdots(n+rM)\frac{\|k\|^{r+1}}{r!}\int_0^t(t-s)^r ds\\
&\quad= n(n+M)\cdots(n+rM)\frac{(\|k\|\,t)^{r+1}}{(r+1)!}.
\end{aligned}$$

Thus (6. 11) holds for any $r\geqq 1$ because it stands for $r=1$.

Now, by (6. 4) and (6. 5), we have

$$\begin{aligned}
U_t|\widetilde{1\cdot 2}|([x,0,0]) &= 1+\sum_{r=1}^{\infty}U_t^{(r)}|\widetilde{1\cdot 2}|([x,0,0])\\
&\leqq 1+\sum_{r=1}^{\infty}(1+M)\,(1+2M)\cdots(1+(r-1)M)\frac{(\|k\|\,t)^r}{r!}\\
&\leqq 1+\sum_{r=1}^{\infty}(M\|k\|\,t)^r, \qquad x\in E.
\end{aligned}$$

This shows that $U_t|\widetilde{1\cdot 2}|([x,0,0])$ is finite for any $0\leqq t<1/M\|k\|=\delta_0$ and $x\in E$.

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Next we prove the finiteness of $U_t|\widetilde{1\cdot 2}|([\boldsymbol{x}, \boldsymbol{p}, j])$ for any $t<\delta_0$ and $[\boldsymbol{x}, \boldsymbol{p}, j] \in \tilde{\boldsymbol{S}}$. As in the proof of (5. 9), we may obtain

$$U_t^{(r)}|\widetilde{1\cdot 2}|([\boldsymbol{x}, \boldsymbol{0}, 0]) = \sum_{r_1+r_2+\cdots+r_n=r} \prod_{i=1}^{n} U_t^{(r_i)}|\widetilde{1\cdot 2}|([x_i, 0, 0]),$$
$$r \geqq 0, \quad \boldsymbol{x} = [x_1, x_2, \cdots, x_n], \quad t<\delta_0.$$

Applying (6. 11) to this equation, we have

$$U_t^{(r)}|\widetilde{1\cdot 2}|([\boldsymbol{x}, \boldsymbol{0}, 0]) \leqq \sum_{r_1+r_2+\cdots+r_n=r} \prod_{i=1}^{n} (M\|k\|t)^{r_i}, \qquad t<\delta_0.$$

Hence it follows from (6. 4) and (6. 12) that

$$\begin{aligned} U_t|\widetilde{1\cdot 2}|([\boldsymbol{x}, \boldsymbol{p}, j]) &= 2^{|\boldsymbol{p}|} U_t|\widetilde{1\cdot 2}|([\boldsymbol{x}, \boldsymbol{0}, 0]) \\ &= 2^{|\boldsymbol{p}|} \sum_{r=0}^{\infty} U_t^{(r)}|\widetilde{1\cdot 2}|([\boldsymbol{x}, \boldsymbol{0}, 0]) \\ &\leqq 2^{|\boldsymbol{p}|} \{\sum_{r=0}^{\infty} (M\|k\|t)^r\}^n \\ &< \infty, \qquad 0 \leqq t<\delta_0, \ [\boldsymbol{x}, \boldsymbol{p}, j] \in \tilde{\boldsymbol{S}}, \end{aligned}$$

as was to be proved. Q.E.D.

Next we shall consider the following integral equation which turns out to (6. 1):

$$\text{(6. 13)} \quad \begin{aligned} u(t, x) &= \int_{R^d} \left(\frac{1}{2\pi t}\right)^{\frac{d}{2}} e^{-\frac{\|y-x\|^2}{2t}} f(y) dy \\ &+ \int_0^t ds \int_{R^d} \left(\frac{1}{2\pi s}\right)^{\frac{d}{2}} e^{-\frac{\|y-x\|^2}{2s}} k(y) F(y; u(t-s, y)) dy, \\ &\qquad t \geqq 0, \ x \in R^d, \end{aligned}$$

where kF is bounded and satisfies (6. 2) for all $x, x' \in R^d$ and $\xi, \xi' \in R^1$ and also (6. 3). Further set

$$u_0(t, x) = \int_{R^d} \left(\frac{1}{2\pi t}\right)^{\frac{d}{2}} e^{-\frac{\|y-x\|^2}{2t}} f(y) dy$$

$$u_{n+1}(t, x) = u_0(t, x) + \int_0^t \int_{R^d} \left(\frac{1}{2\pi s}\right)^{\frac{d}{2}} e^{-\frac{\|y-x\|^2}{2s}} k(y) F(y; u_n(t-s, y)) dy,$$
$$t \geqq 0, \ x \in R^d, \ n \geqq 0.$$

Then the following result is well known.[24)]

LEMMA 6.2. *For a given* $f \in \bar{C}^*(R^d)^+$, *the following holds:*

(i) *Let* $u(t, x; f)$ *be the unique solution of* (6.13) *with initial value* f. *Then we have*

$$0 \leqq u(t, x; f) \leqq 1, \qquad t \geqq 0, \ x \in R^d. \tag{6.14}$$

(ii) *For any positive constant* T, $u_n(t, x)$ *defined above converges to* $u(t, x; f)$ *uniformly in* $(t, x) \in [0, T] \times R^d$.

Let Z_t be a signed branching Markov process with age on $\tilde{S}$ satisfying Condition 3 with (Q) and let U_t be the semi-group induced by Z_t. If we consider the integral equation (6.13), where kF is given by

$$k(x)F(x; \xi) = \sum_{n=1}^{M} \{q_n^+(x) - q_n^-(x)\} \xi^n, \qquad x \in R^d, \ \xi \in R^1,$$

then it follows from the uniqueness of the bounded solution of (6.13), Lemma 6.1 and Theorem 4.1 that

$$(U_t \widetilde{f \cdot 2})|_E(x) = u(t, x; f), \qquad f \in \bar{C}^*(R^d)^+, \ 0 \leqq t < \delta_0, \ x \in R^d,$$

where $u(t, x; f)$ denotes the solution of (6.13) with initial value f. On the other hand, by Lemma 6.2, the solution $\tilde{u}(t, x; f)$ of the integral equation (6.13) where kF is replaced by

$$k(x)F_1(x; \xi) = \begin{cases} \sum_{n=1}^{M} \{q_n^+(x) - q_n^-(x)\} \xi^n, & x \in R^d, \ \xi \in [0, 1], \\ 0 \qquad\qquad\qquad\qquad\quad , & \text{otherwise,} \end{cases}$$

satisfies '$0 \leqq \tilde{u}(t, x; f) \leqq 1$' for $f \in \bar{C}^*(R^d)^+$, because kF_1 satisfies the condition (6.3) and (6.2) for all $x, x' \in R^d$ and $\xi, \xi' \in R^1$. Since $F(x; \xi) = F_1(x; \xi)$ for $\xi \in [0, 1]$, we have $u(t, x; f) = \tilde{u}(t, x; f) \in \bar{C}^*([0, \infty) \times R^d)^+$. Hence, using Lemma 6.1 again, we can consider the following:

$$\begin{aligned} U_t(U_s \widetilde{f \cdot 2})([x, 0, 0]) &= U_t(\widetilde{(U_s \widetilde{f \cdot 2})|_E \cdot 2})([x, 0, 0]) \\ = u(t, x; u(s, \cdot\,; f)) &= u(t + s, x; f), \\ &\qquad f \in \bar{C}^*(R^d)^+, \ 0 \leqq t, s < \delta_0, \ x \in R^d. \end{aligned}$$

But, in general, we can not express the left hand side of the above equation

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24) cf. Kolmogoroff-Petrovsky-Piscounoff [8], theorems 1, 4 and 6.

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by $(U_{t+s}\widetilde{f\cdot 2})|_E(x)$ because it may happen that $E_{[x,0,0]}[|\widetilde{f\cdot 2}|(Z_t)]=\infty$. Even so, still we have the following

THEOREM 6. 1. *Let* $Z=\{Z_t=[X_t,N_t,J_t],\ \zeta,\ \mathscr{B}_t,\ P_{[\boldsymbol{x},\boldsymbol{p},j]};\ [\boldsymbol{x},\boldsymbol{p},j]\in\tilde{\boldsymbol{S}}\}$ *be a signed branching Markov process with age on* $\tilde{\boldsymbol{S}}$ *satisfying Condition* 3 *with* (Q) *and let* U_t *be the semi-group on* $\boldsymbol{B}(\tilde{\boldsymbol{S}})$ *induced by* Z. *Then, for* $f\in\bar{\boldsymbol{C}}^*(R^d)^+$, *we can define* $\hat{U}_t\widetilde{f\cdot 2}([x,0,0])$ *with the following properties*:

(i) $\hat{U}_t\widetilde{f\cdot 2}([x,0,0])=(U_t\widetilde{f\cdot 2})|_E(x)$ *if* $U_t\widetilde{f\cdot 2}$ *exists*, $f\in\bar{C}^*(R^d)^+$.

(ii) $u(t,x)=\hat{U}_t\widetilde{f\cdot 2}([x,0,0])$ *is the unique solution of* (6. 13) *with initial value* $f\in\bar{\boldsymbol{C}}^*(R^d)^+$, *where* kF *is given by*

$$k(x)F(x;\xi)=\sum_{n=1}^{M}\{q_n^+(x)-q_n^-(x)\}\,\xi^n,\qquad x\in R^d,\ \xi\in R^1.$$

Proof. According to Lemma 6. 1, there exists $\delta_0>0$ such that $U_t\widetilde{f\cdot 2}([\boldsymbol{x},\boldsymbol{p},j])$ exists for $f\in\bar{\boldsymbol{C}}^*(R^d)^+$, $0\leqq t<\delta_0$ and $[\boldsymbol{x},\boldsymbol{p},j]\in\tilde{\boldsymbol{S}}$. Set

$$\hat{U}_t\widetilde{f\cdot 2}([x,0,0])=U_t\widetilde{f\cdot 2}([x,0,0]),\ \ f\in\bar{\boldsymbol{C}}^*(R^d)^+,\ \ 0\leqq t<\delta_0,\ \ x\in E.$$

Since $(U_t\widetilde{f\cdot 2})|_E(x)=u(t,x;f)\in\bar{\boldsymbol{C}}^*([0,\delta_0)\times R^d)^+$ as was mentioned already, $(\hat{U}_t\widetilde{f\cdot 2})|_E(x)$ belongs to $\bar{\boldsymbol{C}}^*([0,\delta_0)\times R^d)^+$ and also $(\hat{U}_t\widetilde{f\cdot 2})|_E(\infty)=0$. Using Lemma 6. 1 again, set

$$\hat{U}_{t+s}\widetilde{f\cdot 2}([x,0,0])=(U_t\widetilde{(\hat{U}_s\widetilde{f\cdot 2})|_E\cdot 2})|_E(x),\ \ f\in\bar{\boldsymbol{C}}^*(R^d)^+,\ \ 0\leqq t,s<\delta_0,\tag{6. 15}$$

because the right hand side of the above equation is equal to

$$u(t,x;(U_s\widetilde{f\cdot 2})|_E)=u(t+s,x;f),$$

and hence the right hand side of (6·15) depends only on $t+s$ for given $f\in\bar{\boldsymbol{C}}^*(R^d)^+$ and $x\in E$. Repeating this procedure, we can see that $\hat{U}_t\widetilde{f\cdot 2}([x,0,0])$ can be defined for all $t\geqq 0$ and it is the unique solution of (6. 13) with initial value $f\in\bar{\boldsymbol{C}}^*(R^d)^+$. The property (i) of the theorem is evident by the definition of $\hat{U}_t$ and the semi-group property of U_t. Q.E.D.

In the sequel of this section, we shall use the notation U_t instead of $\hat{U}_t$ because $(U_t\widetilde{f\cdot 2})|_E(x)=\hat{U}_t\widetilde{f\cdot 2}([x,0,0])$ if $U_t\widetilde{f\cdot 2}$ exists.

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Let $\{k_i(x), (q_{i,n}^+(x), q_{i,n}^-(x)); n=1, 2, \cdots, M_i < \infty\}$, $i=1,2,3,\cdots$, be systems satisfying Condition (Q) and $Z_t^{(i)}$ be signed branching Markov processes with age on $\tilde{S}$ satisfying Condition 3 with (Q) for given $\{k_i(x), (q_{i,n}^+(x), q_{i,n}^-(x)); n=1,2,\cdots, M_i\}$. Let also $U_{i,t}$ be the semi-group induced by $Z_t^{(i)}$ and set

$$F_i(x;\xi) = \sum_{n=1}^{M_i} \frac{\{q_{i,n}^+(x) - q_{i,n}^-(x)\}}{k_i(x)} \xi^n, \qquad i=1,2,3,\cdots. \tag{6.16}$$

According to Theorem 6.1, if k_iF_i satisfies

$$|k_i(x)F_i(x;\xi) - k_i(x')F_i(x';\xi')| \leqq K\{\|x-x'\| + |\xi-\xi'|\}, \quad x, x' \in R^d, \ \xi, \xi' \in [0,1], \tag{6.17}$$

where K is a positive constant independent of i, then $u^{(i)}(t,x) = (U_{i,t}\widetilde{f\cdot 2})|_E(x)$ is the solution of the integral equation (6.13) with initial value $f \in \bar{C}^*(R^d)^+$ where kF is replaced by k_iF_i. Then we have

THEOREM 6.2. *Let $\{k_i(x), (q_{i,n}^+(x), q_{i,n}^-(x)); n=1, 2, \cdots, M_i\}$ be systems satisfying Condition* (Q) *and let $k_i(x)F_i(x;\xi)$ given in* (6.16), $i=1,2,3,\cdots$, *satisfy* (6.17). *If k_iF_i converges to kF considered in* (6.13) *uniformly in $(x,\xi) \in R^d \times [0,1]$, then $\{u^{(i)}(t,x) = (U_{i,t}\widetilde{f\cdot 2})|_E(x); i=1,2,3,\cdots\}$ is a uniformly convergent sequence in $(t,x) \in [0,T] \times R^d$ for any given $T>0$. Moreover, $u(t,x) = \lim_{i\to\infty} u^{(i)}(t,x)$ is the unique solution of the integral equation* (6.13) *with initial value $f \in \bar{C}^*(R^d)^+$.*

Proof. According to (i) of Lemma 6.2, we may regard $k_i(x)F_i(x;\xi) = k(x)F(x;\xi) = 0$ for $\xi \notin [0,1]$ so far as we consider the solution of integral equation of type (6.13) with initial value $f \in \bar{C}^*(R^d)^+$, because F_i and F satisfy (6.3) and (6.17), and hence we may apply Lemma 6.2 in the present case.

