INVERSE SEMIGROUPS WHOSE FULL INVERSE SUBSEMIGROUPS FORM A CHAIN

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The structure of semigroups whose subsemigroups form a chain under inclusion was determined by Tamura [**9**]. If we consider the analogous problem for inverse semigroups it is immediate that (since idempotents are singleton inverse subsemigroups) any inverse semigroup whose inverse subsemigroups form a chain is a group. We will therefore, continuing the approach of [**5, 6**], consider inverse semigroups whose *full* inverse subsemigroups form a chain: we call these *inverse* ∇*-semigroups.*

In §1 we show that the non-trivial $\mathcal{J}$-classes of an inverse ∇-semigroup form a chain, the associated principal factors being either cyclic or quasi-cyclic p-groups with zero (p a prime) or isomorphic to B_5, the five-element combinatorial Brandt semigroup. Inverse ∇-semigroups are then characterized by these properties together with (C): for any non-idempotents x and y with $J_x < J_y$, $x = xx^{-1}y^n$ for some non-zero integer n.

In §2 the property (C) is used to further elucidate the properties of inverse ∇-semigroups. It is shown, for instance, that each element of such a semigroup S has index at most 2. If S has an infinite subgroup G, then G contains all the non-idempotents of S; if S has a non-trivial subgroup G of prime-power, but not prime, order, then $x^2 \in G$ for every non-idempotent x of S.

Meakin [8] described a very special class of inverse ∇-semigroups: those with no proper inverse subsemigroups whatsoever. In the present paper the characterization given by the author in [**6**] of those inverse semigroups whose lattice of full inverse subsemigroups is distributive is clearly relevant.

1. A characterization. We begin with some terminology and notation, and a summary of the results from [**5**] which we will use. For background on lattices of full inverse subsemigroups the reader is referred to [**5, 6**].

Denote by $\mathcal{LF}(S)$, or just $\mathcal{LF}$, the lattice of full inverse subsemigroups of the inverse semigroup S (that is, the lattice of those inverse subsemigroups of S containing the semilattice E of idempotents of S). If $A \subseteq S$, denote by $\langle A\rangle$ the inverse subsemigroup of S generated by A, and by $\langle E, A\rangle$ the *full* inverse subsemigroup generated by A, that is, $\langle E \cup A\rangle$. In general we use the notation of [**4**].

Result 1.1 ([**5**, Corollary 1.2, Proposition 1.3]). *For each $\mathcal{J}$-class J of S the relation γ_J on $\mathcal{LF}$, defined by $A\gamma_J B$ if $A \cap J = B \cap J$, is a congruence. In the lattice of congruences on $\mathcal{LF}$, $\bigwedge\{\gamma_J : J \in S/\mathcal{J}\} = 0$. Hence $\mathcal{LF}$ is a subdirect product of the lattices $\mathcal{LF}/\gamma_J$, $J \in S/\mathcal{J}$. Moreover $\mathcal{LF}/\gamma_J \cong \mathcal{LF}(\mathrm{PF}(J))$, where $\mathrm{PF}(J)$ is the principal factor associated with J, under the isomorphism $A\gamma_J \to (A \cap J) \cup \{0\}$.*

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COROLLARY 1.2. *The lattice $\mathcal{LF}$ is distributive if and only if $\mathcal{LF}$(PF(J)) is distributive for each $\mathcal{J}$-class J.*

RESULT 1.3 ([**5,** Corollary 3.6]). *If S is a completely 0-simple inverse semigroup with $\mathcal{LF}$ distributive then S is either a group with zero or is isomorphic to B_5, the combinatorial Brandt semigroup with five elements.*

(Note: as in [**5**] we define $\mathrm{PF}(J) = J \cup \{0\}$, the product of two elements of J being their product in S if it lies in J, and all other products being zero. Thus PF(J) is always 0-simple).

