

Indecomposable representations of certain commutative quivers

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This paper gives a detailed classification of the indecomposable representations of one non-trivial "commutative quiver", gives brief details of certain generalisations, and uses methods likely to be more widely applicable.

Throughout this paper K denotes a field whilst all modules are left modules of finite dimension over K . The indecomposable modules are classified for the two K -algebras whose presentations follow. In both cases $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6\}$ denotes a complete set of primitive orthogonal idempotents whilst a generator ψ_{ij} is understood to satisfy the relationship $\psi_{ij} = \epsilon_i \psi_{ij} \epsilon_j$:

$$R_1 = K\langle \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \gamma_{51}, \gamma_{52}, \gamma_{53}, \gamma_{54}, \delta_{16}, \delta_{26}, \delta_{36}, \delta_{46} \mid \gamma_{51}\delta_{16} = \gamma_{52}\delta_{26}, \gamma_{53}\delta_{36} = \gamma_{54}\delta_{46} \rangle,$$

$$R_2 = K\langle \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \gamma_{15}, \gamma_{25}, \gamma_{35}, \gamma_{45}, \alpha_{56}, \beta_{56} \mid \gamma_{15}\alpha_{56} = \gamma_{25}\alpha_{56} = \gamma_{35}\beta_{56} = \gamma_{45}\beta_{56} = 0 \rangle.$$

These algebras are best understood as quiver algebras in the sense of [3]. For example, a R_1 -module M is just a sextuple of vector spaces $M_1 (= \epsilon_1 M), \dots, M_6 (= \epsilon_6 M)$ together with the eight linear mappings $\gamma_{5i} : M_i \rightarrow M_5, \delta_{i6} : M_6 \rightarrow M_i$ satisfying the relations shown above. The

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significance of these algebras lies in the fact that their representation theory is quite complex but still tame in the sense of [3]. This will be proved in this paper. It will be noted that in the above interpretation of R_1 -modules the special cases

$$(1) \quad M_5 = 0 \quad \text{and}$$

$$(2) \quad M_6 = 0$$

correspond to the non-trivial '4-subspace problem' solved in [2], [5], and elsewhere. Likewise the classification of R_2 -modules with

$$(3) \quad M_6 = 0 \quad \text{and}$$

$$(4) \quad M_1 = M_2 = M_3 = M_4 = 0$$

correspond to different 'extended Dynkin diagram' problems, to wit the 4-subspace problem and the 'Kronecker' problem. In Section 7 the class of problems of this type accessible to the methods of this paper is discussed.

Section 1 contains a discussion of two functors, the first of which establishes a virtual equivalence between the above two classification problems and the second of which simplifies the latter problem. Section 2 briefly describes the matrix regress method of Nazarova and Roiter which is used in Sections 4 and 5. Section 3 gives the classification of indecomposable 4-subspace systems. Section 4 gives a direct determination of the indecomposable R_2 -modules whilst Section 5 gives an alternative approach more suitable for the construction of canonical forms. The quadratic form phenomenon of [3] is discussed in Section 6.

1. The functors used in the classification

The contents of this section are special cases of work of Auslander, Platzeck, Reiten [1], which has recently been extended by Butler. The functors discussed below both have evident adjoints which may be used to give direct proofs of the properties mentioned.

The functor $\Psi : \text{mod}(R_1) \rightarrow \text{mod}(R_2)$ is defined as follows, using the notation of the introduction. If $M \in \text{mod}(R_1)$, $(\Psi M)_i = M_i$ for $i \neq 5$,

$(\Psi M)_5 = \ker(M_1 \perp\!\!\!\perp M_2 \perp\!\!\!\perp M_3 \perp\!\!\!\perp M_4) \rightarrow M_5$; the new γ mappings are the projections and $\alpha_{56}(m_6) = (0, 0, \delta_{36}m_6, -\delta_{46}m_6)$, $\beta_{56}(m_6) = (\delta_{16}m_6, \delta_{26}m_6, 0, 0)$ for $m_6 \in M_6$. It establishes a bijection between the full subcategory of $\text{mod}(R_1)$ whose objects are the R_1 -modules without direct summands isomorphic to the projective module $R_1\varepsilon_5$, and a full subcategory of $\text{mod}(R_2)$.