Let us set

$$u_0^{(i)}(t,x) = \int_{R^d} \left(\frac{1}{2\pi t}\right)^{\frac{d}{2}} e^{-\frac{\|y-x\|^2}{2t}} f(y)dy,$$

$$u_{n+1}^{(i)}(t,x) = u_0^{(i)}(t,x) + \int_0^t ds \int_{R^d} \left(\frac{1}{2\pi s}\right)^{\frac{d}{2}} e^{-\frac{\|y-x\|^2}{2s}} k_i(y)F_i(y; u_n^{(i)}(t-s,y))dy,$$
$$n \geqq 0, \qquad i=1,2,3,\cdots.$$

For any given $\varepsilon>0$, we take also N_0 so large as

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(6. 18) $$|k_i(x)F_i(x\,;\xi)-k_j(x)F_j(x\,;\xi)|<\varepsilon\,, \qquad x\in R^d,\ \ \xi\in R^1,$$ [25]

holds for any $i,j\geqq N_0$. Noting that $u_0^{(i)}(t,x)$ is independent of i, we can see by (6. 18)

$$|u_1^{(i)}(t,x)-u_1^{(j)}(t,x)|\leqq\int_0^t ds\int_{R^d}\Big(\frac{1}{2\pi s}\Big)^{\frac{d}{2}}e^{-\frac{\|y-x\|^2}{2s}}$$

$$\cdot\,|k_i(y)F_i(y\,;u_0^{(i)}(y,t-s,y))-k_j(y)F_j(y\,;u_0^{(j)}(t-s,y))|dy\leqq\varepsilon t,$$

$$x\in R^d,\ \ i,j\geqq N_0\,.$$

Assume

(6. 19) $$|u_n^{(i)}(t,x)-u_n^{(j)}(t,x)|\leqq\varepsilon t\sum_{p=0}^{n-1}\frac{(Kt)^p}{p\,!}\,, \qquad x\in R^d,\ \ i,j\geqq N_0\,,$$

and it follows from (6. 17), (6. 18) and (6. 19) that

$$|u_{n+1}^{(i)}(t,x)-u_{n+1}^{(j)}(t,x)|$$

$$\leqq\int_0^t ds\int_{R^d}\Big(\frac{1}{2\pi s}\Big)^{\frac{d}{2}}e^{-\frac{\|y-x\|^2}{2s}}\{|k_i(y)F_i(y\,;u_n^{(i)}(t-s,y))-k_j(y)F_j(y\,;u_n^{(j)}(t-s,y))|dy$$

$$\leqq\int_0^t ds\int_{R^d}\Big(\frac{1}{2\pi s}\Big)^{\frac{d}{2}}e^{-\frac{\|y-x\|^2}{2s}}\{|k_i(y)F_i(y\,;u_n^{(i)}(t-s,y))-k_j(y)F_j(y\,;u_n^{(i)}(t-s,y))|$$

$$+\,|k_j(y)F_j(y\,;u_n^{(i)}(t-s,y))-k_j(y)F_j(y\,;u_n^{(j)}(t-s,y))|\}dy$$

$$\leqq\varepsilon t+\int_0^t ds\int_{R^d}\Big(\frac{1}{2\pi s}\Big)^{\frac{d}{2}}e^{-\frac{\|y-x\|^2}{2s}}K\,|u_n^{(i)}(t-s,y)-u_n^{(j)}(t-s,y)|dy$$

$$\leqq\varepsilon t+\varepsilon t\sum_{p=1}^{n}\frac{(Kt)^p}{p\,!}$$

$$=\varepsilon t\sum_{p=0}^{n}\frac{(Kt)^p}{p\,!}\,, \qquad x\in R^d,\ \ i,j\geqq N_0\,.$$ [26]

Therefore, by induction, (6. 19) holds for any $n\geqq 0$. Then (ii) of Lemma 6. 2 shows that

$$|u^{(i)}(t,x)-u^{(j)}(t,x)|\leqq\varepsilon te^{Kt}\,, \qquad i,j\geqq N_0\,.$$

Since $\varepsilon>0$ is arbitrary, the above inequality proves the first half of the theorem.

---

25) We regard $k_i(x)F_i(x\,;\xi)=k(x)F(x\,;\xi)=0$ for $\xi\notin[0,\ 1]$.

26) Assume $K\geqq 1$, if necessary.

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On the other hand, $u^{(i)}(t,x)$ is the solution of

$$u^{(i)}(t,x)=u_0^{(i)}(t,x)+\int_0^t ds\int_{R^d}\left(\frac{1}{2\pi s}\right)^{\frac{d}{2}}e^{-\frac{\|y-x\|^2}{2s}}k_i(y)F_i(y;u^{(i)}(t-s,y))dy\,.$$

Letting i tend to infinity in the above equation, we can see that $u(t,x)=\lim_{i\to\infty}u^{(i)}(t,x)$ is the unique solution of (6. 13) with initial value $f\in\bar{C}^*(R^d)^+$.

Q.E.D.

Transforming (6. 13) into the corresponding differential equation, we have

COROLLARY 6. 1. *Let $k_i(x)F_i(x,u), u^{(i)}(t,x)$ and $k(x)F(x,u)$ be functions as in Theorem* 6. 3. *Then $u(t,x)=\lim_{i=\infty}u^{(i)}(t,u)$ is the bounded solution of the parabolic equation*

$$\frac{\partial u(t,x)}{\partial t}=\frac{1}{2}\Delta u(t,x)+k(x)F(x;u(t,x))\,,$$

with initial value $f\in\bar{C}^*(R^d)^+$.

In the following corollary, we consider the case where $k(x)$ is a positive constant and $F(x;\xi)$ is a function of ξ alone.

COROLLARY 6. 2. *Let $F(\xi)$ be a function which is continuously differentiable on* $[0,1]$ *and* $F'(0)>0$. *Let also $F(\xi)$ satisfies the condition:*

(6. 20) $$F(0)=F(1)=0 \text{ and } 0<F(\xi) \text{ for } 0<\xi<1\,.$$

Then the unique solution $u(t,x;f)$ of the parabolic equation

(6. 21) $$\frac{\partial u(t,x)}{\partial t}=\frac{1}{2}\Delta u(t,x)+F(u)\,,$$

with initial value $f\in C^*(R^d)^+$ is expressed as the limit of $u^{(i)}(t,x)$ of the type which appeared in Theorem 6. 2.

Proof. Since $F'(\xi)$ is continuous on $[0,1]$, there exists a sequence of polynomials $g_i'(\xi)$ converging to $F'(\xi)$ uniformly on $[0,1]$. Set

$$G_i(\xi)=\int_0^\xi g_i'(s)ds+c_i\xi\,,\qquad i=1,2,3,\cdots,$$

where c_i is chosen so that $G_i(1)=0$. Then c_i tends to zero as i increases, because $F(1)=0$ and $g_i'(\xi)$ converge to $F'(\xi)$ uniformly on $[0,1]$. Hence the polynomials $G_i(\xi)$ converge to $F(\xi)$ uniformly on $[0,1]$ and $G_i'(\xi)$ is uniformly bounded. Moreover, $\xi_i = \inf\{\xi > 0; G_i(\xi) = 0\}$ tends to 1 as i increases because $F'(0) > 0$ and $F(\xi) > 0$ for $0 < \xi < 1$. Expressing $G_i(\xi)$ in the following form:

$$G_i(\xi) = \sum_{n=1}^{M_i} (q^+_{i,n} - q^-_{i,n})\xi^n, \qquad i = 1, 2, 3, \cdots,\text{[27]}$$

where $q^+_{i,n}$ and $q^-_{i,n}$ are non-negative constants such that $q^+_{i,n} q^-_{i,n} = 0$ and $\sum_{n=1}^{M_i} (q^+_{i,n} - q^-_{i,n}) = 0$, we set

$$\left.\begin{aligned} k_i &= \sum_{n=1}^{M_i} (q^+_{i,n} + q^-_{i,n}), \\ F_i(\xi) &= \frac{1}{k_i} \sum_{n=1}^{M_i} (q^+_{i,n} - q^-_{i,n})\xi^n, \end{aligned}\right\} \quad i = 1, 2, 3, \cdots.$$

Since $k_i F_i = G_i$, $k_i F_i(\xi)$ converges to $F(\xi)$ uniformly on $[0,1]$, and there exists a positive constant K such that

$$|k_i F_i(\xi) - k_i F_i(\xi')| \leqq K |\xi - \xi'|, \qquad \xi, \xi' \in [0,1], \ i = 1, 2, 3, \cdots.$$

Also it holds that

$$\left.\begin{aligned} &0 < F_i(\xi), \quad 0 < \xi < \xi_i, \\ &k_i F_i(0) = k_i F_i(\xi_i) = 0, \end{aligned}\right\} \quad i = 1, 2, 3, \cdots.$$

Now let $Z^{(i)}$ be signed branching Markov process with age on $\tilde{S}$ satisfying Condition 3 with (Q) for $\{(q^+_{i,n}, q^-_{i,n}); n = 1, 2, \cdots, M_i\}$ given above where the condition $0 < F(x; \xi)$, $0 < \xi < 1$, is replaced by $0 < F_i(\xi)$ for $0 < \xi < \xi_i$, and let $U_{i,t}$ be the semi-group induced by $Z^{(i)}$. Then, by Theorem 6. 1, $u^{(i)}(t, x) = (U_{i,t}\widehat{f \cdot 2})|_E(x)$ is the solution of the integal equation of (6. 13), where kF is replaced by $k_i F_i$, with initial value $f \in C^*(R^d)^+$ whose norm $\|f\|$ is less than ξ_i. Moreover, it holds that

$$0 \leqq u^{(i)}(t, x) \leqq \xi_i \leqq 1.\text{[28]}$$

Since $k_i F_i(\xi)$ converge to $F(\xi)$ uniformly on $[0,1]$, we can see, as in the proof of the convergence of $u^{(i)}(t, x)$ in Theorem 6. 2, that $u^{(i)}(t, x)$ converges to the solution of (6. 21) with initial value $f \in C^*(R^d)^+$, where $\|f\| \leqq \inf \{\xi_i; i = 1, 2, 3, \cdots\}$, because the integral equation of type (6. 13) is equiva-

27) Since $G_i(0) = 0$, the constant term of $G_i(\xi)$ is zero.

28) $k_i F_i(0) = k_i F(\xi_i) = 0$ and $k_i F(\xi) > 0$ for $0 < \xi < \xi_i$. So we may consider ξ_i instead of 1 in Theorem 6.1.

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lent to (6. 21) in the present case. On the other hand, ξ_i tends to one as i increases, and hence the same assertion holds for any $f \in C^*(R^d)^+$.

Q.E.D.

§ 7. Construction of signed branching Markov processes with age (I). (Non-branching part.)

In this and in the next section, we shall construct the process discussed in the previous sections. Although such a process can be constructed by continuation of sample paths,[29] we shall here construct them by an analytic method originated by J.E. Moyal [10].

In this section, we shall deal with a process corresponding to a non-branching part. For this purpose, we construct a process which is able to describe the creation of mass, i.e. using the process, we can interpret probabilistically the parabolic equation:

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\Delta u(t,x) + k(x)u(t,x), \qquad x \in R^d, \tag{7. 1}$$

where $k(x)$ is a bounded continuous continuous function on R^d.

First, we shall state some known results which are useful for the construction of our processes. Let S be a locally compact Hausdorff space satisfying the second axiom of countability and let $\chi_0(t,x,\cdot)$ and $\Psi(x;t,\cdot)$ be measures on $\mathscr{B}(S)$ for fixed $x \in S$ and $t \geqq 0$. Let also $\chi_0(t,\cdot,B)$ and $\Psi(\cdot\,; t,B)$ be Borel measurable functions for fixed t and $B \in (S)$. Let the pair of χ_0 and Ψ also satisfy the following conditions:

$$\chi_0(t+s,x,B) = \int_S \chi_0(t,x,dy)\chi_0(s,y,B), \tag{7. 2}$$

$$\lim_{t\to\infty} \Psi(x;t,S) = 1 - \lim_{t\to\infty} \chi_0(t,x,B), \tag{7. 3}$$

$$\Psi(x;t+s,B) = \Psi(x;t,B) + \int_S \chi_0(t,x,dy)\Psi(y;s,B), \tag{7. 4}$$

$$\Psi(x;t,S) \text{ is continuous in } t, \tag{7. 5}$$

$$x \in S,\ B \in \mathscr{B}(S),\ t,s \geqq 0.$$

Then it is said that χ_0 and Ψ satisfy the $\chi_0\Psi$-condition.[30] When χ_0 and Ψ

29) cf. M. Nagasawa [13].

30) Moyal's $\chi_0\Psi$–condition is stated for non stationary Markov processes. The condition stated here is the one for the stationary case and is strengthened in the part of (7.5). (cf. Moyal [10].)

satisfy the $\chi_0\Psi$-condition, by (7. 4), $\Psi(x; t, B)$ is nomotone non-decreasing in t. Let $\Psi(x; dt, B)$ be the measure induced by $\Psi(x; t, B)$ for fixed x and B. We define Ψ_r and χ_r by

$$\Psi_1(x; dt, B) = \Psi(x; dt, B),$$

(7. 6) $$\Psi_{r+1}(x; dt, B) = \int_0^t\int_S \Psi_r(x; ds, dz)\Psi(z; d(t-s), B), \qquad r \geqq 0,$$

$$\Psi_r(x; t, B) = \int_0^t \Psi_r(x; ds, B), \qquad r \geqq 1,$$

(7. 7) $$\chi_r(t, x, B) = \int_0^t\int_S \Psi_r(x; ds, dz)\chi_0(t-s, z, B), \qquad r \geqq 1.$$

Then we have the following

Lemma 7. 1[31] (*J.E. Moyal*) *If the $\chi_0\Psi$-condition is satisfied, then it holds that:* (i)

(7. 8) $$\Psi_{r+r'}(x; dt, B) = \int_0^t\int_S \Psi_r(x; ds, dy)\Psi_{r'}(y; d(t-s), B), \qquad r, r' \geqq 1,$$

(7. 9) $$\chi_{r+r'}(t, x, B) = \int_0^t\int_S \Psi_r(x; ds, dy)\chi_{r'}(t-s, y, B), \qquad r \geqq 1,\ r' \geqq 0,$$

(7. 10) $$\chi_r(t+s, x, B) = \sum_{r'=0}^{r}\int_S \chi_{r'}(t, x, dy)\chi_{r-r'}(s, y, B), \qquad r \geqq 0,$$

(7. 11) $$\sum_{r=0}^{\infty}\chi_r(t, x, S) = 1 - \lim_{r\to\infty}\Psi_r(x; t, S), \qquad x \in S,\ B \in \mathscr{B}(S),\ t, s \geqq 0.$$

(ii) *The function χ defined by*

(7. 12) $$\chi(t, x, B) = \sum_{r=0}^{\infty}\chi_r(t, x, B), \qquad x \in S,\ B \in \mathscr{B}(S),\ t \geqq 0,$$

satisfies

(7. 13) $$\chi(t+s, x, B) = \int_S \chi(t, x, dy)\chi(s, y, B)$$

and

(7. 14) $$\chi(t, x, B) = \chi_0(t, x, B) + \int_0^t\int_S \Psi(x; ds, y)\chi(t-s, y, B).$$

[31] cf. J.E. Moyal [9], theorems in §§2–8.

