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ON THE PASS-EQUIVALENCE OF LINKS

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We give a simple geometric proof that the Jones polynomial at the value i of an oriented link is invariant under pass-equivalence.

0. Introduction

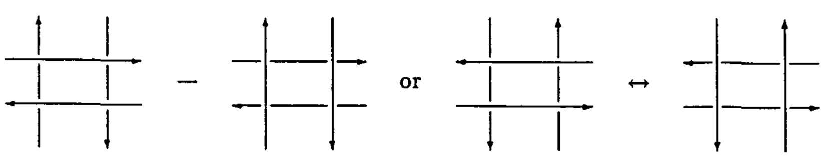
The Arf invariant of a knot or more generally a proper oriented link was introduced in [6]. It was shown by Murakami in [5] that the value $V_L(i)$ of the Jones polynomial at i is a suitable generalisation of the Arf invariant for an arbitrary oriented link L. In fact Murakami computed that $V_L(i)$ equals $\left(-\sqrt{2}\right)^{c(L)-1}(-1)^{Arf(L)}$ if L is a proper oriented link and equals zero if L is not proper, where $c(L)$ denotes the number of components of L. On the other hand, Kauffman introduced in [2] the concept of pass-equivalence or equivalently Γ-equivalence of links. It was shown in [2] that any oriented link is pass-equivalent to either the unlink, the unlink disjoint union a trefoil or the unlink disjoint union a connected sum of Hopf links. This together with Murakami's result implies the following:

Theorem. *Two oriented links L and L' are pass-equivalent if and only if $V_L(i) = V_{L'}(i)$, $c(L) = c(L')$ and $n(L) = n(L')$, where $n(L)$ is the number of components K of L such that $lk(K, L - K)$ is odd.*

In this paper we shall give a direct geometric proof of the fact that $V_L(i)$ is invariant under pass-equivalence and hence the above result and Murakami's result. All the links considered are oriented.

1. Pass-equivalence and Γ-equivalence

A pass-move on a link diagram is a move of one of the following two forms:



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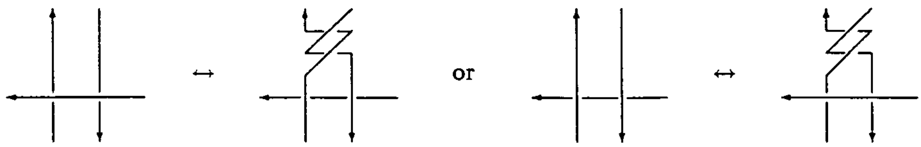


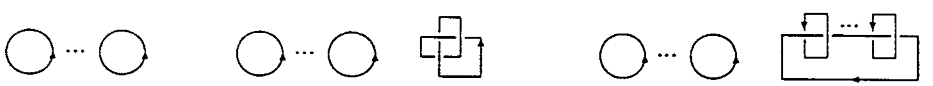
A Γ-move on a link diagram is a move of one of the above two forms:

DEFINITION 1.1: Two links are pass-equivalent (Γ-equivalence) if one can be obtained from the other by a finite combination of pass-moves (Γ-moves) and ambient isotopies.

We shall use $\sim$ to denote pass-equivalence and $\equiv$ to denote the equivalence of being ambient isotopic. Next we recall the following two results from [**2**] and [**3**].

PROPOSITION 1.2. *Two links are pass-equivalent if and only if they are Γ-equivalent.*

PROPOSITION 1.3. *Any link is pass-equivalent to one of the following three forms:*



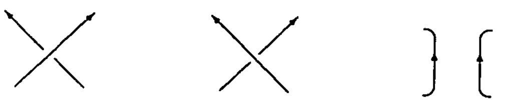
(a) (b) (c)

where in (a) the Arf invariant is 0, *in (b) the Arf invariant is* 1 *and in (c) the number of components minus the number of unknots is even.*

2. THE INVARIANT $V_L(i)$ OF A LINK L

$V_L(i)$ is the value of the Jones polynomial of L at i. It satisfies the following two axioms:

(i) $V_{\text{unknot}}(i) = 1$;
(ii) $V_{L_+}(i) + V_{L_-}(i) = -\sqrt{2}V_{L_0}(i)$, where L_+, L_0 are three links identical except within a ball where they have a projection as follows:



L_+ L_- L_o

In fact (i) and (ii) uniquely determine the numerical invariant $V(i)$.