The functor $\Phi : \text{mod}(R_2) \rightarrow \text{mod}(R_2)$ takes M to M' where $M'_i = M_i$ for $i = 1, 2, 3, 4$, $M'_6 = M_5$, $M'_5 = \text{coker}(M_6 \rightarrow (M_5 \perp\!\!\!\perp M_5))$; $\alpha'_{56}m = \text{class}(0, m)$, $\beta'_{56}m = \text{class}(m, 0)$, $\gamma_{i5} \text{class}(m, m') = \gamma_{i5}m$ ($i = 1, 2$), $= \gamma_{i5}m'$ ($i = 3, 4$). The functor establishes an equivalence between the full subcategory of $\text{mod}(R_2)$ whose objects are the R_2 -modules without direct summands isomorphic to the injective module $R_2\varepsilon_6/(\text{rad } R_2)\varepsilon_6$, and the full subcategory of $\text{mod}(R_2)$ whose objects have no simple quotients isomorphic to $R_2\varepsilon_5/(\text{rad } R_2)\varepsilon_5$. We need, therefore, only classify the indecomposable R_2 -modules in the image of Φ to obtain a classification of indecomposable modules in $\text{mod}(R_1)$ and $\text{mod}(R_2)$.

2. Matrix regression

The Nazarova–Roiter regression method of [6] is briefly described here, and will be used later.

A *regression scheme* T consists of two finite linearly ordered sets $R = \{r_1, \dots, r_m\}$, $C = \{c_1, \dots, c_n\}$, and a symmetric relation ρ on their disjoint union which is such that for any $x \in R \perp\!\!\!\perp C$ there is at most one y related to x .

A T -matrix is a block matrix whose block rows are parametrised in order by the set R' obtained from R by replacing each r_i related to itself by two adjacent elements. The block columns are parametrised in order by the analogous set C' . If $\rho(r_i, r_j) = \rho(c_i, c_j)$, $i \neq j$, then the corresponding block rows (columns) have the same number of rows

(columns) and if $\rho(r_i, c_j)$ the block row has the same number of rows as the number of columns in the block column. Block rows or columns may have zero width.

Two T -matrices are said to be isomorphic if one can be transformed into the other by a sequence of operations of the following types.

1. A block row may be added to any subsequent block row, except that the two block rows corresponding to r_i related to itself may not be added to each other.

2. A block row may be multiplied on the left by any invertible matrix, subject to

(a) if $\rho(r_i, r_j)$, $i \neq j$, the corresponding block rows may be simultaneously multiplied by the same matrix,

(b) if $\rho(r_i, c_j)$, multiplication of the corresponding row on the left by an invertible matrix entails multiplication of the corresponding column on the right by its inverse.

3. The analogues of 1, 2 (a) for columns.

The direct sum of two T -matrices is defined in the usual way, and the Krull-Schmidt Theorem applies. Nazarova and Roiter [6] have shown that the indecomposable T -matrices are isomorphic either to a discrete canonical T -matrix, parametrized by combinatorial invariants independent of the field K , or to continuous canonical T -matrices parametrized by combinatorial invariants and a monic irreducible polynomial. The proof is a complicated induction for all regression schemes simultaneously.

3. The four-subspace classification

Reference [5] is not widely available and it deals only with the case when the field K is algebraically closed. It may therefore be convenient to use Brenner's note [2], which contains the general case including the extension to skew fields, and also gives other references. Indeed, the entire contents of this paper may be extended to the skew field case with only a few small formal changes.

The symbol λ refers to an element of K in the case when the field

is algebraically closed. This is equivalent to a linear monic polynomial: in general the symbol refers to an arbitrary monic irreducible polynomial.

The classification of 4-subspace systems obtained by Gelfand and Ponomarev in [5] will be needed in the next sections. The construction of indecomposable R_2 -modules uses the natural definition of a 4-subspace system as a quintuple of vector spaces V_1, \dots, V_5 with linear maps $\gamma_i : V_i \rightarrow V_5$. At the same time it is convenient to think of the submodule of an R_2 module M obtained by ignoring $M_6, \alpha_{56}, \beta_{56}$ as a 4-subspace system $\ker \gamma_i \rightarrow M_5$. If M has no direct summands isomorphic to $R_2 \varepsilon_i$, $i = 1, \dots, 4$, the structure of this 4-subspace submodule is completely determined by the system $\ker \gamma_i \rightarrow M_5$.