Finally, we require a characterization of ∇-groups, (that is groups whose subgroups form a chain under inclusion).

RESULT 1.4 ([**10,** Theorem 5]). *A group is a ∇-group if and only if it is a cyclic or quasi-cyclic p-group, for some prime p.*

(The notation $Z(p^\infty)$ is often used for quasi-cyclic p-groups. The reader is referred to [**3**] for their properties).

Since $\mathcal{LF}(B_5)$ is a two-element chain (by Theorem 3.2 of [**5**]), it is immediate from Result 1.3 that the completely 0-simple ∇-semigroups are just B_5 and the ∇-groups with zero adjoined. We now show that these are the *only* 0-simple inverse ∇-semigroups; thus by Result 1.1, every inverse ∇-semigroup is *completely semisimple.*

Suppose S is an inverse ∇-semigroup which is 0-simple but not completely 0-simple. Clearly $\mathcal{LF}(S)$ is distributive. It was shown in [**6**] that such a semigroup is in fact a simple semigroup $S^*(=S\backslash 0)$ with zero adjoined and that $\mathcal{LF}(S^*) \cong \mathcal{LF}(S)$. Further S^* is E-unitary (that is $ex = e$, $e \in E$, implies that $x \in E$) and if σ denotes the least group congruence on S^* then the morphism $\sigma^\natural$ of S^* upon S^*/σ induces a lattice morphism of $\mathcal{LF}(S^*)$ upon $\mathcal{LF}(S^*/\sigma)$, the lattice of subgroups of S^*/σ. Since S is a ∇-semigroup, S^*/σ is a ∇-group, whence, by Result 1.4, a p-group for some prime p. But this is impossible, for (since S is not completely 0-simple) S^* contains an element of infinite order ([**1,** Theorem 2.54]), whose image in S^*/σ again has infinite order.

We have proved the necessity of the property (B) in the following characterization of inverse ∇-semigroups.

THEOREM 1.5. *An inverse semigroup S is a ∇-semigroup if and only if*

(A) *the non-trivial $\mathcal{J}$-classes of S form a chain,*

(B) *each non-trivial $\mathcal{J}$-class is either a cyclic or quasi-cyclic p-group for some prime p, or has principal factor isomorphic to B_5,*

(C) *for each $x, y \in S\backslash E_S$ with $J_x < J_y$ there is a non-zero integer n such that $x = xx^{-1}y^n$.*

Proof. Suppose S is a ∇-semigroup, and put $E = E_S$. Let $x, y \in S\backslash E$. Either $\langle E, x\rangle \subseteq \langle E, y\rangle$ so that $x \in \langle E, y\rangle$ and $J_x \leq J_y$, or $\langle E, y\rangle \subseteq \langle E, x\rangle$ so that $y \in \langle E, x\rangle$ and $J_y \leq J_x$. This proves (A); (B) has already been shown.

Again let $x, y \in S\backslash E$, with $J_x < J_y$. Clearly $y \notin \langle E, x\rangle$, so $x \in \langle E, y\rangle$. By expressing x as a product involving E and y and permuting idempotents if necessary (c.f. Lemma 2.1 of

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[**6**]), we may write $x = ey^n$ for some $e \in E$ and non-zero integer n. Then $xx^{-1} \leqslant e$, so

$$x = ey^n = xx^{-1}ey^n = xx^{-1}y^n,$$

proving (C).

Conversely, suppose S is an inverse semigroup satisfying (A), (B) and (C), and let A, $B \in \mathcal{LF}(S)$, where $A \neq E$, $B \neq E$. Suppose $A \nsubseteq B$ and let $a \in A \backslash B$. Put $J = J_a$. From (B) and the comments following Result 1.4, PF(J) is an inverse ∇-semigroup. Now from Result 1.1 the map $C \to (C \cap J) \cup \{0\}$ is a lattice morphism of $\mathcal{LF}(S)$ upon $\mathcal{LF}(\mathrm{PF}(J))$. Hence since $a \in (A \cap J) \backslash (B \cap J)$, so that

$$(A \cap J) \cup \{0\} \nsubseteq (B \cap J) \cup \{0\},$$

we have

$$(B \cap J) \cup \{0\} \subseteq (A \cap J) \cup \{0\},$$

that is, $B \cap J \subseteq A \cap J$.

