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ON SIEVED ORTHOGONAL POLYNOMIALS II: RANDOM WALK POLYNOMIALS

JAIRO CHARRIS AND MOURAD E. H. ISMAIL

1. Introduction. A birth and death process is a stationary Markov process whose states are the nonnegative integers and the transition probabilities

$$(1.1)\qquad p_{mn}(t) = Pr\{X(t) = n|X(0) = m\}$$

satisfy

$$(1.2)\qquad p_{mn}(t) = \begin{cases} \beta_m t + 0(t) & n = m + 1 \\ \delta_m t + 0(t) & n = m - 1 \\ 1 - (\beta_m + \delta_m)t + o(t) & n = m, \end{cases}$$

as $t \to 0$. Here we assume $\beta_n > 0$, $\delta_{n+1} > 0$, $n = 0, 1, \dots$, but $\delta_o \geqq 0$. Karlin and McGregor [**10**], [**11**], [**12**], showed that each birth and death process gives rise to two sets of orthogonal polynomials. The first is the set of birth and death process polynomials $\{Q_n(x)\}$ generated by

$$Q_0(x) = 1,\ Q_1(x) = (\beta_0 + \delta_0 - x)/\beta_0,$$
$$-xQ_n(x) = \beta_n Q_{n+1}(x) + \delta_n Q_{n-1}(x) - (\beta_n + \delta_n)Q_n(x), \qquad n > 0.$$

In this case there exists a positive measure $d\alpha$ supported on $[0, \infty)$ such that

$$\int_0^\infty Q_n(x)Q_m(x)d\alpha(x) = \delta_{m,n}/\pi_n, \quad m, n = 0, 1, \dots,$$

holds where

$$\pi_n = \beta_0\beta_1 \dots \beta_{n-1}/\{\delta_1\delta_2 \dots \delta_n\}, \quad n > 0,\ \pi_0 = 1.$$

The second set is the set of random walk polynomials. They arise when one studies a random walk on the state space. The random walk polynomials $\{R_n(x)\}$ satisfy the recursion

$$(1.3)\qquad xR_n(x) = B_n R_{n+1}(x) + D_n R_{n-1}(x), \quad n > 0$$

and the initial conditions

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$$(1.4) \quad R_0(x) = 1, \quad R_1(x) = x/B_0,$$

with

$$(1.5) \quad B_n = \beta_n/(\beta_n + \delta_n), \quad D_n = \delta_n/(\beta_n + \delta_n).$$

Clearly $B_n + D_n = 1$. The random walk polynomials are orthogonal with respect to a positive measure supported in $[-1, 1]$. In fact

$$(1.6) \quad \int_{-1}^{1} R_m(x)R_n(x)dR(x) = \Lambda_n\delta_{mn},$$

where

$$(1.7) \quad \Lambda_0 = 1, \quad \Lambda_n = \{D_1D_2 \dots D_n\}/\{B_0B_1 \dots B_{n-1}\}, \quad n > 0.$$

The ultraspherical (Gegenbauer) polynomials $\{C_n^\lambda(x)\}$ are random walk polynomials with

$$(1.8) \quad B_n = \frac{1}{2}(n + 1)/(n + \lambda), \quad D_n = \frac{1}{2}(n + 2\lambda - 1)/(n + \lambda),$$

([**15**] and [**17**]). On the other hand the random walk polynomials associated with

$$(1.9) \quad B_n = \frac{1}{2}(n + 2\lambda)/(n + \lambda), \quad D_n = \frac{1}{2}n/(n + \lambda),$$

are $\{n!C_n^\lambda(x)/(2\lambda)_n\}$, where

$$(1.10) \quad (\sigma)_0 = 1, \quad (\sigma)_n = \sigma(\sigma + 1) \dots (\sigma + n - 1), \quad n > 0.$$

Al-Salam, Allaway and Askey [**1**] observed that two limiting cases of the Rogers continuous q-ultraspherical polynomials, [**2**], [**4**], are interesting. In both cases q approached $\exp(2\pi i/k)$, k is a given positive integer, $k > 1$. This led them to define the sieved ultraspherical polynomials of the first kind by

$$(1.11) \quad \begin{cases} c_0^\lambda(x; k) = 1, \ c^\lambda(x; k) = x, \\ (m + 2\lambda)c_{mk+1}^\lambda(x; k) = 2x(m + \lambda)c_{mk}^\lambda(x; k) \\ \qquad\qquad - mc_{mk-1}^\lambda(x; k), \ m > 0, \\ c_{n+1}^\lambda(x; k) = 2xc_n^\lambda(x; k) - c_{n-1}^\lambda(x; k), \ k \nmid n, \ n > 0 \end{cases}$$

and the sieved ultraspherical polynomials of the second kind via

$$(1.12) \quad \begin{cases} B_0^\lambda(x; k) = 1, \ B_1^\lambda(x; k) = 2x, \\ mB_{mk}^\lambda(x; k) = 2x(m + \lambda)B_{mk-1}^\lambda(x; k) \\ \qquad\qquad - (m + 2\lambda)B_{mk-2}^\lambda(x; k), \ m > 0 \\ B_{n+1}^\lambda(x; k) = 2xB_n^\lambda(x; k) - B_{n-1}^\lambda(x; k), \ k \nmid n + 1, \ n > 0. \end{cases}$$

In this work, we generalize the sieved ultraspherical polynomials to a fairly large class of random walk polynomials. This is done by starting with a set of random walk polynomials $\{R_n(x)\}$ satisfying (1.3) and (1.4). We shall also assume that $k > 1$ is a given integer and

(1.13) $B_n + D_n = 1, \quad 0 < B_n < 1, n = 0, 1, \ldots .$

The sieved random walk polynomials of the first kind are generated by

(1.14) $r_0(x) = 1, r_1(x) = x,$

$$xr_n(x) = d_{n-1}r_{n+1}(x) + b_{n-1}r_{n-1}(x), \; n > 0,$$

while the sieved random walk polynomials of the second kind are defined recursively by

(1.15) $s_0(x) = 1, s_1(x) = 2x,$

$$xs_n(x) = b_n s_{n+1}(x) + d_n s_{n-1}(x), \; n > 0,$$

where

(1.16) $b_n = d_n = \dfrac{1}{2}$ if $k \nmid n + 1$, $b_{nk-1} = B_{n-1}$, $d_{nk-1} = D_{n-1}$.