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(iii) *For given* χ_0 *and* Ψ, χ *is the minimal non-negative solution of* (7. 14), *and if*

(7. 15) $$\lim_{r\to\infty} \Psi_r(x; t, S) = 0$$

holds, then χ *is the unique solution of* (7. 14).

Now let us set

(7. 16) $$\begin{aligned} T_t^{(r)} f(x) &= \int_S \chi_r(t, x, dy) f(y), \qquad r \geqq 0, \\ T_t f(x) &= \int_S \chi(t, x, dy) f(y), \end{aligned} \qquad f \in \boldsymbol{C}_0(S).$$

According to (7. 2) and (7. 13), there exist two Markov processes (but we do not assume the right continuity of sample paths here) X_t^0 and X_t whose semi-groups are given by $T_t^{(0)}$ and T_t respectively. When we consider that there exists a Markov time τ of X_t and X_t^0 is the process obtained by the killing of X_t at the time τ, it is expected that $\Psi(x; dt, B)$ denotes $P_x(\tau \in ds,\ X_\tau \in B)$ under certain conditions, where P_x denotes the probability measure of X_t. About this, we quote from Sirao [17] the following

LEMMA 7. 2.[32] *Let* Ψ_r, χ_r *and* χ *be the functions defined by* (7. 6), (7. 7) *and* (7. 12). *Let them also satisfy the following conditions:* (a) $T_t^{(0)}$ *given in* (7. 16) *is strongly continuous on* $\boldsymbol{C}_0(S)$. (b) $T_t^{(r)}$ *given in* (7. 16) *maps* $\boldsymbol{C}_0(S)$ *into itself and also we have*

$$\lim_{t\to 0} \| T_t^{(r)} f \| = 0, \qquad r \geqq 1,\ f \in \boldsymbol{C}_0(S).$$

Then it holds that (i) *there exists a strong Markov process* $X = \{X_t, \zeta, \mathscr{B}_t, P_x;\ x \in S\}$ *corresponding to the semi-group* T_t *given in* (7. 16) *whose sample paths are right continuous and quasi left continuous,*[33] (ii) *there exists a* ($\mathscr{B}_t$-) *Markov time* τ *of* X_t *such that there exists a strong Markov process* $X^0 = \{X_t^0, \tau, \mathscr{B}_t^0, P_x^0;\ x \in S\}$ *corresponding to the semi-group* $T_t^{(0)}$ *and* X^0 *is the killed process of* X *at the time* τ, *and* (iii) *setting*

---

32) cf. [17], Theorem 1.

33) A right continuous strong Markov process X_t on $\mathscr{X}$ is said to be quasi left continuous if, for any monotone non-decreasing sequence $\{\tau_n;\ n \geqq 0\}$ of Markov times,
$$P_x(\lim_{n\to\infty} X_{\tau_n} = X_\tau,\ \tau < \varsigma) = P_x(\tau < \varsigma), \qquad x \in \mathscr{X},$$
holds, where $\tau = \lim_{n\to\infty} \tau_n$ and ς denotes the terminal time of X_t.

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$$\tau_0 = \tau, \ \tau_1 = \tau, \ \tau_{r+1} = \tau_r + \theta_{\tau_r}\tau, \qquad r \geqq 1,^{34)}$$

we have

(7. 17) $$P_x(X_t \in B, \ \tau_r \leqq t < \tau_{r+1}) = \chi_r(t, x, B)$$

(7. 18) $$P_x(X_{\tau_r} \in B, \ \tau_r \in dt) = \Psi_r(x; dt, B),$$
$$x \in S, \ B \in \mathscr{B}(S), \ t \geqq 0, \ r \geqq 0.$$

Now let us apply the above lemmas for our case. In the sequel of this section, let E be a locally compact Hausdorff space[35] satisfying the second axiom of countability and $X' = \{X'_t, \mathscr{B}'_t, P_x; x \in E\}$ be a conservative Feller process. Then the semi-group H_t induced by X'_t is strongly continuous on $C_0(E)$.[36] As in § 2, we shall consider the topological sum $S = \bigcup_{p=0}^{\infty} E \times \{p\} = E \times N$, where $N = \{0, 1, 2, \cdots\}$. Then $S \cup \{\delta\}$, δ being an isolated point, is a locally compact Hausdorff space satisfying the second axiom of countability. A point of S and a Borel sub-set of S are denoted by $[x, p]$ and $[A, p]$ respectively, where $A \in \mathscr{B}(E)$.

Let $k(x)$ be a bounded continuous function on E and let $k(x) = k^+(x) - k^-(x)$ where $k^+(x) = \max(k(x), 0)$ and $k^-(x) = \max(-k(x), 0)$. Then

(7. 19)
$$\varphi_t(w) = \int_0^t |k(X'_s(w))| ds,$$
$$\varphi_t^+(w) = \int_0^t k^+(X'_s(w)) ds,$$
$$\varphi_t^-(w) = \int_0^t k^-(X'_s(w)) ds,$$

are non-negative additive functionals of X' and hence we can consider the $exp(-\varphi_t)$ sub-process of X', which will be denoted by $X^0 = \{X_t^0, \sigma, \mathscr{B}_t^0, P_x^0; x \in E\}$. Then it trivially holds that

(7. 20) $$P_x^0(X_t^0 \in B) = P_x^0(X_t^0 \in B, \ t < \sigma) = E_x[e^{-\varphi_t}; X'_t \in B],$$
$$x \in E, \ B \in \mathscr{B}(E),$$

---

34) τ_r's are Markov times. cf. Ito-McKean [7], p. 87.

35) In this section, we do not assume that E is a compact space, because we consider the equation of type (7.1) with initial value $f \in C(E)$ and the assumption of non-compactness does not cause any difficulty in the discussions of this section.

36) When we consider H_t on $C_0(E)$, H_t may be regarded as the semi-group on $C(E \cup \{\infty\})$ where $E \cup \{\infty\}$ denotes the one-point compactification of E. Then it is known that the convergence of $H_t f(x)$ to $f(x)$ at any $x \in E$ implies the strong convergence of $H_t f$ to f, i.e. $\|H_t f - f\| \to 0$ as $t \to 0$. (For instance, cf. Dynkin [2], Theorem 5.)

where E_x denotes the integral by P_x.

Now let us set

$$(7.21)\qquad \chi_0(t,[x,p],[B,q]) = \begin{cases} \delta_{pq} P_x^0(X_t^0 \in B), & x \in E, \\ 0 & , \text{ if } [x,p]=\delta \text{ and } \delta \notin [B,q], \\ 1 & , \text{ if } [x,p]=\delta \text{ and } \delta \in [B,q], \end{cases}$$

and

$$(7.22)\qquad \Psi([x,p];dt,[B,q]) = \begin{cases} E_x\left[e^{-\varphi_t}\frac{d\varphi_t^+}{dt}dt;\ X_t' \in B\right], & \text{if } x\in E \text{ and } q=p+1, \\ E_x\left[e^{-\varphi_t}\frac{d\varphi_t^-}{dt}dt\right], & \text{if } x \in E \text{ and } \delta \in [B,q], \\ 0 & , \text{ otherwise},^{37)} \end{cases}$$

$$[x,p] \in S,\ [B,q] \in \mathscr{B}(S\cup\{\delta\}),$$

where δ_{pq} denotes Kroncecker's delta. Then χ_0 is a measure on $\mathscr{B}(S\cup\{\delta\})$ with parameters t and $[x,p]\in S\cup\{\delta\}$ and Ψ is a measure on $\mathscr{B}([0,\infty)\times S\cup\{\delta\})$ with parameter $[x,p]\in S\cup\{\delta\}$. Moreover $\chi_0(t,[x,p],\ [B,p+q])$ and $\Psi([x,p];$ $dt,[B,p+q+1])$ are independent of p and vanish for $q\neq 0$. Let also set

$$(7.23)\qquad \Psi([x,p];t,[B,q]) = \int_0^t \Psi([x,p];ds,\ [B,q]),$$

$$[x,p]\in S\cup\{\delta\},\ t\geqq 0,\ [B,q]\in \mathscr{B}(S\cup\{\delta\}].$$

Then we have

LEMMA 7.3. *Let $\chi_0([x,p],\ t,\ \cdot)$ and $\Psi([x,p];t,\cdot)$ be measures given in* (7.21) *and* (7.23) *respectively. Then they satisfy the Moyal's $\chi_0\Psi$-condition.*

Proof. By the definition of χ_0 and Ψ, $\chi_0(t,\delta,\{\delta\})=1$ and $\Psi(\delta;t,S)=0$ for any $t\geqq 0$. So it suffices to show that the conditions (7.2) – (7.5) hold for $[x,p]\in S$.

Since X_t^0 is a Markov process and $\chi_0([x,p],t,[\cdot,p])$ corresponds to the transition function of X_t^0, (7.2) holds evidently.

Combining (7.22) and (7.23), we can see that

$$\Psi([x,p];t,S\cup\{\delta\}) = E_x\left[\int_0^t e^{-\varphi_s}d(\varphi_s^+ + \varphi_s^-)\right]$$

37) $\frac{d\varphi_t}{dt}$, $\frac{d\varphi_t^+}{dt}$ and $\frac{d\varphi_t^-}{dt}$ denote the derivatives of φ_t, φ_t^+ and φ_t^- in the sense of Radon-Nikodym respectively.

$$
\begin{aligned}
&= E_x\Big[\int_0^t(-de^{-\varphi_s})\Big]\\
&= E_x[1-e^{-\varphi_t}]\\
&= 1-\chi_0(t,[x,p],S\cup\{\delta\}),\quad [x,p]\in S,
\end{aligned}
$$

which proves (7. 3).

Now set $B^+=B\cap\{x;k(x)\geqq 0\}$ and $B^-=B\cap\{x;k(x)<0\}$ for any $B\in\mathscr{B}(E)$. Then we have

$$
\begin{aligned}
&\Psi([x,p];\,t+s,\,[B,p+1])\\
&= E_x\Big[\int_0^{t+s}e^{-\varphi_v}I_B((X'_v)\,d\varphi_v^+\Big]\\
&= E_x\Big[\int_0^t e^{-\varphi_v}I_B(X'_v)\,d\varphi_v^+\Big]+E_x\Big[e^{-\varphi_t}E_{X'_t}\Big[\int_0^s e^{-\varphi_v}I_B(X'_v)\,d\varphi_v^+\Big]\Big]\\
&= \Psi([x,p];\,t,\,[B,p+1])+\int_E\chi_0(t,\,[x,p],\,[dy,p])\Psi([y,p];\,s,\,[B,p+1])\\
&= \Psi([x,p];\,t,\,[B,p+1])+\int_{S\cup\{\delta\}}\chi_0(t,[x,p],\,[dy,q])\Psi([y,q];\,s,\,[B,p+1]),\\
&\qquad\qquad\qquad [x,p]\in S,\ \ t,s\geqq 0,\ \ B\in\mathscr{B}(E),
\end{aligned}
$$

where I_B denotes the indicator function of B. Similarly we get

$$
\begin{aligned}
&\Psi([x,p];\,t+s,\,\{\delta\})\\
&= \Psi([x,p];\,t,\,\{\delta\})+\int_{S\cup\{\delta\}}\chi_0(t,[x,p],\,[dy,q])\Psi([y,q];\,s,\,\{\delta\}).
\end{aligned}
$$

The above two equations prove (7. 4), because $\chi_0(t,[x,p],\,[B,q])=\Psi([x,p];\,t,[B,q+1])=0$ for $p\neq q$, $B\in\mathscr{B}(E)$.

Since (7. 5) is evident by the definition of Ψ, we have proved the lemma. Q.E.D.

Now let us set

$$
\begin{aligned}
&\Psi_1([x,p];\,dt,\,[B,q])=\Psi([x,p];\,dt,\,[B,q]),\\
&\Psi_{r+1}([x,p];\,dt,\,[B,q])=\int_0^t\int_{S\cup\{\delta\}}\Psi_r([x,p];\,ds,[dy,p'])\Psi([y,p'];\,d(t-s),[B,q]),\\
&\Psi_r([x,p];\,t,\,[B,q])=\int_0^t\Psi_r([x,p];\,ds,\,[B,q]),
\end{aligned}
$$

(7. 24)

$$
\chi_r(t,[x,p],\,[B,q])=\int_0^t\int_{S\cup\{\delta\}}\Psi_r([x,p];\,ds,\,[dy,p'])\chi_0(t-s,\,[y,p'],\,[B,q]),
$$

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$$\chi(t,[x,p],[B,q]) = \sum_{r=0}^{\infty} \chi_r(t,[x,p],[B,q]),$$

$$r \geqq 1,\ [x,p] \in S \cup \{\delta\},\ [B,q] \in \mathscr{B}(S \cup \{\delta\}),\ t \geqq 0.$$

Then we may apply Lemma 7.1 for our χ_r, Ψ_r and χ.

LEMMA 7.4. *Let Ψ_r be defined in* (7.24). *Then we have*

$$\lim_{r\to\infty} \Psi_r([x,p];t,S \cup \{\delta\}) = 0, \tag{7.25}$$

for any $t \geqq 0$.

Proof. When $[x,p]=\delta$, (7.25) is evident. Let $[x,p] \in S$ and let $B \in \mathscr{B}(E)$. First we shall prove

$$\Psi_r([x,p];t,[B,p+r]) = E_x\Big[\int_0^t\int_{s_1}^t \cdots \int_{s_{r-1}}^t e^{-\varphi_{s_r}} I_B(X'_{s_r}) d\varphi^+_{s_1} d\varphi^+_{s_2} \cdots d\varphi^+_{s_r}\Big], \tag{7.26}$$

$$r \geqq 1.$$

By the definition of Ψ_1, (7.26) holds for $r=1$. Assume that (7.26) holds for r. Then we can obtain from Lemma 7.1 and the strong Markov property of X' that

$$\Psi_{r+1}([x,p];t,[B,p+r+1])$$

$$= \int_0^t\int_0^v\int_{S\cup\{\delta\}} \Psi_1([x,p];ds,[dy,q])\Psi_r([y,q];d(v-s),[B,p+r+1])$$

$$= \int_0^t\int_E \Psi_1([x,p];ds,[dy,p+1])\Psi_r([y,p+1];t-s,[B,p+r+1])$$

$$= E_x\Big[\int_0^t e^{-\varphi_s} d\varphi^+_s E_{X'_s}\Big[\int_0^{t-s}\int_{s_1}^{t-s} \cdots \int_{s_{r-1}}^{t-s} e^{-\varphi_{s_r}} I_B(X'_{s_r}) d\varphi^+_{s_1} d\varphi^+_{s_2} \cdots d\varphi^+_{s_r}\Big]\Big]$$

$$= E_x\Big[\int_0^t\int_{v_1}^t \cdots \int_{v_r}^t e^{-\varphi_{v_{r+1}}} I_B(X'_{v_{r+1}}) d\varphi^+_{v_1} d\varphi^+_{v_2} \cdots d\varphi^+_{v_{r+1}}\Big],$$

which shows the validity of (7.26) for $r+1$. So we can see inductively the validity of (7.26) for any $r \geqq 0$.