On pages 22, 23 there is a complete list of the isomorphism types of indecomposable 4-subspace systems that will be needed. The *dimension* of a system is the quintuple $(\dim V_1, \dots, \dim V_5)$.

The dimension of a non-trivial intersection is always 1. With the exception of $S(2n, 0, \lambda)$, in each case the system is characterised by its dimension and non-trivial intersections. Gelfand and Ponomarev [5] give bases for the spaces that can be used to calculate the homomorphisms of any system into any other, as also does Brenner [2]. Some information about homomorphisms is needed in Section 5, and is recorded there.

4. The representation type of R_2

It is logically redundant to discuss the representation type of R_2 , given that Section 5 describes the construction of the indecomposable R_2 -modules in the image of the functor Φ . Nevertheless, a discussion is given here, on the grounds that the method of obtaining regression schemes to classify R_2 -modules directly may be of some interest.

Let N be any positive integer, and M an R_2 -module whose 4-subspace submodule has indecomposable summands whose components in M_5 have dimension at most $2N$. Any direct summand of the 4-subspace

| Notation | Dimension | Non-trivial intersections of images |
|--|----------------------------|-------------------------------------|
| $S_3(2n+1, -1)$ | $(n, n, n+1, n, 2n+1)$ | - |
| $S_4(2n+1, -1)$ | $(n, n, n, n+1, 2n+1)$ | - |
| $S_1(2n+2, -1)$ | $(n, n+1, n+1, n+1, 2n+2)$ | - |
| $S_2(2n+2, -1)$ | $(n+1, n, n+1, n+1, 2n+2)$ | - |
| $S(2n, 0, \lambda), \lambda \neq 0, 1$ | $(n, n, n, n, 2n)$ | - |
| $S_{12}(2n, 0)$ | $(n, n, n, n, 2n)$ | (V_1, V_2) |
| $S_{13}(2n, 0)$ | $(n, n, n, n, 2n)$ | (V_1, V_3) |
| $S_{14}(2n, 0)$ | $(n, n, n, n, 2n)$ | (V_1, V_4) |
| $S_{23}(2n, 0)$ | $(n, n, n, n, 2n)$ | (V_2, V_3) |
| $S_{24}(2n, 0)$ | $(n, n, n, n, 2n)$ | (V_2, V_4) |
| $S_{34}(2n, 0)$ | $(n, n, n, n, 2n)$ | (V_3, V_4) |
| $S_{12}(2n+1, 0)$ | $(n+1, n+1, n, n, 2n+1)$ | (V_1, V_2) |

| | | |
|-------------------|------------------------------|-----------------------------------|
| $S_{13}(2n+1, 0)$ | $(n+1, n, n+1, n, 2n+1)$ | (V_1, V_3) |
| $S_{14}(2n+1, 0)$ | $(n+1, n, n, n+1, 2n+1)$ | (V_1, V_4) |
| $S_{23}(2n+1, 0)$ | $(n, n+1, n+1, n, 2n+1)$ | (V_2, V_3) |
| $S_{24}(2n+1, 0)$ | $(n, n+1, n, n+1, 2n+1)$ | (V_2, V_4) |
| $S_{34}(2n+1, 0)$ | $(n, n, n+1, n+1, 2n+1)$ | (V_3, V_4) |
| $S_1(2n, 1)$ | $(n+1, n, n, n, 2n)$ | $(V_1, V_i), i = 2, 3, 4 (n > 0)$ |
| $S_2(2n, 1)$ | $(n, n+1, n, n, 2n)$ | $(V_2, V_i), i = 1, 3, 4 (n > 0)$ |
| $S_3(2n, 1)$ | $(n, n, n+1, n, 2n)$ | $(V_3, V_i), i = 1, 2, 4 (n > 0)$ |
| $S_4(2n, 1)$ | $(n, n, n, n+1, 2n)$ | $(V_4, V_i), i = 1, 2, 3 (n > 0)$ |
| $S_1(2n+1, 1)$ | $(n, n+1, n+1, n+1, 2n+1)$ | $(V_i, V_j), i, j \neq 1$ |
| $S_2(2n+1, 1)$ | $(n+1, n, n+1, n+1, 2n+1)$ | $(V_i, V_j), i, j \neq 2$ |
| $S_3(2n+1, 1)$ | $(n+1, n+1, n, n+1, 2n+1)$ | $(V_i, V_j), i, j \neq 3$ |
| $S_4(2n+1, 2)$ | $(n+1, n+1, n+1, n, 2n+1)$ | $(V_i, V_j), i, j \neq 4$ |
| $S(2n+1, 2)$ | $(n+1, n+1, n+1, n+1, 2n+1)$ | (V_i, V_j) all i, j |