In particular, when B_n, D_n are defined by (1.9), $R_n(x) = C_n^\lambda(x)$, the r_n's and s_n's are essentially the c_n^λ's and B_n^λ's of Al-Salam, Allaway and Askey. We shall establish explicit formulas and generating functions for $r_n(x)$ and $s_n(x)$ in terms of $R_n(x)$ and the Chebyshev polynomials, see (2.3), (2.5), (2.6), (3.4) and (3.5). In Section 4 we shall show that such formulas hold only for random walk polynomials. In Section 5 we shall show how to compute the Stieltjes transform of the distribution (spectral) function of $\{r_n(x)\}$ from the asymptotics of the random walk polynomials $\{R_n(x)\}$ and their duals $\{S_n(x)\}$. The dual polynomials $\{S_n(x)\}$ are the random walk polynomials

(1.17) $S_0(x) = 1, S_1(x) = x/D_0, xS_n(x) = D_nS_{n+1}(x) + B_nS_{n-1}(x).$

Karlin and McGregor [**11**] studied random walk polynomials $\{R_n(x)\}$ when $\delta_n = n$ and $\beta_n = b$. Carlitz [**6**] was generalizing earlier work of Tricomi and independently discovered the same set of polynomials at the same time. Chihara [**7**] calls them the Tricomi-Carlitz polynomials but we shall call them the Carlitz-Karlin-McGregor (CKM) polynomials and denote them by $\{r_n(x; b)\}$, as in [**3**]. They are recursively defined by

(1.18) $$\begin{cases} r_0(x; b) = 1, r_1(x; b) = x, \\ x(n + b)r_n(x; b) = br_{n+1}(x; b) + nr_{n-1}(x; b), \; n \geqq 0. \end{cases}$$

Carlitz proved their orthogonality using Euler's identity

(1.19) $$e^{\alpha z} = 1 + \alpha \sum_{n=1}^{\infty} \frac{(\alpha + n)^{n-1}}{n!} (ze^{-z})^n,$$

but Karlin and McGregor used probabilistic methods to compute their distribution function. Their orthogonality relation is

$$(1.20)\quad \sum_{j=0}^{\infty} \sigma_j r_m(x_j;\, b) r_n(x_j;\, b) + \sum_{j=0}^{\infty} \sigma_j r_m(-x_j;\, b) r_n(-x_j;\, b) = h_n \delta_{m,n}$$

where

$$(1.21)\quad \sigma_j = \frac{b/2}{j!}(b + j)^{j-1} \exp(b + j), \quad h_n = n! b^{1-n}/(b + n).$$

These remarkable polynomials are discrete analogues of the Hermite polynomials and their distribution function is a step function (with infinitely many steps) that approximates the integral

$$\int_{-\infty}^{x} \exp(-t^2) dt.$$

To see this let

$$(1.22)\quad q_n(x) = (2b)^{n/2} r_n(x\sqrt{2/b};\, b).$$

It is easy to see that $q_0(x) = 1$, $q_1(x) = 2x$ and

$$2x(1 + n/b) q_n(x) = q_{n+1}(x) + 2n q_{n-1}(x),$$

which when compared with

$$H_0(x) = 1,\ H_1(x) = 2x,\ 2xH_n(x) = H_{n+1}(x) + 2nH_{n-1}(x),$$

[**15**, page 188], shows that

$$\lim_{b\to\infty} q_n(x) = H_n(x),$$

hence

$$(1.23)\quad \lim_{b\to\infty} (2b)^{n/2} r_n(x\sqrt{2/b};\, b) = H_n(x).$$

In Section 6 a sieved analogue of the CKM polynomials will be introduced. We apply the results of Section 3 and 5 to obtain explicit formulas and generating functions for the sieved CKM polynomials. We then apply Theorem 5.1 and compute the distribution function of the sieved CKM polynomials.

2. Sieved polynomials of the second kind. Recall that these polynomials satisfy (1.15) and (1.16). We shall adopt the convention

$$(2.1)\quad U_{-1}(x) = R_{-1}(x) = 0.$$

The elementary trigonometric identity

$$(2.2)\quad U_n(x) = U_{n-2}(x) + 2T_n(x)$$

will be used repeatedly. We now prove:

THEOREM 2.1. *The explicit representations*

(2.3) $$s_{nk+l}(x) = U_l(x)R_n(T_k(x)) + U_{k-l-2}(x)R_{n-1}(T_k(x)),$$

hold for $l = 0, 1, \dots, k - 1$, $n = 0, 1, \dots$.

Proof. Let $s_{nk+l}(x)$ denote the right side of (2.3). These s_n's clearly satisfy the initial conditions in (1.15) so it remains to show that the polynomials also satisfy the recursion in (1.15). It is straightforward to obtain the recursion

$$2xs_{nk+l}(x) = s_{nk+l+1}(x) + s_{nk+l-1}(x), \quad l = 0, 1, \dots, k - 2,$$

from the recurrence relation

(2.4) $$2xU_n(x) = U_{n+1}(x) + U_{n-1}(x).$$

This proves the recursion in (1.15) when $n = mk + l$, $l = 1, 2, \dots, k - 2$. The case $l = 0$ can be similarly proved since (2.4) holds for $n = 0$ and