Similarly we get

$$\Psi_r([x,p];t,\{\delta\})$$

$$= E_x\Big[\int_0^t\int_{s_1}^t \cdots \int_{s_{r-1}}^t e^{-\varphi_{s_r}} d\varphi^+_{s_1} d\varphi^+_{s_2} \cdots d\varphi^+_{s_{r-1}} d\varphi^-_{s_r}\Big], \qquad r \geqq 0.$$

Combining (7.26) with the above equation, we have

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$$\Psi_r([x,p];\, t, S \cup \{\delta\})$$
$$= E_x\Big[\int_0^t\int_{s_1}^t \cdots \int_{s_{r-1}}^t e^{-\varphi_{s_r}} d\varphi^+_{s_1} d\varphi^+_{s_2} \cdots d\varphi^+_{s_{r-1}} d\varphi_{s_r}\Big], \qquad r \geqq 1.$$

Since $k(x)$ is bounded on E, it follows from the definitions of φ_s, φ^+_s, φ^-_s and the above equation that

(7. 27) $$\Psi_r([x,p];\, t, S \cup \{\delta\}) \leqq \frac{(\|k\|\, t)^r}{r!}, \qquad r \geqq 1,$$

which proves the lemma. Q.E.D.

Here we note that

$$\chi(t,[x,p], S \cup \{\delta\}) = 1, \qquad [x,p] \in S \cup \{\delta\},$$

which follows from (7. 11), (7. 12) and (7. 25).

Let us now consider the function space

$$\boldsymbol{C}_0(S \cup \{\delta\}) = \{f;\, f(\delta) = 0,\ f|_S \in \boldsymbol{C}_0(S)\},$$

where $f|_S$ denotes the restricted function of f on S. Let also V_t be the operator defined by

(7. 28) $$V_t f([x,p]) = \int_{S\cup\{\delta\}} \chi(t,[x,p],\, [dy,q]) f([y,q]),$$
$$f \in \boldsymbol{C}_0(S \cup \{\delta\}),\ [x,p] \in S \cup \{\delta\},\ t \geqq 0.$$

Then we have

THEOREM 7. 1. *Let H_t be the semi-group on $\boldsymbol{C}_0(E)$ induced by the Feller process X'_t. Then V_t mentioned above is a strongly continuous and non-negative contraction semi-group on $\boldsymbol{C}_0(S \cup \{\delta\})$.*

Proof. Let us set

(7. 29) $$V^0_t f([x,p]) = \int_{S\cup\{\delta\}} \chi_0(t,[x,p],\, [dy,q]) f([y,q]), \quad f \in \boldsymbol{C}_0(S \cup \{\delta\}),\ t \geqq 0.$$

Then it holds by the definition (7. 21) that

(7. 30) $$V^0_t f([x,p]) = E_x[e^{-\int_0^t |k(X'_s)| ds} f([X'_t, p])], \qquad [x,p] \in S,$$

where $f([x,p])$ is considered as a function on E for fixed p. Since H_t is strongly continuous on $\boldsymbol{C}_0(E)$ and $k(x)$ is bounded continuous on E, the right hand side of (7. 30) belongs to $\boldsymbol{C}_0(E)$ as a function of $x \in E$. Hence

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the semi-group V_t^0 is strongly continuous on $C_0(S\cup\{\delta\})$, because $V_t^0 f(\delta)=f(\delta)$ for $t\geqq 0$.

On the other hand, we can see from Lemma 7.1 and Lemma 7.3 that

$$\|V_t-V_t^0\| = \sup_{[x,p]\in S}\int_0^t\int_{S\cup\{\delta\}}\Psi([x,p];\,ds,\,[dy,q])\chi(t-s,\,[y,q],\,S\cup\{\delta\})$$

$$=\sup_{[x,p]\in S}\sum_{r=1}^{\infty}\int_0^t\int_{S\cup\{\delta\}}\Psi_r([x,p];\,ds,\,[dy,q])\chi_0(t-s,\,[y,q],\,S\cup\{\delta\})$$

$$\leqq \sup_{[x,p]\in S}\sum_{r=1}^{\infty}\Psi_r([x,p];\,t,S\cup\{\delta\}).$$

Applying (7.27) to the right hand side of the above inequality, we have

$$\|V_t-V_t^0\|\leqq\sum_{r=1}^{\infty}\frac{(\|k\|\,t)^r}{r!}=e^{\|k\|t}-1,\qquad t\geqq 0. \tag{7.31}$$

Next we shall prove that V_t maps $C_0(S\cup\}\delta\})$ into itself. Set

$$V_t^{(r)}f([x,p])=\int_{S\cup\{\delta\}}\chi_r(t,[x,p],\,[dy,q])\,f([y,q]),\quad f\in C_0(S\cup\{\delta\}),\ r\geqq 0,\ t\geqq 0.$$

As was proved already, $V_t^{(0)}=V_t^0$ maps $C_0(S\cup\{\delta\})$ into itself. Accordingly, we may use the mathematical induction. Assume that $V_t^{(r)}$ maps $C_0(S\cup\{\delta\})$ into itself. Setting $k^+([x,p])=k^+(x)$ for $x\in E$ and $k^+(\delta)=0$, we can see from (7.9) that

$$\begin{aligned}
&V_t^{(r+1)}f([x,p])\\
&=\int_{S\cup\{\delta\}}\int_0^t\int_{S\cup\{\delta\}}\Psi([x,p];\,ds,\,[dy,\,q])\chi_r(t-s,\,[y,q],\,[dz,q'])f([z,q'])\\
&=\int_0^t\int_{S\cup\{\delta\}}\Psi([x,p];\,ds,\,[dy,q])V_{t-s}^{(r)}f([y,q])\\
&=\int_0^t E_x[e^{-\int_0^s|k(X'_v)|dv}k^+([X'_s,\ p+1])V_{t-s}^{(r)}f([X'_s,\ p+1])ds\\
&=\int_0^t V_s^0(k^+V_{t-s}^{(r)}f)\,([x,p+1])ds,\qquad f\in C_0(S\cup\{\delta\}),\ [x,p]\in S.
\end{aligned}$$

Since $\|V_{t-s}^{(r)}f\|\leqq\|f\|$ and $V_s^0(k^+V_{t-s}^{(r)}f)\in C_0(S\cup\{\delta\})$, the above equation shows that $(V_t^{(r+1)}f)|_S\in C_0(S)$. Also $\Psi(\delta;\cdot,\cdot)=0$, and hence the above equation shows that $V_t^{(r+1)}f\in C_0(S\cup\{\delta\})$. Thus we can see that $V_t^{(r)}f\in C_0(S\cup\{\delta\})$ for any $f\in C_0(S\cup\{\delta\})$ and $r\geqq 1$.

Now the function

$$V_t f([x,p]) = \sum_{r=0}^{\infty} V_t^{(r)} f([x,p]), \qquad f \in \boldsymbol{C}_0(S \cup \{\delta\}), \ t \geqq 0,$$

belongs to $\boldsymbol{C}_0(S \cup \{\delta\})$, because $V_t^{(r)} f \in \boldsymbol{C}_0(S \cup \{\delta\})$ and, by (7. 27),

$$\| V_t^{(r)} f \| \leqq \frac{(\|k\| t)^r}{r!} \| f \| \tag{7. 32}$$

holds for any $r \geqq 1$. Hence the strong continuity of V_t^0 on $\boldsymbol{C}_0(S \cup \{\delta\})$, (7. 31) and (7. 13) prove that V_t is a strongly continuous semi-group on $\boldsymbol{C}_0(S \cup \{\delta\})$.

The non-negative property of V_t follows from the definitions of χ and χ_r and the contractive property of V_t follows from (7. 11). Q.E.D.

New let us consider Markov processes on $S \cup \{\delta\}$. Since V_t^0 and $V_t^{(r)}$ satisfy the conditions (a) and (b) in Lemma 7. 2, there exist two strong Markov processes $Y = \{Y_t = [X_t, N_t], \zeta, \mathscr{B}_t, P_{[x,p]}; [x,p] \in S \cup \{\delta\}\}$ and $Y^0 = \{Y_t^0 = [X_t^0, N_t^0], \zeta^0, \mathscr{B}_t^0, P_{[x,p]}^0; [x,p] \in S \cup \{\delta\}\}$ corresponding to the semi-groups V_t and V_t^0 respectively and a Markov time τ of Y_t such that

$$Y_t^0(w) = \begin{cases} Y_t(w), & \text{if } t < \tau \\ \delta, & \text{if } t \geqq \tau. \end{cases} \tag{7. 33}$$

Also, we may assume that the sample paths of Y_t are right continuous and quasi left continuous and, by Lemma 7. 1 and Lemma 7. 4, Y_t is a conservative Markov process. Let us set

$$\sigma_r(w) = \inf\{t > 0; N_t(w) = N_0(w) + r\}, \qquad r \geqq 0. \tag{7. 34}$$

Then we have

THEOREM 7. 2. *Let χ_r and Ψ_r be measures given in* (7. 2). *Let also $Y = \{Y_t = [X_t, N_t], \mathscr{B}_t, P_{[x,p]}; [x,p] \in S \cup \{\delta\}\}$ be the strong Markov process mentioned above and let σ_r be the Markov time given in* (7. 34). *Then we have*

$$P_{[x,p]}(Y_t \in B, \ \sigma_r \leqq t < \sigma_{r+1}) = \chi_r(t, [x,p], B) \tag{7. 35}$$

$$P_{[x,p]}(Y_{\sigma_r} \in B, \ \sigma_r \in dt) = \Psi_r([x,p]; dt, B), \tag{7. 36}$$

$$[x,p] \in S \cup \{\delta\}, \ B \in \mathscr{B}(S \cup \{\sigma\}), \ r \geqq 0.$$

Proof. If $[x,p] = \delta$, then (7. 35) and (7. 36) hold evidently. So we shall

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prove them for $[x, p] \in S$. Since it follows from the definitions of χ_r and Ψ that

$$\chi_r(t, [x, p],\ [E, p]) = 0, \qquad [x, p] \in S,\ r \geqq 1,$$

we have

$$\begin{aligned} P_{[x,p]}(Y_t \in B,\ t < \sigma_1) &= P_{[x,p]}(Y_t \in B, N_t = p,\ t < \sigma_1) \\ &\leqq \chi_0(t, [x, p], B) \\ &= P_{[x,p]}(Y_t \in B,\ t < \tau), \qquad B \in \mathscr{B}(S \cup \{\delta\}), \end{aligned}$$

and hence

$$(7.\ 37) \qquad P_{[x,p]}(\sigma_1 \leqq \tau) = 1, \qquad [x, p] \in S.$$

On the other hand, $\chi_0(t, [x, p], \cdot)$ vanishes on $S \cup \{\delta\} - E \times \{p\}$ for any fixed $t \geqq 0$. Hence we have

$$P_{[x,p]}(N_t = p, t < \tau) = P^0_{[x,p]}(N^0_t = p, t < \tau) = P_{[x,p]}(t < \tau),$$
$$[x, p] \in S,$$

which means

$$P_{[x,p]}(\tau \leqq \sigma_1) = 1, \qquad [x, p] \in S.$$

Combining (7. 37) with the above equation, we can see that

$$P_{[x,p]}(\tau = \sigma_1) = 1,$$

and accordingly

$$P_{[x,p]}(\tau_r = \sigma_r) = 1, \qquad [x, p] \in S,\ r \geqq 0,$$

where $\tau_0 = 0$, $\tau_1 = \tau$ and $\tau_{r+1} = \tau_r + \theta_{\tau_r}\tau$.

The theorem follows from Lemma 7. 2 immediately. Q.E.D.

Let us now consider the function defined by

$$\widehat{f \cdot \lambda}([x, p]) = \begin{cases} \lambda^p f(x), & [x, p] \in S, \\ 0 \quad , & [x, p] = \delta, \end{cases}$$

for any $f \in \boldsymbol{B}(E)$ and $\lambda \geqq 0$. Then it follows from (7. 12) and (7. 32) that

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$$
(7.38) \qquad \begin{aligned} V_t|\widehat{f\cdot\lambda}|([x,p]) &= \sum_{r=0}^{\infty}\int_{[E,p+r]}\chi_r(t,[x,p],[dy,p+r])|\widehat{f\cdot\lambda}|([y,p+r]) \\ &\leqq \lambda^p\,\|f\|\sum_{r=0}^{\infty}\frac{(\lambda\,\|k\|\,t)^r}{r!} \\ &= \lambda^p\,\|f\|\,e^{\lambda\|k\|t}<\infty\,. \end{aligned}
$$

THEOREM 7.3. *Let V_t^0, V_t and Ψ be semi-groups and measure given in (7.29), (7.28) and (7.22) respectively. Let also $Y=\{Y_t=[X_t,N_t],\ \mathscr{B}_t,\ P_{[x,p]};\ [x,p]\in S\cup\{\delta\}\}$ be a conservative strong Markov process corresponding to V_t. Then the function $u(t,x)=V_t\widehat{f\cdot\lambda}([x,0])$ is a solution of the following integral equation*

$$
(7.39) \qquad u(t,x)=V_t^0\widehat{f\cdot\lambda}+\lambda\int_0^t\!\!\int_E u(t-s,y)\,\Psi([x,0];ds,[dy,1])\,,
$$

$$
x\in E,\ t\geqq 0,\ \lambda\geqq 0,\ f\in \boldsymbol{C}_0(E)\,,
$$

with initial value $u(0+,x)=f(x)$.