submodule such that $\ker \gamma_1 \cap \ker \gamma_2 = 0 = \ker \gamma_3 \cap \ker \gamma_4$ is obviously a direct summand of M , so we shall assume that no such summands occur. The summands then correspond to 4-subspace systems $\ker \gamma_i \rightarrow M_5$ of types $S(2n+1, 2)$, $S_i(m, 1)$, $i = 1, 2, 3, 4$, $S_{12}(m, 0)$, $S_{34}(m, 0)$. One can then define various natural submodules in terms of those summands that appear, as follows:

| Submodule | 4-subspace systems indexing its summands |
|-------------|---|
| $a(k)$ | All $S(2j-1, 2)$, $j \leq k \leq N$; |
| $b_3(k)$ | All $S(2j-1, 2)$, $j \leq N$; $S_1(2j+1, 1)$, $S_3(2j', 1)$, $(j, j' \leq [\frac{1}{2}k])$; $S_2(2j+1, 1)$, $S_4(2j', 1)$, $S_{34}(2j+1, 0)$, $S_{34}(2j', 0)$, $(j, j' < [\frac{1}{2}k], k \leq 2N)$; |
| $c_{34}(k)$ | All $S(2j-1, 2)$, $j \leq N$; $S_1(2j+1, 1)$, $S_3(2j', 1)$, $S_2(2j+1, 1)$, $S_4(2j', 1)$; $S_{34}(2j+1, 0)$, $S_{34}(2j', 0)$ $(j, j' \leq [\frac{1}{2}k], k \leq 2N)$. |

The submodules $b_1(k)$, $b_2(k)$, $b_4(k)$ are obtained by applying permutations (13)(24), (14)(23), (12)(34) to the subscripts in the definition of $b_3(k)$, and the submodule $c_{12}(k)$ by applying (13)(24) to those in the definition of $c_{34}(k)$.

There are inclusions as follows:

- (1) $a(1) \subset a(2) \subset \dots \subset a(N)$,
- (2) $a(N) \subset b_1(1)$, $b_2(1) \subset c_{12}(1) \subset b_1(2)$, $b_2() \subset c_{12}(2) \subset \dots$,
- (3) $a(N) \subset b_3(1)$, $b_4(1) \subset c_{34}(1) \subset b_3(2)$, $b_4(2) \subset c_{34}(2) \subset \dots$.

Now the problem of obtaining simultaneous canonical forms for γ_{15} , γ_{25} , γ_{35} , γ_{45} , and $\beta = \beta_{56}$, is a problem of finite representation type. This is evident from the fact that $\text{im } \beta \subset \ker \gamma_{35} \cap \ker \gamma_{45} \subset c_{34}(2N-1)$, and

from the structure of the filtration of $\ker \gamma_{35} \cap \ker \gamma_{45}$ induced by (1) and (3). The reader may check from Gelfand and Ponomarev [5] the classification of 3-subspace systems needed to calculate filtration (4) below. This classification may be used to check the naturality of the filtrations. The induced filtration of M_6 is:

$$(4) \quad 0 \subset \beta^{-1}a(1) \subset \dots \subset \beta^{-1}a(N) \subset \beta^{-1}b_3(1) , \\ \beta^{-1}b_4(1) \subset \beta^{-1}(b_3(1)+b_4(1)) \subset \dots ,$$

and it may be used to put β into canonical form. A filtration of $X = \ker \gamma_{15} \cap \ker \gamma_{25}$ is likewise induced by (1) and (2). There are isomorphisms $\beta^{-1}a(k+1)/\beta^{-1}a(k) \rightarrow a(k+1)/a(k)$ for $1 \leq k \leq N-1$. Thus the problem of finding canonical forms for α_{56} compatible with those for β_{56} is equivalent to the canonical form problem for the evident regression scheme. As N is arbitrary, this classifies all indecomposable R_2 -modules.