$$s_l(x) = U_l(x), \quad l = 0, 1, \dots, k - 1.$$

The case $l = k - 1$ can be proved as follows. First observe that

$$\begin{aligned} 2xs_{nk+k-1}(x) &= 2xU_{k-1}(x)R_n(T_k(x)) \\ &= \{U_k(x) + U_{k-2}(x)\}R_n(T_k(x)) \\ &= 2\{T_k(x) + U_{k-2}(x)\}R_n(T_k(x)), \end{aligned}$$

in view of (2.2) and (2.4). Now (1.3) and the above relationship yield

$$\begin{aligned} xs_{nk+k-1}(x) &= B_nR_{n+1}(T_k(x)) + D_nR_{n-1}(T_k(x)) \\ &\quad + U_{k-2}(x)R_n(T_k(x)) \\ &= B_n\{R_{n+1}(T_k(x)) + U_{k-2}(x)R_n(T_k(x))\} \\ &\quad + D_n\{U_{k-2}(x)R_n(T_k(x)) + R_{n-1}(T_k(x))\}, \end{aligned}$$

where we used (1.13). Thus (2.3) holds when $n = mk + k - 1$. This identifies the right sides of (2.3) as the polynomials under investigation because both sides satisfy the same second order difference equation and the same initial conditions. The proof is now complete.

COROLLARY 2.2. *The* s_n's *have the generating function*

(2.5) $$\sum_{n=0}^{\infty} s_n(x)t^n = \frac{1 - 2t^kT_k(x) + t^{2k}}{1 - 2xt + t^2} \sum_{n=0}^{\infty} R_n(T_k(x))t^{nk}.$$

Proof. We have

$$\sum_{n=0}^{\infty} s_n(x)t^n = \sum_{l=0}^{k-1} t^l \sum_{m=0}^{\infty} t^{mk} s_{mk+1}(x)$$

$$= \sum_{l=0}^{k-1} t^l \sum_{n=0}^{\infty} t^{nk}[U_l(x)R_n(T_k(x))$$

$$+ U_{k-l-2}(x)R_{n-1}(T_k(x))]$$

$$= \sum_{l=0}^{k-1} t^l\{U_l(x) + t^k U_{k-l-2}(x)\} \sum_{n=0}^{\infty} t^{nk} R_n(T_k(x))$$

$$= \text{the right side of (2.5)},$$

after some simplification, where we used

$$\sum_{l=0}^{k-1} t^l\{U_l(x) + t^k U_{k-l-2}(x)\}$$
$$= \{1 - 2t^k T_k(x) + t^{2k}\}/\{1 - 2xt + t^2\}.$$

This completes the proof.

When $R_n(x) = C_n^{\lambda+1}(x)$, the B_n's and D_n's are given by (1.8) with λ replaced by $\lambda + 1$ and $\{s_n(x)\}$ reduces to $\{B_n^{\lambda}(x; k)\}$. In this case (2.5) gives

$$\sum_{n=0}^{\infty} B_n^{\lambda}(x; k)t^n = \{1 - 2t^k T_k(x) + t^k\}^{-\lambda}/(1 - 2xt + t^2),$$

of [**1**].

Note that in the process of proving Corollary 2.2 we actually proved

COROLLARY 2.3. *The generating relations*

$$\sum_{n=0}^{\infty} s_{nk+l}(x)t^n = \{U_l(x) + tU_{k-l-2}(x)\} \sum_{n=0}^{\infty} R_n(T_k(x))t^n, \tag{2.6}$$

hold for $l = 0, 1, \ldots, k - 1$.

It is easy to show that

COROLLARY 2.4. *Let*

$$\xi_j = \cos(\pi j/k), \quad j = 0, 1, \ldots, k, \tag{2.7}$$

then

$$s_{nk+k-1}(\xi_j) = 0, \quad j = 1, 2, \ldots, k - 1. \tag{2.8}$$

A change of variable in (1.6) gives the following corollary:

COROLLARY 2.5. *The polynomials* $\{s_{nk+k-1}(x)\}$ *satisfy the orthogonality relation*

$$(2.9)\quad \int_{\xi_{j+1}}^{\xi_j} s_{nk+k-1}(x)s_{mk+k-1}(x)\frac{dR(T_k(x))}{U_{k-1}^2(x)} = \Lambda_n\delta_{m,n}.$$

Let the orthogonality relation of $\{s_n(x)\}$ be

$$(2.10)\quad \int_{-1}^{1} s_n(x)s_m(x)d\sigma(x) = \lambda_n\delta_{m,n},\quad \lambda_0 = 1.$$

It is easy to see that

$$(2.11)\quad \lambda_n = \{d_1d_2\ldots d_n\}/\{b_0b_1\ldots b_{n-1}\},\quad n > 0, \lambda_0 = 1,$$

where the b_n's and d_n's are as in (1.16). We now rewrite (2.3) in terms of the orthonormal polynomials. This will be more convenient because the spectral properties of a set of orthogonal polynomials depend on the asymptotic behavior of the orthonormal polynomials. See [**17**], [**8**], and [**13**]. The relationships (1.6) and (2.10) imply

$$(2.12)\quad \lambda_{nk+l} = D_0\Lambda_n/D_n,\quad l < k - 1, \lambda_{nk+k-1} = 2D_0\Lambda_n,$$

and we apply (2.3) to obtain

$$(2.13)\quad \frac{s_{nk+l}(x)}{\sqrt{\lambda_{nk+l}}} = \sqrt{\frac{D_n}{D_0}}U_l(x)\frac{R_n(T_k(x))}{\sqrt{\Lambda_n}} + \sqrt{\frac{B_{n-1}}{D_0}}U_{k-l-2}(x)\frac{R_{n-1}(T_k(x))}{\sqrt{\Lambda_{n-1}}},$$

if $0 \leqq l < k - 1$, and

$$(2.14)\quad \frac{s_{nk+k-1}(x)}{\sqrt{\lambda_{nk+k-1}}} = \frac{U_{k-1}(x)}{\sqrt{2D_0}}\frac{R_n(T_k(x))}{\sqrt{\Lambda_n}}.$$

The following lemma will be very useful.