Proof. By (7.38), $V_t\widehat{f\cdot\lambda}([x,0])$ is bounded on $[0,T]\times\{S\cup\{\delta\}\}$ for any given $T>0$. Then we have

$$
\begin{aligned} V_t\widehat{f\cdot\lambda}([x,0]) &= E_{[x,0]}[\widehat{f\cdot\lambda}(Y_t);\,t<\sigma]+E_{[x,0]}[V_{t-\sigma}\widehat{f\cdot\lambda}(Y_\sigma);\,\sigma\leqq t] \\ &= V_t^0\widehat{f\cdot\lambda}([x,0])+\int_0^t\!\!\int_E\Psi([x,0];ds,\,[dy,\,1])V_{t-s}\widehat{f\cdot\lambda}([y,1]). \end{aligned}
$$

Since $V_t\widehat{f\cdot\lambda}([x,p])=\lambda^pV_t\widehat{f\cdot\lambda}([x,0])$, we can see that $u(t,x)=V_t\widehat{f\cdot\lambda}([x,0])$ satisfies (7.39). Moreover, V_t^0 is a strongly continuous semi-group on $\boldsymbol{C}_0(S\cup\{\delta\})$ and hence we have

$$
(7.40) \qquad \lim_{(t,x)\to(0,x_0)}V_t\widehat{f\cdot\lambda}([x,0])=\lim_{(t,x)\to(0,x_0)}V_t^0\widehat{f\cdot\lambda}([x,0])=\widehat{f\cdot\lambda}([x_0,0])=f(x_0)\,.
$$

Thus we have proved the theorem. Q.E.D.

COROLLARY 7.1. *Let X' be a standard Brownian motion on R^d. If $k(x)$ is a bounded continuous function on R^d, then $u(t,x)=V_t\widehat{f\cdot 2}([x,0])$ is a solution of the following differential equation*

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(7. 41) $$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\Delta u(t,x) + k(x)u(t,x), \qquad x \in R^d, \ t \geqq 0,$$

with initial value $f \in \boldsymbol{C}(R^d)$.

Proof. By the definition of Ψ, we have

$$\Psi([x,0]; dt, [B,1]) = E_x[e^{-\int_0^t |k(X'_s)| ds} k^+(X'_t) I_B(X'_t)]dt,$$
$$x \in R^d, \ B \in \mathscr{B}(R^d),$$

where E_x denotes the integral by the probability measure of a standard Brownian motion X'_t. Then it follows from Theorem 7. 3 that

(7. 42) $$\begin{aligned} u(t,x) &= V_t^0 \widehat{f\cdot 2}([x,0]) + 2\int_0^t \int_{E\times\{1\}} \Psi([x,0]; d(t-s), [dy,1])u(s,y) \\ &= u_0(t,x) + v(t,x), \end{aligned}$$

where

$$u_0(t,x) = E_x[e^{-2\int_0^t |k(X'_s)| ds} f(X'_t)]$$

and

$$v(t,x) = 2\int_0^t E_x[e^{-2\int_0^{t-s} |k(X'_v)| dv} k^+(X'_{t-s}) u(s, X'_{t-s})]ds.$$

On the other hand, by Kac's theorem,[38] we have

$$\frac{\partial u_0(t,x)}{\partial t} = \frac{1}{2}\Delta u_0(t,x) - |k(x)| u_0(t,x),$$

and

$$\begin{aligned} \frac{\partial v(t,x)}{\partial t} = 2k^+(x)u(t,x) + \int_0^t \Delta E_x[e^{-2\int_0^{t-s} |k(X'_v)| dv} k^+(X'_{t-s}) u(s, X'_{t-s})]ds \\ - |k(x)| v(t,x). \end{aligned}$$

Combining (7. 42) with the above two equations, we have

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= \frac{1}{2}\Delta u_0(t,x) - |k(x)| u_0(t,x) + 2k^+(x)u(t,x) + \frac{1}{2}\Delta v(t,x) - |k(x)| v(t,x) \\ &= \frac{1}{2}\Delta u(t,x) + (2k^+(x) - |k(x)|)u(t,x) \\ &= \frac{1}{2}\Delta u + k(x)u(t,x). \end{aligned}$$

---

[38] cf. Ito-McKean [7], pp. 54–55.

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Since $u(0+, x) = f(x)$, the above equation proves the corollary.

Q.E.D.

§ 8. Construction of a signed branching Markov processes with age (II).

According to Theorem 5. 1, a strong Markov process on $\tilde{S}$ satisfying Condition 3 is a signed branching Markov process with age. We shall construct such a process in this section.

Let E be a compact Hausdorff space satisfying the second axiom of countability, and consider $S^{(n)}$, S^n, $\hat{S}$ and $\tilde{S}$ defined in § 2. We shall define the mapping γ from $\bigcup_{n=0}^{\infty} S^{(n)}$ into $\hat{S}$ by

$$\gamma((x_1, p_1), (x_2, p_2), \cdots, (x_n, p_n)) = [[x_1, x_2, \cdots, x_n], [p_1, p_2, \cdots, p_n]] \in S^n.$$

Let $\{(q_n^+(x), q_n^-(x)); n = 0, 1, 2, \cdots\}$ be a system of pairs of non-negative continuous functions on E such that

$$k(x) = \sum_{n=0}^{\infty} \{q_n^+(x) + q_n^-(x)\}, \qquad x \in E, \tag{8. 1}$$

is bounded continuous on E, and

$$q_n^+(x) q_n^-(x) = 0, \qquad x \in E, \ n = 0, 1, 2, \cdots. \tag{8. 2}$$

Further let X'_t be a conservative Feller process on E, H_t be the strongly continuous semi-group on $C(E)$ induced by X'_t, and let $Y = \{Y_t = [X_t, N_t], \mathscr{B}_t, P_{[x,p]}; [x, p] \in S\}$, where $S = E \times N$, be the strong Markov process constructed in § 7 from the system $\{k(x), X'_t\}$. (Since $k(x)$ is non-negative, the extra point δ is not needed.) Then, by (i) of Lemma 7. 2, we may assume that almost all sample paths of Y_t are right continuous and, for any given Markov time $\tau > 0$, they have their left limit $Y_{\tau-}$ at the time τ.

Let us set

$$k([x, p]) = k(x), \qquad [x, p] \in S,$$

and

$$\varphi_t(w) = \int_0^t k(Y_s(w)) ds.$$

We shall denote the $exp(-\varphi_t)$ sub-process of Y_t by $Y^0 = \{Y_t^0 = [X_t^0, N_t^0], \eta, \mathscr{B}_t^0, P_{[x,p]}^0; [x, p] \in S\}$. Let also $Y_{i,t}^0$, $i = 1, 2, \cdots, n$, be Markov processes such that their fundamental spaces are identical to the one for Y_t^0, each of them is stochastically equivalent to Y_t^0 and they are mutually independent to each other. Then the probability measure of the joint process $(Y_{1,t}^0, Y_{2,t}^0, \cdots, Y_{n,t}^0)$ starting from $((x_1, p_1), (x_2, p_2), \cdots, (x_n, p_n)) \in S^{(n)}$ is given by the product measure $P_{[x_1,p_2]}^0 \times P_{[x_1,p_2]}^0 \times \cdots \times P_{[x_n,p_n]}^0$. Using this product measure, we shall define a measure $\chi_0(t, [x, p], \cdot)$ on $\mathscr{B}(\hat{\boldsymbol{S}})$ by

$$\chi_0(t, [\boldsymbol{x}, \boldsymbol{p}], [B, \boldsymbol{q}])$$

(8. 3)
$$= \begin{cases} P_{[x_1,p_2]}^0 \times P_{[x_1,p_2]}^0 \times \cdots \times P_{[x_n,p_n]}^0((Y_{1,t}^0, Y_{2,t}^0, \cdots, Y_{n,t}^0) \in \gamma^{-1}([B, \boldsymbol{q}])), \\ \qquad \text{if } [\boldsymbol{x}, \boldsymbol{p}] \neq \Delta,\ [\partial, p], \\ 1, \quad \text{if } [\boldsymbol{x}, \boldsymbol{p}] = [\partial, p] \text{ and } [\partial, p] \in [B, \boldsymbol{q}], \\ 1, \quad \text{if } [x, p] = \Delta \text{ and } \Delta \in [B, \boldsymbol{q}], \\ 0, \quad \text{otherwise,} \end{cases}$$

where $\boldsymbol{x} = [x_1, x_2, \cdots, x_n]$, $\boldsymbol{p} = [p_1, p_2, \cdots, p_n]$ and $B \in \mathscr{B}(\hat{\boldsymbol{S}})$.

Let us next define a measure $\Psi([\boldsymbol{x}, \boldsymbol{p}, j]); \cdot, \cdot)$ on $\mathscr{B}([0, \infty) \times \tilde{\boldsymbol{S}})$. Using a given system $\{(q_n^+(x), q_n^-(x)); n=0,1,2,\cdots\}$, we shall define $\pi([x, p, j], [B, \boldsymbol{q}, j'])$ by (4. 1). Then a measure $\Psi([x, p, j]; dt, [B, \boldsymbol{q}, j'])$ on $\mathscr{B}([0, \infty) \times \tilde{\boldsymbol{S}})$ is defined by

(8. 4)
$$\Psi([x, p, j]; dt, [B, \boldsymbol{q}, j']) = E_{[x,p]}^0(\pi([X_{\eta-}^\circ, N_{\eta-}^\circ, j], [B, \boldsymbol{q}, j']); \eta \in dt)$$
$$[x, p, j] \in S \times J, \ [B, \boldsymbol{q}, j'] \in \mathscr{B}(\tilde{\boldsymbol{S}}),$$

where $Y_t^0 = [X_t^0, N_t^0]$ is the Markov process mentioned above, $E_{[x,p]}^0$ denotes the integral by the probability measure $P_{[x,p]}^0$ of Y_t^0 and $J = \{0, 1, 2, 3\}$. Then we shall extend the parameter space of Ψ to $\tilde{\boldsymbol{S}}$ as follows:

(8. 5)
$$\Psi(\Delta; dt, \tilde{\boldsymbol{S}}) = \Psi([\partial, p, j]; dt, \tilde{\boldsymbol{S}}) = 0, \qquad [\partial, p] \in S^0,$$

and for $[\boldsymbol{x}, \boldsymbol{p}] = [[x_1, x_2, \cdots, x_n], [p_1, p_2, \cdots, p_n]] \in S^n$

(8. 6)
$$\Psi([\boldsymbol{x}, \boldsymbol{p}, j]; dt, [B, \boldsymbol{q}, j'])$$
$$= \sum_{i=1}^n \int_S \Psi([x_i, p_i, j]; dt, [d\boldsymbol{y}, \boldsymbol{p}', j'])\chi_0(t, [\boldsymbol{x}_i', \boldsymbol{p}_i'], [B_y, \boldsymbol{q}']),$$
$$[B, \boldsymbol{q}] \in \mathscr{B}(\hat{\boldsymbol{S}}), \ j, j' \in J,$$
$$\Psi([\boldsymbol{x}, \boldsymbol{p}, j]; dt, \{\Delta\}) = 0,$$

where χ_0 is given in (8. 3), $\boldsymbol{x}_i' = [x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n]$, $\boldsymbol{p}_i' = [p_1, \cdots,$

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$p_{i-1}, p_{i+1}, \cdots, p_n]$, and $[B_y, \boldsymbol{q}']$ denotes the Borel set $\{[\boldsymbol{z}, \boldsymbol{r}] \in \hat{\boldsymbol{S}}; \gamma(\gamma^{-1}([\boldsymbol{y}, \boldsymbol{p}'])\times \gamma^{-1}(\boldsymbol{z}, \boldsymbol{r}] \in [B, \boldsymbol{q}]\}$.

Now we shall define χ_0, χ_r, χ and Ψ_r by

$$\Psi_1([\boldsymbol{x}, \boldsymbol{p}, j]; dt, [B, \boldsymbol{q}, j']) = \Psi([\boldsymbol{x}, \boldsymbol{p}, j]; dt, [B, \boldsymbol{q}, j'])$$

$$\Psi_{r+1}([\boldsymbol{x}, \boldsymbol{p}, j]; dt, [B, \boldsymbol{q}, j']) = \int_0^t \int_{\tilde{S}} \Psi_r([\boldsymbol{x}, \boldsymbol{p}, j]); ds, [d\boldsymbol{y}, \boldsymbol{p}', i])\Psi([\boldsymbol{y}, \boldsymbol{p}', i]; d(t-s), [B, \boldsymbol{q}, j']),$$

$$\Psi_r([\boldsymbol{x}, \boldsymbol{p}, j]; t, [B, \boldsymbol{q}, j']) = \int_0^t \Psi_r([\boldsymbol{x}, \boldsymbol{p}, j]; ds, [B, \boldsymbol{q}, j']),$$

(8. 7) $$\chi_0(t, [\boldsymbol{x}, \boldsymbol{p}, j], [B, \boldsymbol{q}, j']) = \delta_{j,j'}\chi_0(t, [\boldsymbol{x}, \boldsymbol{p}], [B, \boldsymbol{q}]),$$

$$\chi_r(t, [\boldsymbol{x}, \boldsymbol{p}, j], [B, \boldsymbol{q}, j']) = \int_0^t \int_{\tilde{S}} \Psi_r([\boldsymbol{x}, \boldsymbol{p}, j]; ds, [d\boldsymbol{y}, \boldsymbol{p}', i]) \chi_0(t-s, [\boldsymbol{y}, \boldsymbol{p}', i], [B, \boldsymbol{q}, j']),$$

$$\chi(t, [\boldsymbol{x}, \boldsymbol{p}, j], [B, \boldsymbol{q}, j']) = \sum_{r=0}^{\infty} \chi_r(t, [\boldsymbol{x}, \boldsymbol{p}, j], [B, \boldsymbol{q}, j']),$$

$$[\boldsymbol{x}, \boldsymbol{p}, j] \in \tilde{\boldsymbol{S}}, \ [B, \boldsymbol{q}, j'] \in \mathscr{B}(\tilde{\boldsymbol{S}}), \ r \geqq 1, \ t \geqq 0.$$

Then we have

LEMMA 8. 1. *χ_0 and Ψ mentioned above satisfy the Moyal's $\chi_0\Psi$-condition, i.e. it holds that for any $[\boldsymbol{x}, \boldsymbol{p}, j] \in \tilde{\boldsymbol{S}}$, $[B, \boldsymbol{q}, j'] \in \mathscr{B}(\tilde{\boldsymbol{S}})$ and $t, s \geqq 0$*

(8. 8) $$\chi_0(t+s, [\boldsymbol{x}, \boldsymbol{p}, j], [B, \boldsymbol{q}, j']) = \int_{\tilde{S}} \chi_0(t, [\boldsymbol{x}, \boldsymbol{p}, j], [d\boldsymbol{y}, \boldsymbol{p}', i]) \cdot \chi_0(s, [\boldsymbol{y}, \boldsymbol{p}', i], [B, \boldsymbol{q}, j']),$$

(8. 9) $$\Psi([\boldsymbol{x}, \boldsymbol{p}, j]; \infty, \tilde{\boldsymbol{S}}) = 1 - \lim_{t\to\infty} \chi_0(t, [\boldsymbol{x}, \boldsymbol{p}, j], \tilde{\boldsymbol{S}}),$$

(8. 10) $$\Psi([\boldsymbol{x}, \boldsymbol{p}, j]; t+s, [B, \boldsymbol{q}, j']) = \Psi([\boldsymbol{x}, \boldsymbol{p}, j]; t, [B, \boldsymbol{q}, j']) + \int_{\tilde{S}} \chi_0(t, [\boldsymbol{x}, \boldsymbol{p}, j], [d\boldsymbol{y}, \boldsymbol{p}', i])\Psi([\boldsymbol{y}, \boldsymbol{p}', i]; s, [B, \boldsymbol{q}, j']),$$

(8. 11) *$\Psi([\boldsymbol{x}, \boldsymbol{p}, j]; t, \tilde{\boldsymbol{S}})$ is continuous in t.*

Proof. Since (8. 8) is evident from the definition of χ_0 and also (8. 9)-(8. 11) are evident when $[\boldsymbol{x}, \boldsymbol{p}, j] = \Delta$ or $[\boldsymbol{x}, \boldsymbol{p}, j] = [\partial, p, j]$, we shall prove (8. 9)-(8. 11) for $[\boldsymbol{x}, \boldsymbol{p}, j] \in \tilde{\boldsymbol{S}} - (S^0 \times J) \cup \{\Delta\}$.