5. Construction of indecomposable modules

Let $M \in \underline{\mathcal{C}}$ where $\underline{\mathcal{C}}$ is the full subcategory of the image of ϕ whose objects have no direct summands isomorphic to any of the simple projective modules $R_2\varepsilon_i$ ($i = 1, \dots, 4$). Write α for α_{56} , β for β_{56} , γ_i for γ_{i5} . Since $M_5 = \ker \gamma_1 \cap \ker \gamma_2 + \ker \gamma_3 \cap \ker \gamma_4$, the 4-subspace submodule of M can have direct summands only of the types corresponding to the systems $S_{34}(1, 0)$, $S_1(1, 1)$, $S_2(1, 1)$, $S_{12}(1, 0)$, $S_3(1, 1)$, $S_4(1, 1)$, or $S(1, 2)$. Decomposition of M_5 into a direct sum of subspaces W_i , $i = 1, \dots, 7$, corresponding to summands of each type (in the given order) allows one to obtain block matrices in canonical form for the $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ as follows:

$$\gamma_1 \begin{bmatrix} \bar{1} & 0 & 0 & 0 & 0 & 0 & \bar{0} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} ,$$

$$\begin{aligned} \gamma_2 & \begin{bmatrix} \underline{1} & 0 & 0 & 0 & 0 & 0 & \underline{0} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \gamma_3 & \begin{bmatrix} \underline{0} & 0 & 0 & 1 & 0 & 0 & \underline{0} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \\ \gamma_4 & \begin{bmatrix} \underline{0} & 0 & 0 & 1 & 0 & 0 & \underline{0} \\ 0 & 0 & 0 & 0 & 0 & 1 & \underline{0} \end{bmatrix}. \end{aligned}$$

Given that, for any such M , $\text{im } \beta \leq W_1 + W_2 + W_3 + W_7$, and $W_1 + W_2 + W_3 \leq \text{im } \beta + W_7$, one can reduce the matrix of β , using operations compatible with the filtration, to the form on the left below, where the marked divisions correspond to the division of the γ_i into columns, except that the last block row is further subdivided. The matrix of α is then in the form on the right below with block rows and columns corresponding to those of β :

$$\beta = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline x & x & x & x & x \\ \hline x & x & x & x & x \\ \hline x & x & x & x & x \\ \hline x & x & x & x & x \\ \hline x & x & x & x & x \end{pmatrix}.$$

Compatibility requirements imply that the permissible operations on the nonzero part of the matrix of α are those permitted by the following regression scheme:

$$\begin{aligned} R &= \{r_1, r_2, r_3, r_4\}, \quad C = \{c_1, c_2, c_3, c_4\}, \\ \rho &= \{(r_2, r_2), (r_4, c_2), (c_2, r_4), (c_3, c_3)\}, \end{aligned}$$

where the rows and columns of α are grouped and ordered so that

r_1 corresponds to block row 4 , c_1 to block column 1 ,
 r_2 to block rows 5 and 6 , c_2 to block column 2 ,
 r_3 to block row 7 , c_3 to block columns 3 and 4 ,
 r_4 to block row 8 , c_4 to block column 5 .

Note that $M \in \underline{\mathbb{C}}$ requires that the blocks corresponding to r_1, r_2, r_3 must have rank equal to the number of rows they contain.

The canonical forms for the matrices α that produce indecomposable modules in $\underline{\mathbb{C}}$ are thus obtainable from the rules given by Nazarova and Roiter in [6]. The rules are abstracted from the process of reduction of each component of a T -matrix to canonical form, which, at each stage, replaces the regression scheme by a new one with finer divisions of the block rows and columns. A brief reinterpretation of the process, as applicable to modules in $\underline{\mathbb{C}}$ is given here.