LEMMA 2.6. *Let* $\{\widetilde{p}_n(x)\}$ *be orthonormal with respect to the positive measure* $d\psi(x)$. *The measure* $d\psi(x)$ *has a discrete mass at* $x = \xi$ *if and only if*

$$\sum_{n=0}^{\infty} |\widetilde{p}_n(\xi)|^2 < \infty,$$

when the corresponding moment problem is determined.

The above lemma is Corollary 2.6, pages 45-46 in [**16**].

THEOREM 2.7. *If* $x = T_k(\xi)$ *is a mass point of* dR *then* ξ *supports a mass of* $d\sigma$.

Proof. The moment problem is determined because the support of both dR and $d\sigma$ lie in $[-1, 1]$. In view of Lemma 2.6 we only need to establish the convergence of

$$\sum_{n=0}^{\infty} s_n^2(x)/\lambda_n.$$

The convergence of the series follows from (2.13), (2.14), Schwartz inequality and the fact that both B_n and D_n lie between zero and one, and the proof is complete.

The following converse to Theorem 2.7 follows trivially from (2.14).

THEOREM 2.8. *Assume that* $x = \xi \neq \xi_j$, $j = 0, 1, \dots, k$, *supports a discrete mass of* $d\sigma$, *then* $T_k(\xi)$ *supports a mass of* dR.

The situation when $x = \xi_j$ is a mass point is covered by the following

THEOREM 2.9. *If* $x = \xi_j$ *supports a discrete mass of* $d\sigma$ *then* $T_k(\xi_j)$ (*which is* ± 1) *does not support a discrete mass* dR.

Proof. Since dR is symmetric it suffices to consider ξ_{2j}, so

$$T_k(\xi_{2j}) = 1.$$

Now (2.3) implies

$$s_{nk+l}(\xi_{2j}) = U_l(\xi_{2j})[R_n(1) - R_{n-1}(1)]. \tag{2.15}$$

The recurrence relation (1.3), when written in the form

$$B_n[R_{n+1}(1) - R_n(1)] = D_n[R_n(1) - R_{n-1}(1)],$$

and then iterated, leads to

$$R_n(1) - R_{n-1}(1) = \frac{D_0 D_1 \dots D_{n-1}}{B_0 B_1 \dots B_{n-1}}, \quad n > 0. \tag{2.16}$$

Now choose l, $l < k - 1$, such that $U_l(\xi_{2j}) \neq 0$. For this l, the identity

$$\left\{\frac{s_{nk+l}(\xi_{2j})}{\sqrt{\lambda_{nk+l}}}\right\}^2 = U_l^2(\xi_{2j})\frac{D_0 \dots D_{n-1}}{B_0 \dots B_{n-1}}$$

follows from (2.12), (2.15) and (2.16). The fact $d\sigma(\xi) > 0$ establishes the convergence of the series

$$\sum_{n=1}^{\infty} \{D_0 \dots D_{n-1}\}/\{B_0 \dots B_{n-1}\},$$

which, when combined with (2.16), shows that

$$\lim_{n\to\infty} R_n(1)$$

exists and is positive. Furthermore, we get

$$\lim_{n\to\infty} \{D_0 \dots D_{n-1}\}/\{B_0 \dots B_{n-1}\} = 0.$$

Finally, the above limit and (1.7) prove that $\Lambda_n \to 0$ as $n \to \infty$ and the series

$$\sum_1^{\infty} R_n^2(1)/\Lambda_n$$

will then diverge, and the proof is complete.

3. The polynomials of the first kind. We again start with a set of random walk polynomials $\{R_n(x)\}$ satisfying (1.3) and (1.4) and an integer $k > 1$. We also assume (1.13). Define the polynomials of the first kind $\{r_n(x)\}$ via

(3.1) $\quad r_0(x) = 1,\ r_1(x) = x,\ a_n r_n(x) = s_n(x) - s_{n-2}(x),\ n > 1,$

where

(3.2) $\quad a_n = 2\{d_0 d_1 \dots d_{n-2}\}/\{b_1 b_2 \dots b_{n-1}\},\ n > 1,\ a_0 = 1,\ a_1 = 2.$

The reason for the above peculiar choice of a_n will become apparent shortly.

THEOREM 3.1. *The polynomials* $\{r_n(x)\}$ *satisfy the recursion*

(3.3) $\quad xr_n(x) = d_{n-1}r_{n+1}(x) + b_{n-1}r_{n-1}(x), \quad n > 0,$

and b_n *and* d_n *are as in* (1.16).

Proof. Clearly (1.15) and (3.1) give for $n > 1$,

$$\begin{aligned} xa_n r_n(x) &= b_n s_{n+1}(x) + d_n s_{n-1}(x) - b_{n-2}s_{n-1}(x) \\ &\quad - d_{n-2}s_{n-3}(x) \\ &= b_n a_{n+1} r_{n+1}(x) + \{b_n + d_n - b_{n-2}\}s_{n-1}(x) \\ &\quad - d_{n-2}s_{n-3}(x), \end{aligned}$$

where $s_{-1}(x)$ is interpreted as 0. Observe that (1.13) and (1.16) guarantee

$$b_n + d_n = 1, \quad n = 0, 1, \dots .$$

Hence, when $n > 1$, we have

$$\begin{aligned} xa_n r_n(x) &= b_n a_{n+1} r_{n+1}(x) + d_{n-2}\{s_{n-1}(x) - s_{n-3}(x)\} \\ &= b_n a_{n+1} r_{n+1}(x) + d_{n-2}a_{n-1}r_{n-1}(x). \end{aligned}$$

The above recurrence relation and (3.2) prove (3.3).