Let $Y^0_{i,t}$, $i = 1, 2, \cdots, n$, be Markov processes and let $P^0_{[x_1, p_1]} \times P^0_{[x_2, p_2]} \times \cdots \times P^0_{[x_n, p_n]}$ be the probability measure used in (8. 3). Then it follows from (8. 6) that

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$$\Psi([\boldsymbol{x}, \boldsymbol{p}, j]; dt, \tilde{\boldsymbol{S}})$$
$$= \sum_{i=1}^{n} \int_{\tilde{S}} \Psi([x_i, p_i, j]; dt, [d\boldsymbol{y}, \boldsymbol{q}, j'])\chi_0(t, [\boldsymbol{x}'_i, \boldsymbol{p}'_i, 0], \tilde{\boldsymbol{S}})$$
$$= -d_t P^0_{[x_1, p_1]} \times \cdots \times P^0_{[x_n, p_n]}((Y^0_{1,t}, \cdots, Y^0_{n,t}) \in \gamma^{-1}\hat{\boldsymbol{S}}),$$
$$[\boldsymbol{x}, \boldsymbol{p}] = [x_1, x_2, \cdots, x_n], [p_1, p_2, \cdots, p_n] \in S^n, \quad n \geqq 1.$$

Hence we have

$$\Psi([\boldsymbol{x}, \boldsymbol{p}, j]; t, \tilde{\boldsymbol{S}}) = -\int_0^t d_s P^0_{[x_1, p_1]} \times \cdots \times P^0_{[x_n, p_n]}((Y^0_{1,t}, \cdots, Y^0_{n,t}) \in \gamma^{-1}\hat{\boldsymbol{S}})$$
$$= 1 - \chi_0(t, [\boldsymbol{x}, \boldsymbol{p}, j], \tilde{\boldsymbol{S}}), \qquad [\boldsymbol{x}, \boldsymbol{p}, j] \in \tilde{\boldsymbol{S}}, \ t \geqq 0,$$

which proves (8. 9).

We shall next show (8. 10). Considering the process $(Y^0_{1,t}, \cdots, Y^0_{n,t})$ mentioned above, we have for $[\boldsymbol{x}, \boldsymbol{p}] = [[x_1, x_2, \cdots, x_n], [p_1, p_2, \cdots, p_n]]$

$$\Psi([\boldsymbol{x}, \boldsymbol{p}, j]; t + s, [B, \boldsymbol{q}, j'])$$
$$= \Psi([\boldsymbol{x}, \boldsymbol{p}, j]; t, [B, q, j']) + \int_t^{t+s} \Psi([\boldsymbol{x}, \boldsymbol{p}, j]; dv, [B, \boldsymbol{q}, j'])$$
$$= \Psi([\boldsymbol{x}, \boldsymbol{p}, j]; t, [B, \boldsymbol{q}, j'])$$
$$+ \int_t^{t+s} \sum_{i=1}^{n} \int_{\hat{S}} \Psi([x_i, p_i, j]; dv, [d\boldsymbol{y}, \boldsymbol{p}', j'])\chi_0(v, [\boldsymbol{x}'_i, \boldsymbol{p}'_i, j'], [B_y, \boldsymbol{q}', j'])$$
$$= \Psi([\boldsymbol{x}, \boldsymbol{p}, j]; t, [B, \boldsymbol{q}, j']) + \int_0^s \sum_{i=1}^{n} \int_{\hat{S}} E^0_{[x_i, p_i]}[\Psi([X^0_{i,t}, N^0_{i,t}, j]; dv, [d\boldsymbol{y}, \boldsymbol{p}', j'])]$$
$$\cdot E^0_{[\boldsymbol{x}_i', \boldsymbol{p}_i']}[\chi_0(v, [X'^0_{i,t}, N'^0_{i,t}, j'], [B_y, \boldsymbol{q}', j'])],$$

where $E^0_{[\boldsymbol{x}, \boldsymbol{p}]}$ denotes the integral by $P^0_{[x_1, p_1]} \times \cdots \times P^0_{[x_n, p_n]}$ for $[\boldsymbol{x}, \boldsymbol{p}] = [[x_1, x_2, \cdots, x_n], [p_1, p_2, \cdots, p_n]]$ and $[X'^0_{i,t}, N'^0_{i,t}]$ denotes $[Y^0_{1,t}, \cdots, Y^0_{i-1,t}, Y^0_{i+1,t}, \cdots, Y^0_{n,t}]$. Then the right hand side of the above equation is equal to

$$\Psi([\boldsymbol{x}, \boldsymbol{p}, j]; t, [B, \boldsymbol{q}, j']) + \int_0^s E^0_{[\boldsymbol{x}, \boldsymbol{p}]}[\sum_{i=1}^{n} \int_{\hat{S}} \Psi([X^0_{i,t}, N^0_{i,t}, j]; dv, [d\boldsymbol{y}, \boldsymbol{p}', j'])$$
$$\cdot \chi_0(v, [X'^0_{i,t}, N'^0_{i,t}, 0], [B_y, \boldsymbol{q}', 0])]$$
$$= \Psi([\boldsymbol{x}, \boldsymbol{p}, j]; t, [B, \boldsymbol{q}, j'])$$
$$+ \int_0^s E^0_{[\boldsymbol{x}, \boldsymbol{p}]}[\Psi([[X^0_{1,t}, X^0_{2,t}, \cdots, X^0_{n,t}], [N^0_{1,t}, N^0_{2,t}, \cdots, N^0_{n,t}], j]; dv, [B, \boldsymbol{q}, j'])]$$
$$= \Psi([\boldsymbol{x}, \boldsymbol{p}, j]; t, [B, \boldsymbol{q}, j'])$$

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$$+\int_{\tilde{S}}\chi_0(t,[\boldsymbol{x},\boldsymbol{p},j],[d\boldsymbol{y},\boldsymbol{p}',i])\Psi([\boldsymbol{y},\boldsymbol{p},i];s,[B,\boldsymbol{q},j']).$$

So we have (8. 10) for any $[\boldsymbol{x},\boldsymbol{p},j]\in\tilde{\boldsymbol{S}}$.

Now $k(x)$ is bounded continuous and hence $\Psi([x,p,j];\cdot,\tilde{\boldsymbol{S}})$ is absolutely continuous with respect to the Lebesgue measure on $[0,\infty)$. Then (8. 6) proves (8. 11) for any $[\boldsymbol{x},\boldsymbol{p},j]\in\tilde{\boldsymbol{S}}$. Q.E.D.

Now we shall consider the linear operators $U_t^{(r)}$ and U_t on $\boldsymbol{B}(\tilde{\boldsymbol{S}})$ defined by

$$U_t^{(r)}h([\boldsymbol{x},\boldsymbol{p},j])=\int_{\tilde{S}}\chi_r(t,[\boldsymbol{x},\boldsymbol{p},j],[d\boldsymbol{y},\boldsymbol{q},j'])h([\boldsymbol{y},\boldsymbol{q},j']),$$

(8. 12) $$[\boldsymbol{x},\boldsymbol{p},j]\in\tilde{\boldsymbol{S}}.$$

$$U_t h([\boldsymbol{x},\boldsymbol{p},j])=\int_{\tilde{S}}\chi(t,[\boldsymbol{x},\boldsymbol{p},j]),[d\boldsymbol{y},\boldsymbol{q},j'])h([\boldsymbol{y},\boldsymbol{q},j']),$$

Further set

$$\widehat{f\cdot\lambda}([x,p,j])=\widetilde{f\cdot\lambda}([x,p,0]),\qquad f\in\boldsymbol{B}(E),\ [\boldsymbol{x},\boldsymbol{p},j]\in\tilde{\boldsymbol{S}}.$$

Then we have

Lemma 8. 2. *Let $U_t^{(0)}$ be the operator defined above. Then $U_t^{(0)}$ is strongly continuous on $\boldsymbol{C}_0(\tilde{\boldsymbol{S}})$.*

Proof. By Theorem 7. 1, the semi-group V_t corresponding to the process $Y_t=[X_t,N_t]$ on S is strongly continuous on $\boldsymbol{C}_0(S)$, while $k([x,p])=k(x)$ $(\geqq 0)$ is bounded and continuous on S. Hence $U_t^{(0)}$ is strongly continuous on $\boldsymbol{C}_0(S)$.

Now suppose $h\in\boldsymbol{C}_0(\tilde{\boldsymbol{S}})$ and set

$$h|_{S^n\times\{j\}}([\boldsymbol{x},\boldsymbol{p},j])=\begin{cases}h([\boldsymbol{x},\boldsymbol{p},j]), & \text{if } [\boldsymbol{x},\boldsymbol{p}]\in S^n,\ n\geqq 0,\\ 0\quad, & \text{otherwise.}\end{cases}$$

Then we have

(8. 13) $$U_t^{(0)}h([\boldsymbol{x},\boldsymbol{p},j])=U_t^0(h|_{S^n\times J})([\boldsymbol{x},\boldsymbol{p},j]),\quad [\boldsymbol{x},\boldsymbol{p},j]\in S^n\times\{j\},\ n\geqq 0.$$

On the other hand, the linear hull of $\{\widehat{f\cdot\lambda};f\in\boldsymbol{C}^*(E),\ 0\leqq\lambda<1\}$ is dense in $\boldsymbol{C}_0(\hat{\boldsymbol{S}})$ and $U_t^{(0)}\widehat{f\cdot\lambda}|_{S^n\times\{j\}}\in\boldsymbol{C}_0(S^n\times\{j\})$ which follows from $U_t^{(0)}\widehat{f\cdot\lambda}|_{S^n\times\{j\}}=(U_t^{(0)}\widehat{f\cdot\lambda})|_E\cdot\lambda|_{S^n\times\{j\}}$ and $U_t^{(0)}\widehat{f\cdot\lambda}|_E\in\boldsymbol{C}_0(S)$. So, for any $\varepsilon>0$, $n\geqq 0$ and $j\in J$, we can find constants α_i, $f_i\in\boldsymbol{C}^*(E)$ and $0\leqq\lambda_i<1$, $i=1,2,\cdots,i_n$, which may depend on ε, n and j, such that

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$$\| h - \sum_{i=1}^{i_n} \alpha_i (\widehat{f_i \cdot \lambda_i}) \|_{S^n \times \{j\}} < \varepsilon^{39)}. \tag{8. 14}$$

Then (8. 13) and the contraction property of $U_t^{(0)}$ imply that $U_t^{(0)} h \in \boldsymbol{C}_0(\tilde{\boldsymbol{S}})$.

Next we shall show the strong continuity of $U_t^{(0)}$ on $\boldsymbol{C}_0(\tilde{\boldsymbol{S}})$. As was stated already, $U_t^{(0)}$ is strongly continuous on $\boldsymbol{C}_0(S)$. Hence we have

$$\| U_t^{(0)} \widehat{f \cdot \lambda} - \widehat{f \cdot \lambda} \|_{S \times J} \to 0 \text{ as } t \to 0, \qquad f \in \boldsymbol{C}^*(E), \ 0 \leqq \lambda < 1. \tag{8. 15}$$

Then, for any $[\boldsymbol{x}, \boldsymbol{p}] = [[x_1, x_2, \cdots, x_n], [p_1, p_2, \cdots, p_n]]$ and $j \in J$, it follows from the definitions of $U_t^{(0)}$ and χ_0 that

$$\begin{aligned} & | U_t^{(0)} \widehat{f \cdot \lambda}([\boldsymbol{x}, \boldsymbol{p}, j]) - \widehat{f \cdot \lambda}([\boldsymbol{x}, \boldsymbol{p}, j]) | \\ & \leqq | \prod_{i=1}^{n} U_t^{(0)} \widehat{f \cdot \lambda}([x_i, p_i, 0]) - \prod_{i=1}^{n} \widehat{f \cdot \lambda}([x_i, p_i, 0]) | \\ & = | \sum_{r=1}^{n} \{ U_t^{(0)} \widehat{f \cdot \lambda}([x_r, p_r, 0]) - \widehat{f \cdot \lambda}([x_r, p_r, 0]) \} \\ & \qquad \cdot \prod_{i=1}^{r-1} \widehat{f \cdot \lambda}([x_i, p_i, 0)] \prod_{i=r+1}^{n} U_t^{(0)} \widehat{f \cdot \lambda}([x_i, p_i, 0]) | \\ & \leqq C(f, \lambda) \| U_t^{(0)} \widehat{f \cdot \lambda} - \widehat{f \cdot \lambda} \|_{S \times J}, \qquad f \in \boldsymbol{C}^*(E), \ 0 \leqq \lambda < 1, \end{aligned} \tag{8. 16}$$

where $C(f, \lambda)$ is a constant defined by

$$C(f, \lambda) = \sup \{ n \| \widehat{f \cdot \lambda} \|_{S \times J}^{n-1} ; \ n = 1, 2, 3, \cdots \}.$$

Combining (8. 15) and (8. 16), we have

$$\lim_{t \to 0} \| U_t^{(0)} \widehat{f \cdot \lambda} - \widehat{f \cdot \lambda} \|_{\tilde{S}} = 0, \qquad f \in \boldsymbol{C}^*(E), \ 0 \leqq \lambda < 1, \tag{8. 17}$$

because $U_t^{(0)} \widehat{f \cdot \lambda}([\partial, p, j]) = \widehat{f \cdot \lambda}([\partial, p, j])$ and $U_t^{(0)} \widehat{f \cdot \lambda}(\varDelta) = \widehat{f \cdot \lambda}(\varDelta) = 0$. Then it follows from (8. 14) and (8. 17) that for any fixed $n \geqq 0$ and $j \in J$

$$\lim_{t \to 0} \| U_t^{(0)} h |_{S^n \times \{j\}} - h |_{S^n \times \{j\}} \|_{\tilde{S}} = 0, \qquad h \in \boldsymbol{C}_0(\tilde{\boldsymbol{S}}). \tag{8. 18}$$

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39) For any function f on a topological space $\mathscr{X}$, we denote in the sequel $\sup \{ | f(x) | ; x \in A \subset \mathscr{X} \}$ by $\| f \|_A$.