On the other hand if $b \in B \backslash E$ and $b \notin J$, then by (A), either $J_b > J$ or $J_b < J$. But if $J_b > J$ then, using (C), $a = (aa^{-1})b^n \in B$, for some $n \neq 0$, a contradiction. Thus $J_b < J$ and, using (C) again, $b = (bb^{-1})a^n \in A$, for some $n \neq 0$. Therefore $B \subseteq A$.

Hence S is a ∇-semigroup.

2. Some consequences. Throughout this section S will be an inverse ∇-semigroup, with $E = E_S$. The properties (A), (B) and (C) will be those in Theorem 1.5.

We consider first the restrictions that (B) places on a non-idempotent x of S. Clearly if x belongs to a subgroup, that is, x has index 1 (in the terminology of [**4**, §1.2]), it has prime-power period. Suppose now that x does not belong to a subgroup. Then J_x has precisely 4 elements: $J_x = \{x, x^{-1}, xx^{-1}, x^{-1}x\}$. Consider the *monogenic inverse subsemigroup* $\langle x \rangle$ of S: since S is completely semisimple so is $\langle x \rangle$, and, further each non-group $\mathcal{J}$-class of $\langle x \rangle$ has at most four elements; from the description of all monogenic inverse semigroups (given in, for example, [**2**]) it is apparent that x has index 2, that is, x^2 lies in the *kernel* K_x of $\langle x \rangle$. From (B), again, K_x is a (cyclic) group of order p^k, for some prime p and some $k \geqslant 0$. (If $k = 0$, $K_x = \{x^2\}$). Thus

$$x^2 = x^{2+p^k},$$

that is, x has period p^k. The identity f of K_x is x^{p^k} if $k \geqslant 1$, or x^2 if $k = 0$.

Let J be the $\mathcal{J}$-class of S containing K_x. Let $z \in S$ be such that $z\mathcal{R}f$. Then, by (C), there is a non-zero integer n such that $z = fx^n$. Thus $z \in K_x$. Thus J is a group and in fact $J = K_x$. We have thus established

PROPOSITION 2.1. *In an inverse ∇-semigroup S every element x which is not in a subgroup of S has index 2 and period p^k for some prime p and some $k \geqslant 0$. The kernel K_x of $\langle x \rangle$ is an entire group $\mathcal{J}$-class of S, with identity x^{p^k} if $k \geqslant 1$, or x^2 if $k = 0$.*

It is easily verified that any monogenic inverse semigroup generated by an element of index 2 and period p^k, $k \geqslant 0$, satisfies (A), (B) and (C) and is therefore a ∇-semigroup.

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We now show that the property (C) imposes major restrictions on the permissible combinations of non-trivial $\mathcal{J}$-classes of S. First, however, a technical lemma, whose proof is routine, is needed.

LEMMA 2.2. *Let T be any inverse semigroup, G a group $\mathcal{J}$-class of T, with identity e, and U an inverse subsemigroup of T such that $e \leq f$ for all $f \in E(U)$. Then the map $u \mapsto eu$ is a morphism of U into G.*

Now let G and H be non-trivial group $\mathcal{J}$-classes of S with identities e and f, respectively, such that $e < f$ (so $G < H$ as $\mathcal{J}$-classes). From (C) it follows that the morphism $\phi_{f,e}: u \mapsto eu$ of H into G, defined in the lemma, is surjective. In fact $K\phi_{f,e} = G$ for *every non-trivial subgroup K of H*. In particular this implies $\ker \phi_{f,e} = \{f\}$, so $\phi_{f,e}$ is a bijection.