Theorem 3.1, (3.1) and the explicit formulas (2.3) imply

COROLLARY 3.2. *The polynomials of the first kind are explicitly given by*

$$(3.4)\quad \begin{cases} a_{nk+l}(x)r_{nk+l}(x) = 2T_l(x)R_n(T_k(x)) \\ \qquad\qquad - 2T_{k-l}(x)R_{n-1}(T_k(x)),\ l > 0,\ n \geqq 0, \\ a_{nk}r_{nk}(x) = R_n(T_k(x)) - R_{n-2}(T_k(x)),\ n \geqq 0. \end{cases}$$

Our next results provide generating functions for $\{r_n(x)\}$.

THEOREM 3.3. *We have*

$$(3.5)\quad \sum_0^\infty a_n r_n(x)t^n = (1 - t^2)\frac{(1 - 2t^k T_k(x) + t^{2k})}{1 - 2xt + t^2}\sum_{n=0}^\infty t^{nk}R_n(T_k(x))$$

and

$$(3.6)\quad \sum_{n=0}^\infty a_{nk}r_{nk}(x)t^n = (1 - t^2)\sum_{n=0}^\infty t^n R_n(T_k(x)).$$

Proof. From (3.1) we obtain

$$\sum_0^\infty a_n r_n(x)t^n = a_0 + a_1 xt + \sum_2^\infty t^n[s_n(x) - s_{n-2}(x)]$$

$$= (1 - t^2)\sum_{n=0}^\infty s_n(x)t^n.$$

Now (3.5) follows from the above identity and (2.5). The generating function (3.6) immediately follows from the second formula of (3.4). This completes the proof.

4. A characterization theorem. One way of looking at the results of Section 2 is the following. We started with a given set of orthogonal polynomials $\{P_n(x)\}$ (the R_n's in Section 2) and defined polynomials $\{p_n(x)\}$ by

$$(4.1)\quad p_{nk+l}(x) = U_l(x)P_n(T_k(x)) + U_{k-l-2}(x)P_{n-1}(T_k(x)),$$
$$n \geqq 0,\ 0 \leqq l < k,$$

with $P_{-1}(x) = 0$. We then required the polynomials $\{p_n(x)\}$ to be also orthogonal. As we saw in Section 2 this is always possible when the P_n's are random walk polynomials. We now show that this is the only possible case.

THEOREM 4.1. *The polynomials* $\{p_n(x)\}$ *and* $\{P_n(x)\}$ *are orthogonal if and only if* $\{P_n(x)\}$ *is a set of random walk polynomials and* $\delta_0 > 0$.

Proof. We need only to show that it is necessary for the P_n's to be random walk polynomials. Let

$$(4.2) \quad xp_n(x) = \xi_n p_{n+1}(x) + \eta_n p_n(x) + \xi_n p_{n-1}(x),$$

and

$$(4.3) \quad xP_n(x) = B_n P_{n+1}(x) + C_n P_n(x) + D_n P_{n-1}(x)$$

be the three term recurrence relations satisfied by $\{p_n(x)\}$ and $\{P_n(x)\}$. The recursion

$$2xp_{nk+l}(x) = p_{nk+l-1}(x) + p_{nk+l-1}(x), \quad 0 \leqq l < k - 1,$$

follows from (4.1) and (2.4). Thus

$$\xi_n = \zeta_n = \frac{1}{2}, \eta_n = 0 \quad \text{if } k \nmid n + 1.$$

On the other hand (2.2) and (2.4) imply

$$\begin{aligned} xp_{nk+k-1}(x) &= \frac{1}{2}[U_k(x) + U_{k-2}(x)]P_n(T_k(x)) \\ &= [T_k(x) + U_{k-2}(x)]P_n(T_k(x)) \\ &= B_n P_{n+1}(T_k(x)) + [C_n + U_{k-2}(x)]P_n(T_k(x)) \\ &\quad + D_n P_{n-1}(T_k(x)) \\ &= B_n p_{nk+k}(x) + [C_n + (1 - B_n)U_{k-2}(x)] \\ &\quad \times P_n(T_k(x)) + D_n P_{n-1}(T_k(x)), \end{aligned}$$

where we used (4.1) and (4.3). By equating the coefficients of x^{nk+k} in the above relationship we see that ξ_{nk+k-1} of (4.2) is B_n. Thus

$$\begin{aligned} &\eta_{nk+k-1}p_{nk+k-1}(x) + \zeta_{nk+k-1}p_{nk+k-2}(x) \\ &= [C_n + (1 - B_n)U_{k-2}(x)]P_n(T_k(x)) + D_n P_{n-1}(T_k(x)). \end{aligned}$$

This shows that the η_{nk+k-1} vanish since the right side of the above equation is a polynomial of degree $nk + k - 2$. Therefore

$$\begin{aligned} \zeta_{nk+k-1}p_{nk+k-2}(x) = (1 - B_n)&p_{nk+k-2}(x) \\ &+ C_n P_n(T_k(x)) + (B_n + D_n - 1) \\ &\times P_{n-1}(T_k(x)). \end{aligned}$$

Equating coefficients of x^{nk+k-2} we get

$$\zeta_{nk-1} = 1 - B_{n-1}, \quad n > 0.$$

Finally this gives

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$$C_n P_n(T_k(x)) + (B_n + D_n - 1)P_{n-1}(T_k(x)) = 0,$$

and equating coefficients of the highest power of x forces C_n to be zero and $B_n + D_n$ to be 1. One then has to go back and treat the case $n = 0$ separately to see that $D_0 > 0$, so $\delta_0 > 0$.