For any two links L_1 and L_2, $lk(L_1, L_2)$ denotes the total linking number of L_1 and L_2. We say that a link L is proper if $lk(K, L-K)$ is even for every component

K in L, otherwise it is said to be non-proper. For example the link in Proposition 1.2 (c) is non-proper. Next we recall the following result in [1].

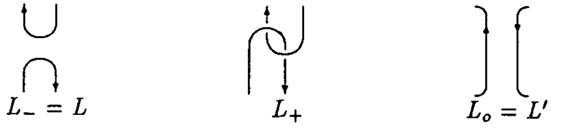
JONES REVERSING RESULT: If L' is obtained from L by reversing the orientation of one component that has linking number m with the remaining components of L, then $V_{L'}(t) = t^{-3m}V_L(t)$.

Suppose L is a non-proper link. Let K be a component of L such that $lk(K, L-K) = m$ is odd. Then by reversing the orientation of K, we have $V_{L'}(i) = i^{-3m}V_L(i) = \pm iV_L(i)$. From axiom (ii), $V_L(i)$ is always a real number. Hence $V_L(i) = 0$.

To prove the main theorem we need two lemmas which are also use in [4] to prove Murakami's result.

LEMMA 2.1. *Let L be an oriented link and L' the link constructed by banding together two distinct components of L. If L is proper, then L' is proper and $V_{L'}(i) = -\sqrt{1/2}V_L(i)$.*

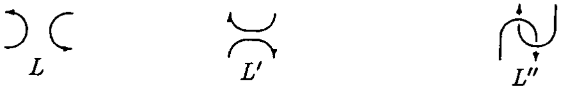
PROOF: A calculation of linking numbers shows that if L is proper, then L' is proper. By cutting the band within a 3-ball surrounding the band, we have the following skein triple.



Since L_- is non-proper, $V_{L_-}(i) = 0$. Hence $V_{L'}(i) = -\sqrt{1/2}V_L(i)$. ∎

Similarly one can prove the next lemma.

LEMMA 2.2. *Let L, L' and L'' be three oriented links identical except within a ball where they have a projection as shown below:*



where the two strings in L belong to the same component. Suppose L is proper. Then precisely one of L' and L'' is proper. Furthermore if $L^* \in \{L', L''\}$ is proper, then $V_{L^*}(i) = -\sqrt{2}V_L(i)$.

LEMMA 2.3. $V_{L\#\text{Trefoil}}(i) = -V_L(i)$.

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PROOF: $V_{L\#\text{Trefoil}}(i) = V_{\text{Trefoil}}(i)V_L(i) = -V_L(i)$. ∎

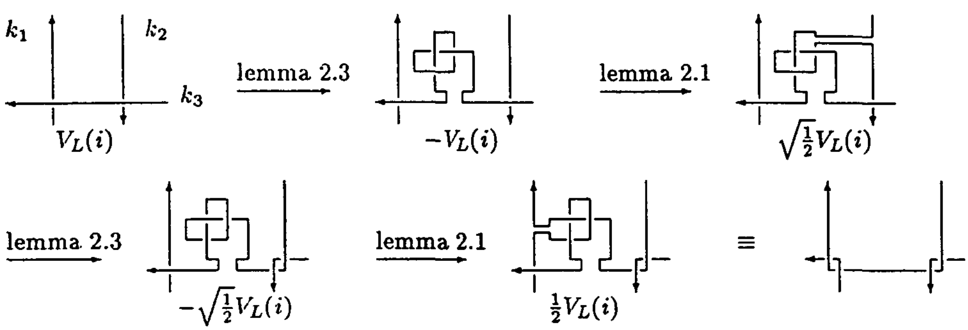
3. Proof of the theorem

PROPOSITION 3.1. *If L and L' are Γ-equivalent, then $V_L(i) = V_{L'}(i)$.*

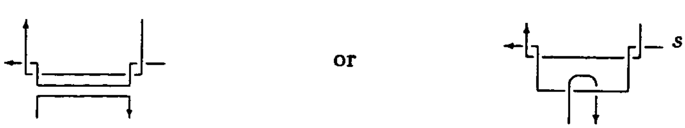
PROOF: It suffices to show that $V(i)$ does not change with the two Γ-moves. For this, we shall show that if L and L' are links identical except within a ball where they have a projection as shown below,

then $V_L(i) = V_{L'}(i)$. Notice that Γ-moves or pass-moves preserve properness. Therefore in the case of non-proper links, $V_L(i) = V_{L'}(i)$. Hence we only need to consider proper links. There are three cases.