Furthermore H, being a cyclic or quasi-cyclic p-group, certainly contains a subgroup of order p. Thus $|G| = |H| = p$. Applying (A) we therefore have

PROPOSITION 2.3. *If S is an inverse ∇-semigroup with more than one non-trivial maximal subgroup then there is a prime p such that every non-trivial subgroup of S has order precisely p.*

Now let J be a non-trivial $\mathcal{J}$-class of S containing an element x of index 2, and let G be a non-trivial group $\mathcal{J}$-class of S, with identity e, such that $G < J$. The case $K_x = G$ was covered in Proposition 2.1. By Exercise 3, §8.4 of [**1**], $e < xx^{-1}$, $e < x^{-1}x$ and so $e \leq (xx^{-1})(x^{-1}x) = f$, the identity of K_x. Hence $K_x \geq G$ and there is a morphism of $\langle x \rangle$ upon G (using (C)), as defined in Lemma 2.2, whose restriction to K_x is the identity if $K_x = G$, and is the bijection $\phi_{f,e}$ defined above if $K_x > G$. (Note that the $\mathcal{J}$-class K_x cannot be trivial, for if so, we have $x^2 = x^3$, from which, using (C), it follows that $z^2 = z^3$ for any $z \in G \backslash \{e\}$, a contradiction). Summing up, we have

PROPOSITION 2.4. *Let x be an element of S of index 2 and let G be a group $\mathcal{J}$-class of S, with identity e, such that $G < J_x$. Then $G \leq K_x$ and the map $u \mapsto eu$ $(u \in \langle x \rangle)$ is a morphism of $\langle x \rangle$ upon G whose restriction to K_x is a bijection upon G. Thus if K_x is trivial so is every group $\mathcal{J}$-class $G < J_x$.*

When the order of a group and a non-group $\mathcal{J}$-class is reversed the situation is rather different.

PROPOSITION 2.5. *Let x be an element of S of index 2 and let G be a non-trivial group $\mathcal{J}$-class of S, with identity e, such that $G > J_x$. Then $|G| = 2$ and $x = xx^{-1}z$, where z is the involution of G. In that case x has period at most 2.*

Proof. Since $J_x < G$, we have $x = (xx^{-1})z^n$ for some $n \neq 0$, for any $z \in G \backslash \{e\}$. Thus $xx^{-1} < e$ and for any such z,

$$xx^{-1}z \mathcal{R} xx^{-1}e = xx^{-1}.$$

But $R_{xx^{-1}} = \{xx^{-1}, x\}$ and if $xx^{-1}z = xx^{-1}$ then $xx^{-1}z^n = xx^{-1}$ for every non-zero integer n, contradicting (C). Hence $xx^{-1}z = x$ for every z in $G \backslash \{e\}$.

Suppose some element z of G has order $l > 2$. Then

$$x = xx^{-1}z = xx^{-1}z^2,$$

so $xz = x$ and $xz^n = x$ for all $n \geq 1$. But then

$$x = xz^{l-1} = (xx^{-1}z)z^{l-1} = xx^{-1}z^l = xx^{-1}e = xx^{-1},$$

a contradiction. So every non-identity element of G has order 2. Since G is cyclic or quasi-cyclic, $|G| = 2$.

The last statement is an application of Propositions 2.3 and 2.1.

An interesting application of these results is the following.

COROLLARY 2.6. *If an inverse* ∇*-semigroup* S *contains a quasi-cyclic maximal subgroup* G*, then* G *constitutes the only non-trivial* $\mathcal{J}$*-class of* S*.*

If S *contains a maximal subgroup* G *of prime-power, but not prime order then* G *is the only non-trivial maximal subgroup of* S *and* $G = K_x$ *for every element* x *of* S *of index* 2.