5. The distribution function of the r_n's. Recall that the polynomials dual to $\{R_n(x)\}$ are defined by (1.17). Let $\{r_n^*(x)\}$ be the numerator polynomials of the r_n's. The r_n^*'s satisfy the initial conditions $r_0^*(x) = 0$, $r_1^*(x) = 1$ and the second order difference equation

$$xr_n^*(x) = d_{n-1}r_{n+1}^*(x) + b_{n-1}r_{n-1}^*(x), \quad n > 0.$$

This, (1.16) and (1.15) identify $\{r_{n+1}^*(x)\}_0^\infty$ as the sieved polynomials of the second kind $\{s_n(x)\}_0^\infty$ associated with the dual random walk polynomials $\{S_n(x)\}$. Let

$$\int_{-1}^{1} r_n(x)r_m(x)d\rho(x) = \rho_n\delta_{m,n}, \quad \rho_0 = 1, \tag{5.1}$$

be the orthogonality relation of $\{r_n(x)\}$. Markov's theorem [**17**, p. 57] establishes

$$\int_{-1}^{1} \frac{d\rho(t)}{x - t} = \lim_{n\to\infty} r_{nk}^*(x)/r_{nk}(x), \quad x \notin [-1, 1]. \tag{5.2}$$

THEOREM 5.1. *The continued fraction* $\chi(x)$ *whose denominators are* $\{r_n(x)\}$ *is given by*

$$\chi(x) = \int_{-1}^{1} \frac{d\rho(t)}{x - t} = \lim_{n\to\infty} \frac{a_{nk}U_{k-1}(x)S_{n-1}(T_k(x))}{R_n(T_k(x)) - R_{n-2}(T_k(x))}, \tag{5.3}$$

where

$$a_{nk} = \{D_0D_1 \ldots D_{n-2}\}/\{B_0B_1 \ldots B_{n-1}\}, \quad n > 1. \tag{5.4}$$

Proof. Combine (2.3), (3.4), (3.2) and (5.3).

We now apply Theorem 5.1 to the case of sieved ultraspherical polynomials. We choose

$$B_n = \frac{1}{2}(n + 1)/(n + \lambda + 1), \quad D_n = 1 - B_n, \tag{5.5}$$

hence, [**15**, p. 279] and (1.11) give

$$R_n(x) = C_n^{\lambda+1}(x), \quad r_n(x) = c_n^\lambda(x; k).$$

Now the S_n's satisfy

$$2(n + \lambda + 1)xS_n(x) = (n + 2\lambda + 1)S_{n+1}(x) + (n + 1)S_{n-1}(x).$$

In order to identify the S_n's we set

$$S_n(x) = (n + 1)!P_{n+1}(x)/(2\lambda + 1)_n,$$

and observe that the P_n's will then satisfy the recursion relation

$$2(n + \lambda)xP_n(x) = (n + 1)P_{n+1}(x) + (2\lambda + n - 1)P_{n-1}(x), \quad n > 0,$$

and the initial conditions $P_0(x) = 0$ and $P_1(x) = 1$. This and (1.3) of page 279 in [**15**] identify the P_n's and S_n's as

$$P_n(x) = \frac{1}{2}C_n^{*\lambda}(x)/\lambda \quad \text{and } S_n(x) = (n + 1)!C_{n+1}^{*\lambda}(x)/(2\lambda)_{n+1}.$$

In the present case we have

$$a_{nk} = \frac{(2\lambda)_n(n + \lambda)}{\lambda(n!)},$$

and

$$R_n(x) - R_{n-2}(x) = C_n^{\lambda+1}(x) - C_{n-2}^{\lambda+1}(x) = \frac{\lambda + n}{\lambda} C_n^{\lambda}(x),$$

[**15**, page 283]. The calculations enable us to reduce (5.3) to

$$(5.6) \quad \int_{-1}^{1} \frac{d\rho(t)}{x - t} = U_{k-1}(x) \lim_{n\to\infty} C_n^{*\lambda}(T_k(x))/C_n^{\lambda}(T_k(x)).$$

Recall that the ultraspherical (Gegenbauer) polynomials are orthogonal on $[-1, 1]$ with respect to $(1 - x^2)^{\lambda-1/2}dx$, hence

$$C\int_{-1}^{1} \frac{(1 - t^2)^{\lambda-1/2}}{x-t} dt = \lim_{n\to\infty} C_n^{*\lambda}(x)/C_n^{\lambda}(x)$$

follows from Markov's theorem, where C is a normalization constant that makes

$$C\int_{-1}^{1} (1 - t^2)^{\lambda-1/2}dt = 1.$$

It is easy to see that

$$C = \Gamma(\lambda + 1)/\left\{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\lambda + \frac{1}{2}\right)\right\}.$$

This and the Perron Stieltjes inversion formula

$$(5.7) \quad F(z) = \int_{-\infty}^{\infty} \frac{d\alpha(t)}{z - t}$$

if and only if

$$\alpha(t_2) - \alpha(t_1) = \lim_{\epsilon \to 0+} \int_{t_1}^{t_2} \frac{F(t - i\epsilon) - F(t + i\epsilon)}{2\pi i} dt,$$

imply

$$d\rho(x) = \frac{\Gamma(\lambda + 1)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\lambda + \frac{1}{2}\right)} |U_{k-1}(x)| \{1 - T_k^2(x)\}^{\lambda - 1/2} dx,$$

$$-1 < x < 1,$$

that is

$$(5.8) \quad \frac{d\rho(x)}{dx} = \frac{\Gamma(\lambda + 1)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\lambda + \frac{1}{2}\right)} (1 - x^2)^{\lambda - 1/2} |U_{k-1}(x)|^{2\lambda},$$

$$-1 < x < 1.$$

Combining (5.1), (5.8) and (1.7) we establish the orthogonality relation

$$(5.9) \quad \int_{-1}^{1} r_n(x) r_m(x) (1 - x^2)^{\lambda - 1/2} |U_{k-1}(x)|^{2\lambda} dx = \lambda_n \delta_{m,n}$$

with

$$(5.10) \quad \lambda_0 = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\lambda + \frac{1}{2}\right)}{\Gamma(\lambda + 1)}, \quad \lambda_1 = 2b_0\lambda_0,$$

$$\lambda_n = \frac{2b_0 b_1 \dots b_{n-1}}{d_0 d_1 \dots d_{n-2}} \lambda_0, \quad n > 1.$$