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On the other hand, h and $U_t^{(0)}h$ are elements of $\boldsymbol{C}_0(\tilde{\boldsymbol{S}})$. Hence there exists an n_0 such that

$$\| U_t^{(0)}h - h \|_{\tilde{S}} \leqq \max_{n \leqq n_0, j \in J} \| U_t^{(0)}h \mid_{S^n \times \{j\}} - h_{S^n \times \{j\}} \|_{\tilde{S}} .$$

Therefore we can see from (8. 18) that $U_t^{(0)}$ is strongly continuous on $\boldsymbol{C}_0(\tilde{\boldsymbol{S}})$.
Q.E.D.

Now let $Y_{i,t} = [X_{i,t}, N_{i,t}]$, $i = 1, 2, \cdots, n$, be Markov processes on S such that their fundamental spaces are identical to the one of Y_t, each of them is stochastically equivalent to $Y_t = [X_t, N_t]$ and mutually independent to each other. Then the probability measure of the joint process $(Y_{1,t}, Y_{2,t}, \cdots, Y_{n,t})$ is given by the product measure $P_{[x_1, p_1]} \times P_{[x_2, p_2]} \times \cdots \times P_{[x_n, p_n]}$. The integral by the probability measure $P_{[x_1, p_1]} \times P_{[x_2, p_2]} \times \cdots \times P_{[x_n, p_n]}$ is denoted by $E_{(\boldsymbol{x}, \boldsymbol{p})}$ when $\boldsymbol{x} = [x_1, x_2, \cdots, x_n]$ and $\boldsymbol{p} = [p_1, p_2, \cdots, p_n]$. Then the process $Y_{i,t}^0$ mentioned already can be considered as the $exp(-\varphi_t)$ subprocess of $Y_{i,t}$.

We shall next define the set D_n by

$$D_n = \{\boldsymbol{x};\, \boldsymbol{x} = [x, x, \cdots, x] \in E^n\}.$$

Then, by (8. 4), $\Psi([x, p, 0]; ds, [\cdot, \boldsymbol{q}, \cdot])$ vanishes outside of $(\bigcup_{n=0}^{\infty} D_n) \times \{1, 3\}$. Hence it follows from (8. 6) and (8. 7) that for $[\boldsymbol{x}, \boldsymbol{p}] = [[x_1, x_2, \cdots, x_n], [p_1, p_2, \cdots, p_n]]$

$$\begin{aligned}
&\Psi([\boldsymbol{x}, \boldsymbol{p}, 0]; dt, [B, \boldsymbol{q}, 1]) \\
&= \sum_{i=1}^{n} \int_{\hat{S}} \Psi([x_i, p_i, 0]; dt, [d\boldsymbol{y}, \boldsymbol{p}', 1]) \chi_0(t, [\boldsymbol{x}_i', \boldsymbol{p}_i', 0], [B_y, \boldsymbol{q}', 0]) \\
&= \sum_{i=1}^{n} \int_{\hat{S}} E^0_{[x_i, p_i]} [\pi([Y^0_{i,\eta-}, N^0_{i,\eta-}, 0], [d\boldsymbol{y}, \boldsymbol{p}', 1]); \eta \in dt] \\
&\quad \cdot P^0_{[x_1, p_1]} \times \cdots \times P^0_{[x_{i-1}, p_{i-1}]} \times P^0_{[x_{i+1}, p_{i+1}]} \times \cdots \times P^0_{[x_n, p_n]} \\
&\quad ((Y^0_{1,t}, \cdots, Y^0_{i-1,t}, Y^0_{i+1,t}, \cdots, Y^0_{n,t}) \in \gamma^{-1}([B_y, \boldsymbol{q}'])) \qquad (8.\ 19) \\
&= \sum_{i=1}^{n} \int_{\hat{S}} E_{(x_i, p_i)} [e^{-\int_0^t k(Y_{i,s}) ds} k(Y_{i,t}) \pi([X_{i,t}, N_{i,t}, 0], [d\boldsymbol{y}, \boldsymbol{p}', 1])] \\
&\quad \cdot E_{(\boldsymbol{x}_i', \boldsymbol{p}_i')} [e^{-\sum_{r \neq i} \int_0^t k(Y_{r,s}) ds} \\
&\qquad \cdot I_{\gamma^{-1}([B_y, \boldsymbol{q}'])} (Y_{1,t}, \cdots, Y_{i-1,t}, Y_{i+1,t}, \cdots, Y_{n,t})] dt
\end{aligned}$$

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$$= \int_{\hat{S}} E_{(\boldsymbol{x},\boldsymbol{p})}[e^{-\sum_{i=1}^{n}\int_0^t k(Y_{i,s})ds} \sum_{i=1}^{n} \{ \sum_{m=0}^{\infty} q_m^+(X_{i,t})\delta_m(Y_{i,t},[d\boldsymbol{y}, \boldsymbol{p}'])$$

$$\cdot I_{\gamma^{-1}([B_y,\boldsymbol{q}'])}(Y_{1,t}, \cdots, Y_{i-1,t}, Y_{i+1,t}, \cdots, Y_{n,t})\}]dt ,$$

where $[\boldsymbol{x}_i', \boldsymbol{p}_i'] = [[x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n], [p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_n]]$ and $[B_y, \boldsymbol{q}'] = \{[\boldsymbol{z}, \boldsymbol{r}] \in \hat{\boldsymbol{S}} ; \gamma(\gamma^{-1}([\boldsymbol{y}, \boldsymbol{p}']) \times \gamma^{-1}([\boldsymbol{z}, \boldsymbol{r}]) \in [B, \boldsymbol{q}]\}$. Similarly, we have

$$\Psi([\boldsymbol{x}, \boldsymbol{p}, 0]; dt, [B, \boldsymbol{q}, 3])$$

$$= \int_{\hat{S}} E_{(\boldsymbol{x},\boldsymbol{p})}[e^{-\sum_{i=1}^{n}\int_0^t k(Y_{i,s})ds} \cdot \sum_{i=1}^{n} \{ \sum_{m=0}^{\infty} q_m^-(X_{i,t})\delta_m(Y_{i,t}, [d\boldsymbol{y}, \boldsymbol{p}']) \tag{8. 20}$$

$$\cdot I_{\gamma^{-1}([B_y,\boldsymbol{q}'])}(Y_{1,t}, \cdots, Y_{i-1,t}, Y_{i+1,t}, \cdots, Y_{n,t})\}]dt .$$

Then we have

LEMMA 8. 3. *Let* $U_t^{(r)}$ *be the operator on* $\boldsymbol{B}(\tilde{\boldsymbol{S}})$ *given in* (8. 12). *Then* $U_t^{(r)}$ *maps* $\boldsymbol{C}_0(\tilde{\boldsymbol{S}})$ *into itself. Moreover it holds that*

$$\lim_{t\to 0} \| U_t^{(r)}h \| = 0 , \qquad h \in \boldsymbol{C}_0(\tilde{\boldsymbol{S}}), \ r \geqq 1 . \tag{8. 21}$$

Proof. We shall first prove that $U_t^{(r)}h \in \boldsymbol{C}_0(\tilde{\boldsymbol{S}})$ for any $h \in \boldsymbol{C}_0(\tilde{\boldsymbol{S}})$. By Lemma 8. 2, $U_t^{(0)}$ is strongly continuous on $\boldsymbol{C}_0(\tilde{\boldsymbol{S}})$. So it suffices to prove that for $h \in \boldsymbol{C}_0(\tilde{\boldsymbol{S}})$ $U_t^{(r+1)}h$ is continuous in $(t,[\boldsymbol{x},\boldsymbol{p},j])$ as a function on $[0,\infty) \times \tilde{\boldsymbol{S}}$ and $U_t^{(r+1)}h \in \boldsymbol{C}_0(\tilde{\boldsymbol{S}})$ for any fixed $t \geqq 0$ under the assumption that $U_t^{(r)}h$ satisfies the same properties.

Now, by (8. 7) and (8. 12), we have

$$U_t^{(r+1)}h([x,p,0]) = \int_0^t \int_{\tilde{S}} \Psi([\boldsymbol{x},\boldsymbol{p},0]; ds, [d\boldsymbol{y},\boldsymbol{q},j])U_{t-s}^{(r)}h([\boldsymbol{y},\boldsymbol{q},j]) .$$

Applying (8. 19) and (8. 20) to the right hand side of the above equation, we can see that for $[\boldsymbol{x},\boldsymbol{p}] = [[x_1, x_2, \cdots, x_n], [p_1, p_2, \cdots, p_n]]$

$$U_t^{(r+1)}h([\boldsymbol{x},\boldsymbol{p},0])$$

$$= \int_0^t E_{(\boldsymbol{x},\boldsymbol{p})}[e^{-\sum_{i=1}^{n}\int_0^s k([X_{i,v},N_{i,v}])dv} \tag{8. 22}$$

$$\cdot \sum_{i=1}^{n} \sum_{m=0}^{\infty} \{q_m^+(X_{i,s})U_{t-s}^{(r)}h([X_{1,s}, \cdots, X_{i-1,s}, \underbrace{X_{i,s}, \cdots, X_{i,s}}_{m}, X_{i+1,s}, \cdots, X_{n,s}],$$

$$[N_{1,s}, \cdots, N_{i-1,s}, N_{i,s}, \underbrace{0, 0, \cdots, 0}_{m-1}, N_{i+1,s}, \cdots, N_{n,s}], 1])$$

$$+ q_m^-(X_{i,s})U_{t-s}^{(r)}h([[X_{1,s}, \cdots, X_{i-1,s}, \underbrace{X_{i,s}, \cdots, X_{i,s}}_{m}, X_{i+1,s}, \cdots, X_{n,s}],$$

$$[N_{1,s}, \cdots, N_{i-1,s}, N_{i,s}, \underbrace{0, 0, \cdots, 0}_{m-1}, N_{i+1,s}, \cdots, N_{n,s}], 3])\}]ds.$$

To prove the right hand side of (8. 22) is continuous in $(t, [\boldsymbol{x}, \boldsymbol{p}])$ and also belongs to $\boldsymbol{C}_0(\hat{\boldsymbol{S}})$ for any fixed $t \geqq 0$, we consider the following function

$$g(s; [\boldsymbol{x}, \boldsymbol{p}])$$

$$= \sum_{i=1}^{n} \sum_{m=0}^{\infty} \{q_m^+(x_i)U_s^{(r)}h([[x_1, \cdots, x_{i-1}, \underbrace{x_i, \cdots, x_i}_{m}, x_{i+1}, \cdots, x_n],$$

(8. 23) $$[p_1, \cdots, p_{i-1}, p_i, \underbrace{0, \cdots, 0}_{m-1}, p_{i+1}, \cdots, p_n], 1])$$

$$+ q_m^-(x_i)U_s^{(r)}h([[x_1, \cdots, x_{i-1}, \underbrace{x_i, \cdots, x_i}_{m}, x_{i+1}, \cdots, x_n],$$

$$[p_1, \cdots, p_{i-1}, p_i, \underbrace{0, \cdots, 0}_{m-1}, p_{i+1}, \cdots, p_n], 3])\},$$

where $[\boldsymbol{x}, \boldsymbol{p}] = [[x_1, x_2, \cdots, x_n], [p_1, p_2, \cdots, p_n]]$. By the assumption of induction, $U_s^{(r)}h$ is bounded and continuous on $[0, T] \times \tilde{\boldsymbol{S}}$ for any given $T > 0$ and belongs to $\boldsymbol{C}_0(\tilde{\boldsymbol{S}})$ for any fixed $t \geqq 0$. On the other hand, $\sum_{m=0}^{\infty} \{q_m^+(x) + q_m^-(x)\}$ converges to $k(x)$ uniformly on the compact space E because $\{q_m^+(x) + q_m^-(x)\} \geqq 0$ and $k(x)$ is continuous. Hence the right hand side of (8. 23) is the sum of uniformly convergent series of continuous functions, and accordingly $g(s; [\boldsymbol{x}, \boldsymbol{p}])$ is continuous in $(s, [\boldsymbol{x}, \boldsymbol{p}])$. Moreover we can see that $g(s; [\boldsymbol{x}, \boldsymbol{p}])$ belongs to $\boldsymbol{C}_0(\hat{\boldsymbol{S}})$ for any fixed $t \geqq 0$.

Now we have from (8. 22) and (8. 23)

$$U_t^{(r+1)}h([\boldsymbol{x}, \boldsymbol{p}, 0])$$

$$= \int_0^t E_{(\boldsymbol{x}, \boldsymbol{p})} [e^{-\sum_{i=1}^{n} \int_0^s k([X_{i,v}, Y_{i,v}])dv}$$

$$\cdot g(t-s; [[X_{1,s}, X_{2,s}, \cdots, X_{n,s}], [N_{1,s}, N_{2,s}, \cdots, N_{n,s}]])]ds.$$

Since the semi-group V_t corresponding to $Y_{i,t} = [X_{i,t}, N_{i,t}]$ is strongly continuous on $\boldsymbol{C}_0(S)$ and $g(s; [\boldsymbol{x}, \boldsymbol{p}])$ is bounded and continuous on $[0, T] \times \hat{\boldsymbol{S}}$, the integrand of the right hand side of the above equation is also continuous on $[0, T] \times \hat{\boldsymbol{S}}$. Hence $U_t^{(r+1)}h([\boldsymbol{x}, \boldsymbol{p}, 0])$ is continuous in $(t, [\boldsymbol{x}, \boldsymbol{p}])$ and belongs to $C_0(\hat{S})$ for any fixed $t \geqq 0$ because $g(t, [x, p]) \in C_0(\hat{S})$ for fixed $t \geqq 0$. Similarly, we can see that $U_t^{(r+1)}h([x, p, j])$, $j \in J$, are continuous in $(t, [x, p])$ and belong to $C_0(\hat{S})$ for any fixed $t \geqq 0$. Hence $U_t^{(r+1)}h$ is continuous in $(t, [x, p, j])$ and belongs to $C_0(\tilde{S})$ for any fixed $t \geqq 0$.