CASE 1. (k_1, k_2 and k_3 belong to the same components of L.) We can represent a Γ-move by a sequence of taking connected sums of the trefoils or the components of L. By keeping track of the value $V_L(i)$, we will show that $V_L(i) = V_{L'}(i)$. This is shown as follows.



Here the three strings in the last diagram belong to a single component. By taking a connected sum or a connected sum with a twist of the two lower strings, we get two possibilities:



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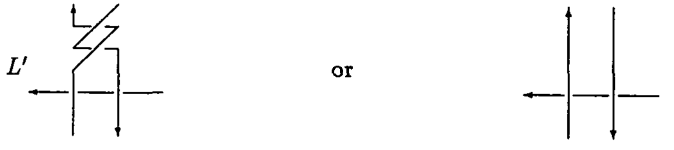
But in the second case the linking number of s and the rest of the other components is odd so that it is not proper. By Lemma 2.2 the first link is proper and its value of $V(i)$ is equal to $-\sqrt{1/2}V_L(i)$. That is

and its value of $V_L(i)$ equals $-\sqrt{1/2}V_L(i)$.

Again we take a connected sum or a connected sum with a twist of the upper and lower strings. We then have the cases:



Since the latter case gives a non-proper link, we perform the operation of taking a connected sum with a twist to get L'. By Lemma 2.2 $V_{L'}(i) = V_L(i)$.

CASE 2. (Only two of the strings belong to the same component.) By taking a connected sum or a connected sum with a twist of the two strings, say k_1 and k_2 of the same component outside the ball, we get two links. By Lemma 2.2 precisely one of them is proper and for that one L^*, $V_{L^*}(i)$ equals $-\sqrt{2}V_L(i)$. Inside the ball we still have the same link diagram but the three strings now belong to different components of L^*. Hence we can apply the result of Case 1 to conclude that if L^{**} is the link obtained by performing a Γ-move on L^* within the ball, then $V_{L^{**}}(i) = V_{L^*}(i) = -\sqrt{2}V_L(i)$. Now we take a connected sum of the knots k_1 and k_2. We get a proper link which is L' and by Lemma 2.1, $V_{L'}(i) = -\sqrt{1/2}V_{L^{**}}(i) = V_L(i)$.

CASE 3. (All three strings belong to the same component.) We can apply the same argument as in Case 2 to two of the strings and reduce this case to Case 2. This completes the proof. $\square$

PROOF OF THE THEOREM: $(\Rightarrow)$ That $V_L(i) = V_{L'}(i)$ is proved in Proposition 3.1. Since pass-equivalence does not change the number of components K such that $lk(K, L-K)$ is odd, we have $n(L) = n(L')$. Obviously $c(L) = c(L')$.

$(\Leftarrow)$ By Proposition 1.3, any link is pass-equivalent to one of the form (a), (b) and (c) as shown in Proposition 1.3. If both L and L' are not proper, then they are pass-equivalent to a link of the form (c). Since $c(L) = c(L')$ and $n(L) = n(L')$, we must have $L \sim L'$. If L and L' are both proper, then they are pass-equivalent to a link of the form (a) and (b). By Proposition 3.1, we have

$$V_L(i) = \begin{cases} \left(-\sqrt{2}\right)^{c(L)-1} & \text{if } L \sim (a) \\ -\left(-\sqrt{2}\right)^{c(L)-1} & \text{if } L \sim (b). \end{cases}$$

Since $V_L(i) = V_{L'}(i)$, we must have L and L' both pass-equivalent to either the form (a) or (b). Hence $L \sim L'$. This completes the proof of the theorem. ☐

Corollary. (Murakami [5]) *For any oriented link* L,

$$V_L(i) = \begin{cases} \left(-\sqrt{2}\right)^{c(L)-1}(-1)^{Arf(L)} & \text{if } L \text{ is proper} \\ 0 & \text{if } L \text{ is non-proper.} \end{cases}$$

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