In this case B_n and D_n are as in (5.5), b_n and d_n are related to B_n and D_n via (1.16). The orthogonality relation (5.9) is mentioned in [**1**]. Note that one can actually evaluate the right side of (5.6) without knowing the weight function of the ultraspherical polynomials. All is needed is to apply Darboux's method, [**14**, Section 8.9] to the generating function

$$\sum_{n=0}^{\infty} C_n^{\lambda}(x) t^n = (1 - 2xt + t^2)^{-\lambda}$$

(or, equivalently use Darboux's formula, [**17**, Section 8.21]) and to the generating function

$$\sum_{n=0}^{\infty} C_n^{*\lambda}(x) t^n = 2\lambda(1 - 2xt + t^2)^{-\lambda} \int_0^t (1 - 2xu + u^2)^{\lambda - 1} du,$$

see e.g. [**5**, Section 3]. The result is

$$(5.11)\quad \int_{-1}^{1} \frac{d\rho(t)}{x-t} = 2\lambda U_{k-1}(x) \int_{0}^{\beta^{k}|} (1 - 2uT_k(x) + u^2)^{\lambda-1} du,$$

where $\beta = x - \sqrt{x^2 - 1}$ and $\sqrt{x^2 - 1}$ is the branch that behaves like x as $x \to \infty$. The relationship (5.11) holds in the complex plane cut along $[-1, 1]$ and the integral on the right side of (5.11) is a Hadamard integral, [**3**, pp. 45-46].

6. Sieved Carlitz-Karlin-McGregor polynomials. We now introduce a sieved analogue of the CKM polynomials. Following the notation in (1.3), (1.4), (1.13), (1.14) and (1.16) we choose

$$(6.1)\quad B_n = (n + 1)/(n + b + 1), \quad D_n = b/(n + b + 1),$$

and denote the corresponding $R_n(x)$ by $R_n^b(x)$, so that

$$(6.2)\quad R_0^b(x) = 1, \quad R_1^b(x) = x(b + 1),$$

$$(6.3)\quad x(b + n + 1)R_n^b(x) = (n + 1)R_{n+1}^b(x) + bR_{n-1}^b(x), \quad n > 0.$$

Let

$$(6.4)\quad R(x, t) = \sum_{0}^{\infty} R_n^b(x)t^n$$

be a generating function of $\{R_n^b(x)\}$. It is straight forward to transform the system (6.2)-(6.3) to the initial value problem

$$R(x, 0) = 1, \quad (1 - xt)\frac{\partial R(x, t)}{\partial t} = [x(b + 1) - bt]R(x, t).$$

Therefore

$$(6.5)\quad \sum_{n=0}^{\infty} R_n^b(x)t^n = R(x, t) = (1 - xt)^{bx^{-2}-b-1} \exp(bt/x).$$

We now apply Darboux's method to the generating function (6.5), [**14**, Section 8.4]. A comparison function is

$$(1 - xt)^{bx^{-2}-b-1} \exp(bx^{-2}).$$

Therefore

$$(6.6)\quad R_n^b(x) \approx \frac{x^n n^{b-bx^{-2}}}{\Gamma(b + 1 - b/x^2)} \exp(b/x^2).$$

Similarly we denote the corresponding dual polynomials $S_n(x)$ by $S_n^b(x)$ and obtain

(6.7) $x(b + n + 1)p_n(x) = (n + 2)p_{n+1}(x) + bp_{n-1}(x), \quad n > 0$

with $p_0(x) = 1, p_1(x) = x(b + 1)/2$, where

(6.8) $p_n(x) = b^n S_n^b(x)/(n + 1)!$.

We again use the generating function

$$p(x, t) = \sum_{n=0}^{\infty} p_n(x)t^n$$

to transform the recurrence relation (6.7) to $p(x, 0) = 0$ and

$$t(1 - xt)\frac{\partial}{\partial t}p(x, t) + [1 + bt^2 - tx(b + 1)]p(x, t) = 1.$$

Solving the above initial value problem we get

(6.9) $$\sum_{n=0}^{\infty} t^n b^n S_n^b(x)/(n + 1)! = t^{-1}e^{bt/x}(1 - xt)^{bx^{-2}-b} \times \int_0^t e^{-bu/x}(1 - xu)^{b-bx^{-2}-1}du,$$

where we used (6.8). We now apply Darboux's method to (6.9). The result is

$$S_n^b(x) \approx \frac{x^{n+1}n^{b-bx^{-2}}(n!)}{b^n\Gamma(b - bx^{-2})\exp(-bx^{-2})} \times \int_0^{\overline{1/x}\rceil} e^{-bu/x}(1 - xu)^{b-1-bx^{-2}}du.$$

The integral on the right side is a Hadamard integral. The asymptotic formula for $\{S_n^b(x)\}$ can be expressed in the form

(6.10) $$S_n^b(x) \approx \frac{x^n n^{b-bx^{-2}}(n!)}{b^n\Gamma(b - bx^{-2})\exp(-bx^{-2})} \times \int_0^{\overline{1}\rceil} e^{-bu/x^2}(1 - u)^{b-1-bx^{-2}}du.$$

In the present case (5.4) becomes

(6.11) $a_{nk} = b^{n-1}(n + b)/n! \approx b^{n-1}/(n - 1)!$.