Next we shall prove (8. 21). Let $r \geqq 1$. Since $U_t^{(r)}h$ is continuous on a compact set $[0, T] \times \tilde{S}$ as a function of $(t, [x, p, j])$ and vanishes on $[0, T] \times \{\Delta\}$, it holds that for any $\varepsilon > 0$, there exists an n_0 such that

$$\sup_{n \geqq n_0} \| U_t^{(r)}h \|_{S^n \times J} < \varepsilon, \qquad 0 \leqq t \leqq T. \tag{8. 24}$$

On the other hand, it follows from (8. 22) that

$$\begin{aligned} &|U_t^{(r)}h([x, p, j])| \\ &\leqq \int_0^t E_{(x,p)}[e^{-\sum_{i=1}^{n}\int_0^t k([X_{i,v}, N_{i,v}])dv} \sum_{i=1}^{n} \sum_{m=0}^{\infty} (q_m^+(X_{i,s}) + q_m^-(X_{i,s})) \sup_{0 \leqq s \leqq t} \| U_{t-s}^{(r-1)}h \| ds \\ &= \sup_{0 \leqq s \leqq t} \| U_s^{(r-1)}h \| \int_0^t E_{(x,p)}[e^{-\sum_{i=1}^{n}\int_0^s k([X_{i,v}, N_{i,v}])dv} \sum_{i=1}^{n} k([X_{i,s}, N_{i,s}])]ds \\ &= \sup_{0 \leqq s \leqq t} \| U_s^{(r-1)}h \| (1 - e^{-n\|k\|t}), \qquad [x, p, j] \in \tilde{S}, \ t \geqq 0. \end{aligned} \tag{8. 25}$$

Since $U_s^{(r-1)}h$ is bounded on $[0, T] \times \tilde{S}$, there exists a constant M such that

$$\sup_{0 \leqq s \leqq T} \| U_s^{(r-1)}h \| \leqq M < \infty.$$

Then (8. 24) and (8. 25) show us the following inequality.

$$\| U_t^{(r)}h \| \leqq M(1 - e^{-n_0\|k\|t}) + \varepsilon, \qquad 0 \leqq t \leqq T.$$

This proves (8. 21) because ε is arbitrary. Q.E.D.

We are now in a position to state the following

THEOREM 8. 1. *Let $\{(q_n^+(x), q_n^-(x)); n = 0, 1, 2, \cdots\}$ be a given system of pairs of non-negative continuous functions on E such that*

$$k(x) = \sum_{n=0}^{\infty} (q_n^+(x) + q_n^-(x)), \qquad x \in E,$$

is bounded continuous on E, and

$$q_n^+(x)q_n^-(x) = 0, \qquad n = 0, 1, 2, \cdots.$$

Then there exists a signed branching Markov process with age $Z = \{Z_t = [X_t, N_t, J_t], \zeta, \mathscr{B}_t, P_{[x,p,j]}; [x, p, j] \in \tilde{S}\}$ on $\tilde{S}$ satisfying Condition 3 for a given $\{(q_n^+(x), q_n^-(x)); n = 0, 1, 2, \cdots\}$.

Proof. According to Lemma 7. 2 and Lemma 8. 2-8. 3, there exists a right continuous strong Markov process $Z=\{Z_t=[X_t, N_t, J_t], \zeta, \mathscr{B}_t, P_{[x,p,j]}; [x, p, j] \in \tilde{S}\}$ on $\tilde{S}$ corresponding to the semi-group U_t given in (8. 12) and a $\mathscr{B}_t$-Markov time η such that

$$P_{[x,p,j]}(Z_t \in B, \eta_r \leqq t < \eta_{r+1}) = \chi_r(t, [x, p, j], B),$$

(8. 26)

$$P_{[x,p,j]}(Z_{\eta_r} \in B, \eta_r \in dt) = \Psi_r([x, p, j]; dt, B),$$

$$[x, p, j] \in \tilde{S}, \ B \in \mathscr{B}(\tilde{S}), \ r \geqq 0, \ t \geqq 0,$$

where

$$\eta_0 = 0, \ \eta_1 = \eta, \ \eta_{r+1} = \eta_r + \theta_{\eta_r}\eta, \ r \geqq 1.$$

Let us set

$$\tilde{\eta}(w) = \inf\{t > 0; \ J_t(w) \neq J_0(w) \text{ or } \sup_{s \leqq t} |N_s(w)| = \infty\}.$$

Since we can see from (8. 4), (8. 6) and (8. 26) that

$$P_{[x,p,j]}(J_\eta = J_0 \text{ or } \sup_{s \leqq \eta} |N_s(w)| = \infty) = 0, \qquad [x, p, j] \in \tilde{S},$$

we have

$$P_{[x,p,j]}(\tilde{\eta} > \eta) = 0, \qquad [x, p, j] \in \tilde{S}.$$

On the other hand, $\chi_0(t, [x, p, j], \cdot)$ vanishes outside of $\hat{S} \times \{j\}$. Hence we have

$$P_{[x,p,j]}(J_t \neq J_0, \ t < \eta) = 0,$$

which means

$$P_{[x,p,j]}(J_s = J_0 \text{ for any } s \leqq t < \eta) = P_{[x,p,j]}(t < \eta),$$

because J_t is right continuous. So we have

$$P_{[x,p,j]}(\tilde{\eta} \neq \eta) = 0, \qquad [x, p, j] \in \tilde{S},$$

which means that we may regard η_r is the rth branching time of Z_t.

Now we shall check the conditions in Condition 3 for our process.

(*a*) (i) of Condition 1 follows from (8. 4), (8. 26) and the definition of Y_t^0 which was used to construct χ_0.

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(b) Let X'_t be the conservative Feller process on E and let $Y_t=[X_t, N_t]$ be the strong Markov process on S which are mentioned in the first part of this section. Then it follows from the way of constructions of χ_0 and Ψ, that

$$\begin{aligned} &P_{[x,0,j]}([X_{\eta-}, N_{\eta-}] \in [B, n],\ \eta \in dt) \\ &= E_{[x,0]}[e^{-\int_0^t k([X_s, N_s])ds} k([X_t, N_t]) I_{B\times\{n\}}([X_t, N_t])]dt \\ &= E_x[e^{-2\int_0^t k(X'_s)ds} k(X'_t)\, I_B(X'_t) \frac{\left(\int_0^t k(X'_s)\,ds\right)^n}{n!}]dt,\ x\in E,\ j\in J,\ B\in \mathscr{B}(E). \end{aligned}$$

where $E_{[x,p]}$ and E_x denote the integrals by the probability measures of Y_t and X'_t respectively, and $k([x, p]) = k(x)$. So (3. 3) holds. Similarly, (3. 4) holds. Moreover, by the definitions of χ_0, Ψ and Ψ_r, we have

$$\chi_0(t, [\boldsymbol{x}, \boldsymbol{p}, j],\ [B, \boldsymbol{q}, J]) = \chi_0(t, [\boldsymbol{x}, \boldsymbol{p}, j'],\ [B, \boldsymbol{q}, J]),$$
$$\Psi([\boldsymbol{x}, \boldsymbol{p}, j];\ dt,\ [B, \boldsymbol{q}, J]) = \Psi([\boldsymbol{x}, \boldsymbol{p}, j'];\ dt, [B, \boldsymbol{q}, J]),$$

and hence

$$\Psi_r([\boldsymbol{x}, \boldsymbol{p}, j];\ dt,\ [B, \boldsymbol{q}, J]) = \Psi_r([\boldsymbol{x}, \boldsymbol{p}, j'];\ dt,\ [B, \boldsymbol{q}, J]),$$
$$[\boldsymbol{x}, \boldsymbol{p}] \in \hat{\boldsymbol{S}},\ \ j, j' \in J,\ \ [B, \boldsymbol{q}] \in \mathscr{B}(\hat{\boldsymbol{S}}).$$

So we have

$$\chi(t, [\boldsymbol{x}, \boldsymbol{p}, j],\ [B, \boldsymbol{q}, J]) = \chi(t, [\boldsymbol{x}, \boldsymbol{p}, j'],\ [B, \boldsymbol{q}, J]),$$
$$[\boldsymbol{x}, \boldsymbol{p}] \in \hat{\boldsymbol{S}},\ \ j, j' \in J,\ \ [B, \boldsymbol{q}] \in \mathscr{B}(\hat{\boldsymbol{S}}).$$

Thus our process satisfies (i) of Condition 2.

(c) (4. 2) follows from (8. 26) and (8. 4). (4. 3) follows also from Theorem 7. 2 and (7. 22).

Combining (a) - (c), we can see that our process satisfies (i) of Condition 3.

(d) (ii) of Condition 3 follows from (8. 3), (8. 7) and the definition of "$\sim$".

(e) (iii) of Condition 3 follows from (8. 6).

(f) (iv) of Condition 3 follows from (8. 3), (8. 5) and (8. 7).

(g) By the definition of χ and (8. 26), we have

$$P_{[\boldsymbol{x},\boldsymbol{p},j]}(Z_t \in \tilde{\boldsymbol{S}}) = \sum_{r=0}^{\infty} U_t^{(r)} I_{\tilde{S}}([\boldsymbol{x},\boldsymbol{p},j])$$

$$= \sum_{r=0}^{\infty} P_{[\boldsymbol{x},\boldsymbol{p},j]}(\eta_r \leqq t < \eta_{r+1})$$

$$= P_{[\boldsymbol{x},\boldsymbol{p},j]}(t < \eta_\infty),$$

$$[\boldsymbol{x},\boldsymbol{p},j] \in \tilde{\boldsymbol{S}}.$$

Therefore we may consider that

$$P_{[x,p,j]}(\eta_\infty < \zeta) = 0, \qquad [\boldsymbol{x},\boldsymbol{p},j] \in \tilde{\boldsymbol{S}}.$$

If we consider a new process $\tilde{Z}_t$ defined by

$$\tilde{Z}_t(w) = \begin{cases} Z_t(w), & \text{if } t < \eta_\infty(w) \wedge \zeta(w), \\ \Delta \quad , & \text{if } t \geqq \eta_\infty(w) \wedge \zeta(w), \end{cases}$$

and Borel field $\tilde{\mathscr{B}}_t$ induced naturally from $\mathscr{B}_t$, then $\tilde{Z}_t$ satisfies (v) of Condition 3.

We shall denote $\tilde{Z}_t$ by Z_t again. Then (a) - (g) implies that our process Z_t satisfies Condition 3. Moreover, by Theorem 5. 1, Z_t is a signed branching Markov process with age on $\tilde{\boldsymbol{S}}$. Q.E.D.

Corollary 8. 1. *Let* $\{(q_n^+(x),\ q_n^-(x));\ n = 0,1,2,\cdots\}$ *be a given system of pairs of non-negative continuous functions on* E *such that* $k(x) = \sum_{n=0}^{\infty}(q_n^+(x) + q_n^-(s))$ *is bounded continuous on* E *and* $q_n^+(x)q_n^-(x) = 0,\ n = 0,1,2,\cdots$. *Then there exists a signed branching Markov process* $Z = \{Z_t,\ \zeta,\ \mathscr{B}_t,\ P_{[\boldsymbol{x},\boldsymbol{p},j]};\ [\boldsymbol{x},\boldsymbol{q},j] \in \tilde{\boldsymbol{S}}\}$ *on* $\tilde{\boldsymbol{S}}$ *satisfying Condition* 2 *for a given* $\{(q_n^+(x),\ q_n^-(x));\ n = 0,1,2,\cdots\}$.

Corollary 8. 2. *Let* $\{q_n(x);\ n = 0,2,3,\cdots\}$ *be a given system of non-negative continuous functions on* E *such that* $k(x) = \sum_{n\neq 1} q_n(x)$ *is bounded continuous on* E. *Then there exists a branching Markov process with age* $Y = \{Y_t = [X_t, N_t],\ \zeta,\ \mathscr{B}_t,\ P_{[\boldsymbol{x},\boldsymbol{p}]};\ [\boldsymbol{x},\boldsymbol{p}] \in \hat{\boldsymbol{S}}\}$ *on* $\hat{\boldsymbol{S}}$ *satisfying Condition* 1 *for a given* $\{q_n(x);\ n = 0,2,3,\cdots\}$.

Proof. Let us consider in Theorem 8. 1 the special case where $q_n^-(x)=0$, $n = 0,1,2,\cdots$, and $q_1^+(x) = 0$. Let $Z = \{Z_t = [X_t, N_t, J_t],\ \zeta,\ \mathscr{B}_t,\ P_{[\boldsymbol{x},\boldsymbol{p},j]};\ [\boldsymbol{x},\boldsymbol{p},j] \in \tilde{\boldsymbol{S}}\}$ be the process obtained in Theorem 8. 1 for the present case. Setting

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$$\xi_t(w) = \begin{cases} n\,, & \text{if } [X_t(w),\ N_t(w)] \in S^n,\ n \geqq 0, \\ \infty\,, & \text{if } [X_t(w),\ N_t(w)] = \Delta\,, \end{cases}$$

and

$$\tau(w) = \inf\,\{t > 0;\ \xi_t(w) \neq \xi_0(w) \text{ or } \sup_{s \leqq t} |N_s(w)| = \infty\}\,,$$

we have

$$P_{[\boldsymbol{x},\boldsymbol{p},0]}(\tau \neq \eta) = 0\,, \qquad [\boldsymbol{x},\boldsymbol{p}] \in \hat{\boldsymbol{S}}\,.$$

Also it follows from the definition of Ψ that

$$P_{[\boldsymbol{x},\boldsymbol{p},0]}(J_t = 2 \text{ or } 3) = 0\,, \qquad [\boldsymbol{x},\boldsymbol{p}] \in \hat{\boldsymbol{S}},\ t \geqq 0\,,$$

and hence we have

$$P_{[\boldsymbol{x},\boldsymbol{p},0]}(\widetilde{f \cdot \lambda}(Z_t) = \widehat{f \cdot \lambda}(Z_t),\ t < \zeta) = P_{[\boldsymbol{x},\boldsymbol{p},0]}(t < \zeta)\,, \qquad [\boldsymbol{x},\boldsymbol{p}] \in \hat{\boldsymbol{S}}\,.$$

So, if we disregard J_t in $Z_t = [X_t, N_t, J_t]$ and define $P_{[\boldsymbol{x},\boldsymbol{p}]}$ by

$$P_{[\boldsymbol{x},\boldsymbol{p}]}([X_t, N_t] \in [B, q]) = P_{[\boldsymbol{x},\boldsymbol{p},0]}([Z_t \in [B, q, J])\,,$$

then the process $Y = \{Y_t = [X_t, N_t],\ \zeta,\ \mathscr{B}_t,\ P_{[\boldsymbol{x},\boldsymbol{p}]};\ [\boldsymbol{x},\boldsymbol{p}] \in \hat{\boldsymbol{S}}\}$ satisfies Condition 1 and

$$T_t\widehat{f \cdot \lambda}([\boldsymbol{x},\boldsymbol{p}]) = \widehat{(T_t\widehat{f \cdot \lambda})|_E \cdot \lambda}([\boldsymbol{x},\boldsymbol{p}])\,, \qquad [x,p] \in \hat{\boldsymbol{S}},\ f \in \boldsymbol{C}^*(E),\ 0 \leqq \lambda < 1\,,$$

where T_t denotes the semi-group on $\boldsymbol{B}(\hat{\boldsymbol{S}})$ induced by Y. Q.E.D.

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