Proof. First let G be any non-trivial maximal subgroup of S not of prime order. By Proposition 2.3, G is the only non-trivial maximal subgroup of S. Suppose $x \in S$ has index 2. By Proposition 2.5, $G \not> J_x$. Thus $G < J_x$, so that by Proposition 2.4, $G \leq K_x$ and $|G| = |K_x|$. Since G is non-trivial, so is K_x. By Proposition 2.1, K_x is an entire $\mathcal{J}$-class of S and hence $G = K_x$. Again by Proposition 2.1, K_x is finite so G cannot be quasi-cyclic, proving the first statement.

We consider, finally, the relationship between two non-group $\mathcal{J}$-classes $J < J'$ of S.

Let $x \in J \backslash E$ and $y \in J' \backslash E$. From Proposition 2.1 we have that K_y is a group $\mathcal{J}$-class of S, so $K_y \not\leq J$. Consider the case $K_y > J$ and suppose K_y is trivial, that is, $K_y = \{y^2\}$. From Exercise 3, §8.4 of [**1**] again, either $y^2 > xx^{-1}$ or $y^2 > x^{-1}x$. In the former case

$$xx^{-1}y^n = xx^{-1}(y^2y^n) = xx^{-1}$$

for any non-zero integer n, contradicting (C). Since the latter case similarly contradicts the obvious dual of (C), K_y is therefore non-trivial. In fact, by Proposition 2.5, $|K_y| = 2$ and $x = xx^{-1}y^3$ (since y^3 is the involution in K_y), so that $xx^{-1} < y^2$ and

$$x = (xx^{-1}y^2)y = xx^{-1}y.$$

Suppose next that $K_y < J$. Then by Proposition 2.4, $K_y \leq K_x$. From (C),

$$x = xx^{-1}y^{\pm 1}$$

(since $J_{y^n} < J_x$ for $|n| \geq 2$). When $x = xx^{-1}y$, we obtain

$$x^2 = (xx^{-1}yxx^{-1}y^{-1})y^2,$$

so $K_y = K_x$. We obtain a similar result when $x = xx^{-1}y^{-1}$.

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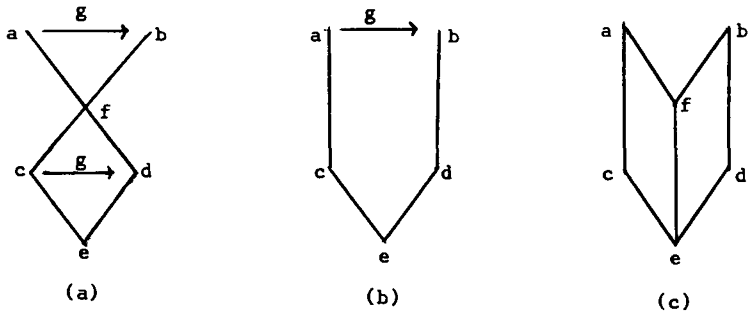

Figure 1

Finally suppose K_y and J are incomparable. Then, by (A), we have that K_y is trivial, so that, by Proposition 2.4, K_x is also. Again we have $x = xx^{-1}y^{\pm 1}$. Summarizing,

PROPOSITION 2.7. *If x and y in S have index 2 and $J_x < J_y$ then either*
(i) *$K_y > J_x$, in which case $|K_y| = 2$ and $|K_x| \leq 2$, or*
(ii) *$K_y < J_x$, in which case $K_y = K_x$, or*
(iii) *K_y and J_x are incomparable, in which case K_x and K_y are trivial.*
In each case $x = xx^{-1}y^{\pm 1}$.

Before continuing, we provide examples to show that each of these cases may occur.