At each stage of the regression, one may regard the block columns (rows) of the matrix of α as being indexed by triples (M, ϕ, Q) $((N, \psi, P))$ where $M (N)$ is an indecomposable module with a distinguished quotient Q (submodule P), and $\phi (\psi)$ is a fixed surjection $M \rightarrow Q$ (injection $P \rightarrow N$). Distinct block columns (rows) related by ρ have the same $M (N)$, while a block column related to a block row has $M \simeq N$.

The block column (M, ϕ, Q) (row (N, ψ, P)) is to the left of (M', ϕ', Q') (above (N', ψ', P')) if and only if there are homomorphisms $\mu : M' \rightarrow M$, $\kappa : Q' \rightarrow Q$ ($\nu : N \rightarrow N'$, $\pi : P \rightarrow P'$) such that $\kappa(\pi)$ is surjective and $\kappa\phi' = \phi\mu$ ($\psi'\pi = \nu\psi$). Two block columns (rows) correspond to the same (self-related) element of the regression scheme if and only if they are indexed by modules having no such homomorphisms between them in either direction.

The module $Q(P)$ is always generated by $\phi_6(\ker \alpha)$ $(\psi_2^{-1}(\text{coker } \alpha \cap \ker \gamma_1 \cap \ker \gamma_2))$ and may always be chosen isomorphic to one of those indexing the columns (rows) in the initial scheme; namely,

$$\begin{aligned}
c_4 : M_4 &= Q_4 = R_2 \varepsilon_6 / \langle \alpha R_2 \varepsilon_6 \rangle , \\
c_3 : M_3 &= Q_3 = M_4 / \langle \gamma_2 M_4 \rangle , \\
\bar{M}_3 &= \bar{Q}_3 = M_4 / \langle \gamma_1 M_4 \rangle , \\
c_2 : M_2 &= M_3 / \langle \gamma_1 M_2 \rangle , \quad Q_2 = M_2 / \langle \beta M_2 \rangle , \\
c_1 : M_1 &= Q_1 = Q_2 , \\
r_1 : N_1 &= P_1 = \langle \alpha R_2 \varepsilon_6 \rangle , \\
r_2 : N_2 &= P_2 = P_1 / \langle \gamma_3 P_1 \rangle , \\
\bar{N}_2 &= \bar{P}_2 = P_1 / \langle \gamma_4 P_1 \rangle , \\
r_3 : N_3 &= P_3 = \langle \beta M_2 \rangle , \\
r_4 : N_4 &= M_3 , \quad P_4 = \langle \beta M_2 \rangle .
\end{aligned}$$

Each step of the regression constructs new indecomposable modules, by specifying linear maps from the kernel of α in the column-modules to the cokernel of α in the row-modules in such a way that the kernel of the map contains all copies of $(\ker \phi)_6$ (or $(\ker \phi)_6$ and $(\ker \bar{\phi})_6$ when the column is self-related) and the image of the map is contained in the space spanned by all copies of $(\text{im } \psi)_5$ (or $(\text{im } \psi)_5$ and $(\text{im } \bar{\psi})_5$ when the row is self-related).

If the column is related to the row, the canonical forms for the matrices of maps producing new indecomposable modules are indecomposable rational canonical forms. Self-relation of a column specifies two distinguished subspaces in the domain of the map, and self-relation of the row specifies two in the codomain. When the column is not related to the row, then, the construction corresponds to the construction of certain indecomposable 2-subspace, 3-subspace, or 4-subspace systems, according to the degree of self-relation present. The systems are restricted by the requirement that the subspaces corresponding to the rows span the junction space and must have trivial intersection if there are two of them. Two-subspace constructions are obvious. To identify the 3- and 4-subspace systems that produce new indecomposable modules, we use the 4-subspace notation from Section 2, assuming that the first two subspaces correspond

to columns and the last two to rows, and that, in the case of 3-subspace systems, an appropriate subspace is zero. The types that occur are then

$$S_{i,j}(m, 0), \{i, j\} = \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\};$$