Now apply (5.3), (6.6), (6.10) and (6.11) to obtain

(6.12) $$\chi(x) = \frac{bU_{k-1}(x)}{T_k(x)}\int_0^{\overline{1}\rceil} \exp[-buT_k^{-2}(x)](1 - u)^{b-1-bT_k^{-2}(x)}du,$$

where $\chi(x)$ is the associated continued fraction

$$(6.13)\quad \chi(x) = \int_{-1}^{1} \frac{d\rho(t)}{x - t};$$

$\rho(x)$ being the distribution function of our sieved polynomials. It is clear that the right side of (6.12) is single valued across the real axis, hence the singularities of the right side of (6.12) are either poles or essential singularities. This and the inversion formula (5.7) show that the measure $d\rho$ of (6.13) is purely discrete. A series representation for $\chi(x)$ is

$$(6.14)\quad \chi(x) = bT_k(x)U_{k-1}(x)\exp[-bT_k^{-2}(x)] \times \sum_{n=0}^{\infty} \frac{b^n[T_k(x)]^{-2n}}{n![(b + n)T_k^2(x) - b]}.$$

Let

$$(6.15)\quad x_{n,j} > 0,\ T_k(x_{n,j}) = \pm\sqrt{b/(b + n)}, \qquad x_{n,1} > x_{n,2} > \dots > x_{n,k},\ n = 0, 1, \dots.$$

Clearly the solutions of

$$(n + b)T_k^2(x) = b$$

are $\pm x_{n,j}$, $j = 1, \dots k$. The series representation (6.14) shows that $\chi(x)$ has simple poles at $x = \pm x_{n,j}$. Recall that

$$(6.16)\quad T_k'(x) = kU_{k-1}(x).$$

The identity (6.16) enables us to express the residue of $\chi(x)$ at $x_{n,j}$ in the form

$$(6.17)\quad \sigma_n(b;\ k) = \operatorname{Res}(\chi(x);\ x_{n,j}) = \frac{b(b + n)^{n-1}}{2k(n!)}\exp(-b - n).$$

Observe that $\sigma_n(b;\ k)$ does not depend on j.

Let us denote the sieved CKM polynomials of the first kind by $\{r_n(x;\ b;\ k)\}$, so that

$$(6.18)\quad r_0(x;\ b;\ k) = 1, \quad r_1(x;\ b;\ k) = x,$$

$$(6.19)\quad \begin{aligned} &2xr_n(x;\ b;\ k) = r_{n+1}(x;\ b;\ k) + r_{n-1}(x;\ b;\ k) \text{ if } k \nmid n,\\ &x(n + b)r_{nk}(x;\ b;\ k) = br_{nk+1}(x;\ b;\ k) + nr_{nk-1}(x;\ b;\ k),\\ &\qquad n > 0.\end{aligned}$$

Now (6.17) gives the orthogonality relation

$$(6.20)\quad \sum_{u=0}^{\infty} \sigma_u(b;\ k)\left\{\sum_{j=1}^{k} r_n(x_{u,j};\ b;\ k)r_m(x_{u,j};\ b;\ k) + r_n(-x_{u,j};\ b;\ k)r_m(-x_{u,j};\ b;\ k)\right\} = \lambda_n\delta_{m,n},$$

with

(6.21) $\lambda_0 = 1, \lambda_1 = b_0, \lambda_n = [b_0 b_1 \dots b_{n-1}]/[d_0 d_1 \dots d_{n-2}], \quad n > 1,$

and

(6.22) $b_n = d_n = \frac{1}{2}$ if $k \nmid n$, $b_{nk-1} = b/(n + b)$, $d_{nk-1} = n/(n + b)$.

We now state generating functions and explicit formulas for $\{r_n(x; b; k)\}$. We first obtain the explicit representation from (6.5)

(6.23) $$R_n^b(x) = \frac{x^{-n}}{n!} {}_2F_0(-n, b + 1 - bx^{-2}; -; x^2),$$

then substitute in (3.4) to obtain an explicit representation for $r_n(x; b; k)$ as a combination of two ${}_2F_0$'s. Theorem 3.3, (6.5), and (6.11) give

(6.24) $$\sum_{n=0}^{\infty} b^{n-1} \frac{(n + b)}{n!} r_{nk}(x; b; k) t^n$$
$$= (1 - t^2)(1 - xt)^{bx^{-2}-b-1} \exp(t/x), \ x = T_k(y)$$

and

(6.25) $$\sum_{n=0}^{\infty} t^n a_n r_n(x; b; k) = (1 - t^2) \frac{(1 - 2t^k T_k(x) + t^{2k})}{1 - 2xt + t^2}$$
$$\times (1 - t^k T_k(x))^{-b-1+b/T_k^2(x)} \exp(t^k / T_k(x)).$$

7. Concluding remarks. A sequence $\{\xi_n\}_{n=1}^{\infty}$ is called a *chain sequence* if there exists a sequence $\{\eta_n\}_{n=0}^{\infty}$ such that

$$\xi_n = (1 - \eta_{n-1})\eta_n, \ n = 1, 2, \dots,$$
$$0 \leqq \eta_0 < 1, \ 0 < \eta_n < 1, \ n > 0.$$

Chihara [7] uses the monic form of the three term recursion,

$$p_0(x) = 1, \ p_1(x) = x - c_1,$$
$$p_{n+1}(x) = (x - c_{n+1})p_n(x) - \lambda_{n+1} p_{n-1}(x), \quad n > 0.$$

The monic form of $xR_n(x) = B_n R_{n+1}(x) + D_n R_{n-1}(x)$ is

$$R_{n+1}(x) = xR_n(x) - D_n(1 - D_{n-1})R_{n-1}(x),$$

so the class of random walk polynomials coincides with the class of symmetric orthogonal polynomials (i.e., $c_n = 0, n > 0$) when $\{\lambda_{n+1}\}_{n=1}^{\infty}$ is a chain sequence. For additional properties of this class of orthogonal polynomials we refer the interested reader to [7]. Properties of the corresponding continued factions are in [**18**].

Finally one word about the characterization theorem of Section 4. We are saying that the type of explicit formula (2.3) holds only for sieved random walk polynomials of the second kind if $R_n(x)$ is required to be orthogonal. We are not saying that this is the end of easy explicit formulas. As a matter of fact the symmetric sieved Pollaczek polynomials [**9**] satisfy (2.3) but the $\{R_n(x)\}$ are no longer orthogonal.

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The National University of Colombia,
Bogota, Colombia;
Arizona State University,
Tempe, Arizona