First, let E be the semilattice in Fig. 1(a), and let G be a group of order 2, with g its involution and 1 its identity. Let G act on E on the left by order automorphisms so that g acts by "reflection". Let U be the semidirect product of E and G: that is $U = E \times G$, with product

$$(x, h)(x', h') = (x \wedge hx', hh').$$

(In the terminology of [7], $U = P(G, E, E)$). Then U has 2 non-trivial group $\mathcal{J}$-classes, $J_{(f,1)}$ and $J_{(e,1)}$, of order 2, and two non-group J-classes, $J_{(a,g)}$ and $J_{(c,g)}$, each with principal factor isomorphic with B_5. Clearly (A) and (B) are satisfied and (C) is easily verified. Here $K_{(a,g)} > J_{(c,g)}$ and $|K_{(a,g)}| = |K_{(c,g)}| = 2$. By taking the Rees quotient modulo $J_{(e,g)}$, we obtain a similar example with $|K_{(a,g)}| = 2$ and $|K_{(c,g)}| = 1$.

Now let E be the semilattice in Fig. 1(b), let G be as above, g again acting by "reflection", and form the semidirect product V of E and G. In this case

$$K_{(a,g)} = K_{(c,g)} = J_{(e,1)}.$$

(Here $|K_{(a,g)}| = 2$ but examples may be similarly constructed where $K_{(a,\,g)}$ has arbitrary prime-power order).

Finally let E be the semilattice in Fig. 1(c) and let W be the full inverse subsemigroup of T_E (see [4, Chapter V]) generated by the isomorphism y taking aE to bE and fixing f. Then W is an inverse ∇-semigroup. If x is the isomorphism taking cE to dE then it is easily verified that, if we take $S = W$, then S has the properties described in Proposition 2.7.(iii).

We now continue the theme of Corollary 2.6.

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COROLLARY 2.8. *If an inverse* ∇*-semigroup S contains a maximal subgroup of prime order* $p \neq 2$, *then every non-trivial subgroup has order p. Each element of S of index 2 has the same kernel, K say, and the same period p. Further* $G \leq K$ *for each non-trivial subgroup G of S.*

Proof. That every non-trivial subgroup has order p follows from Proposition 2.3. Let G be such a subgroup and let x be any element of S of index 2. By Proposition 2.5, $G \not> J_x$. Hence $G < J_x$, so by Proposition 2.4 we have that $G \leq K_x$ and $|G| = |K_x|$. So x has period p.

If y is another element of index 2 then $|K_y| = p$ also and we have $K_y \leq K_x \leq K_y$, that is, $K_x = K_y = K$, say, and $G \leq K$ for every non-trivial subgroup G of S.

When S has a maximal subgroup of order 2 then we can similarly show that every element of index 2 has period *at most* 2. However the examples above show that not every such element need have period 2, and the kernels may be disjoint.

REFERENCES

1. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Surveys of the Amer. Math. Soc. 7 (Providence, R.I., 1961 (Vol. I) and 1967 (vol. II)).

2. C. Eberhart, Elementary orthodox semigroups, Proc. Symposium on Inverse Semigroups and their Generalizations, Northern Illinois University, 1973, 23–33.

3. L. Fuchs, *Abelian groups* (Pergamon Press, 1960).

4. J. M. Howie, *An introduction to semigroup theory*, (Academic Press, 1976).

5. P. R. Jones, Semimodular inverse semigroups, *J. London Math. Soc.* **17** (1978), 446–456.

6. P. R. Jones, Distributive inverse semigroups, *J. London Math. Soc.* **17** (1978), 457–466.

7, D. B. McAlister, Groups, semilattices and inverse semigroups II, *Trans. Amer. Math. Soc.* **196** (1974), 351–370.

8. J. Meakin, One-sided congruences on inverse semigroups, *Trans. Amer. Math. Soc.* **206** (1975), 67–82.

9. T. Tamura, On a monoid whose submonoids form a chain, *J. Gakugei Tokushima Univ.* **5** (1954), 8–16.

10. T. Tamura, Commutative semigroups whose lattice of congruences is a chain. *Bull. Soc. Math. France* **97** (1969), 369–380.

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