$$S(2n, 0, \lambda); S_{12}(2n, 0); S_3(2n+1, -1), S_4(2n+1, -1);$$

$$S_1(2n, -1), S_2(2n, -1); S_3(2n+1, 1), S_4(2n+1, 1);$$

$$S_1(2n, 1), S_2(2n, 1).$$

New constructions index new block rows and/or columns if and only if they continue to have distinguished quotients generated by elements of $\ker \alpha$ or distinguished submodules generated by elements of $\text{coker } \alpha \cap \ker \gamma_1 \cap \ker \gamma_2$, and the dimensions of these two subspaces indicate how many new index pairs must occur. Thus, for example, when both block row and column used in the construction are self-related, a new construction indexes a new block column if and only if the two subspaces corresponding to columns intersect nontrivially, and indexes a new block row if and only if these two subspaces do not span the junction space.

Relations between, and ordering of the modules indexing new block rows and columns are determined using the rules given above. The special homomorphisms used in ordering are present between new constructions arising from the same block row and column if and only if there are similarly defined special homomorphisms between the subspace systems used in the construction. For example, the new columns obtained from 4-subspace systems are ordered, from left to right, as follows:

$\{S_3(2n-1, 1), S_4(2n-1, 1)\}$ (two block columns corresponding to a self-related element)

$\{S_1(2n, 1), S_2(2n, 1)\} \{S_3(2n+1, 1), S_4(2n+1, 1)\}$

$\dots S_{12}(2n+2, 0), S_{12}(2n, 0) \dots$

The new rows obtained are ordered from top to bottom as follows:

$\{S_1(2n, -1), S_2(2n, -1)\}$ (two block rows corresponding to a self-related element)

$\{S_3(2n+1, -1), S_4(2n+1, -1)\} \{S_1(2n+2, -1), S_2(2n+2, -1)\}$

$\dots S_{12}(2n+2, 0), S_{12}(2n, 0) \dots$

6. Dimensions of indecomposable modules

If M is an R_i -module, the *dimension* of M is the sextuple (m_1, \dots, m_6) , where m_j is the dimension of the vector space M_j .

The algebras R_1 and R_2 are algebras corresponding to quivers with relations, the relations in these two cases stating that certain composites of arrows in the quiver are equal or zero. In [3], Brenner defines the quadratic form $q(x)$ associated with a quiver with relations as

$$q(x) = \sum_{i=1}^n x_i^2 + \sum_{i \neq j} a_{ij} x_i x_j,$$

where the quiver has n points, and $a_{ij} = -k$ if there are k arrows from i to j in the quiver, and $a_{ij} = +k$ if there are k (independent) relations on composites of arrows connecting i and j . For quivers without relations, and also for those quivers with relations whose representation type is known, the representation type of the quiver is finite, tame, or wild (see Brenner [3]) according as the quadratic form is positive definite, positive semidefinite, or indefinite on the cone $\{x \mid x_i \geq 0\}$. The dimensions of indecomposable representations of quivers of finite or tame type are precisely the solutions in non-negative integers to $q(x) = 1$ (finite type), 0 or 1 (tame type) (see our notes [4]). The heuristic argument provided by Brenner in [3] to explain this, views $q(m) - 1$ as measuring the degree to which endomorphisms of a representation of the quiver, in general position, of dimension m , are determined, modulo scalars, by the requirements of compatibility with the linear maps in the representation. The functors of Section 1 induce isometries.

The quadratic form for R_2 is

$$q(x) = \sum_{i=1}^4 [x_i - \frac{1}{2}(x_5 - x_6)]^2,$$

which is clearly semidefinite, of co-rank 2. If $x_1 = x_2 = x_3 = x_4 = 0$ it becomes the quadratic form q_1 associated with the quiver \hat{A}_1 (the 'Kronecker problem'), and if $x_6 = 0$ it becomes the quadratic form q_2

associated with the quiver \hat{D}_4 (the 4-subspace problem). The solutions to $q(x) = 0$ are of form $na + mb$, where $a = (0, 0, 0, 0, 1, 1)$ (corresponding to $q_1(a') = 0$, $a' = (1, 1)$) and $b = (1, 1, 1, 1, 2, 0)$ (corresponding to $q_2(b') = 0$, $b' = (1, 1, 1, 1, 2)$). The solutions to $q(x) = 1$ are of form $na + mb + \kappa$ where κ is either $(0, 0, 0, 0, 0, 1)$ or one of the 2^4 values $(r_1, r_2, r_3, r_4, r_5, 0)$ with $0 \leq r_1 \leq 1$, $q(\kappa) = 1$.

The construction method of Section 5, however, reveals the existence of indecomposable R_2 -modules whose dimension m is such that $q(m) = 2$. Simple calculations of dimensions arising in the course of construction of indecomposable modules in \underline{C} , together with the use of the functor Φ and its adjoint, show that the set of dimensions of indecomposable A_2 -modules splits into three subsets as follows.

(1) Solutions to $q(x) = 0$. All dimensions $na + mb$, $n, m \geq 0$, are in fact realised as dimensions of indecomposable modules. The infinite families of indecomposable modules of the same dimension, parametrised by the monic irreducible polynomials over the field, are all of this dimension type.

(2) Solutions to $q(x) = 1$. Again, the dimensions $na + mb + \kappa$ are all realised for $n, m \geq 0$.

(3) Solutions to $q(x) = 2$. Those that arise as dimensions of indecomposable modules are of form $na + mb + \delta$, where δ is one of the following:

$$\begin{aligned} \delta_1 &= (1, 1, 0, 0, 0, 0), & \delta_2 &= (0, 0, 1, 1, 0, 0), \\ \delta_3 &= (1, 1, 0, 0, 2, 0), & \delta_4 &= (0, 0, 1, 1, 2, 0), \\ \delta_5 &= (2, 0, 1, 1, 2, 0), & \delta_6 &= (0, 2, 1, 1, 2, 0), \\ \delta_7 &= (1, 1, 2, 0, 2, 0), & \delta_8 &= (1, 1, 0, 2, 2, 0). \end{aligned}$$

If $\delta = \delta_i$, $i = 1, \dots, 4$, the dimension is realised for all choices of $n \geq 1$, $m \geq 0$, while if $\delta = \delta_i$, $i = 5, \dots, 8$, it is realised for all $n \geq 3$, $m \geq 0$.

7. Further extensions of this method

It should be noted that our notes [4] on the representation theory of the Dynkin diagrams A_n , D_n , and E_n , and the corresponding extended Dynkin diagrams have been generalised by Diab and Ringel in numerous papers so as to include the other Dynkin diagrams. It is to be expected that analogous generalisation is possible in the context of commutative quivers.

In the absence of a general representation theory of quivers with commutativity and zero-composite relationships, various examples of such problems are discussed below. All have semidefinite quadratic form.

Firstly, for each positive integer n there is a 'factorised \hat{A}_{2n-1} ' quiver to consider. It may be obtained from the R_1 -module problem of this paper by addition of the requirements that each of the vector spaces $\epsilon_1^M, \dots, \epsilon_6^M$ is graded by the integers taken modulo n , that $\gamma_{51}, \gamma_{52}, \gamma_{53}, \gamma_{54}, \delta_{16}, \delta_{26}$ respect this grading whilst δ_{36}, δ_{46} map the i th graded component to the $(i+1)$ th. As before, this gives rise to the corresponding 'graded R_2 -module' problem, which can be handled by the introduction of 'cyclically graded regression schemes'. This concept is a direct generalisation of that described in Section 2.

Secondly, Butler has pointed out, the algebra R_2 may be extended by the adjunction of further idempotents $\epsilon_7, \epsilon_8, \dots$, further generators $\alpha_{67}, \beta_{67}, \alpha_{78}, \beta_{78}, \dots$ subject to relationships $\alpha_{56}\beta_{67} = \beta_{56}\alpha_{67} = 0$, and so on. The modules M for which $\epsilon_1^M = \epsilon_2^M = \epsilon_3^M = \epsilon_4^M = 0$ may be classified by use of the filtration induced by the appropriate 'Brauer graph'. However, for the purposes of this paper, it is more convenient to put the mappings $\alpha_{56}, \alpha_{67}, \alpha_{78}, \dots$ into simultaneous canonical form. (This is a standard finite representation type problem.) The determination of canonical forms for $\beta_{56}, \beta_{67}, \beta_{78}, \dots$ is then a regression scheme calculation of a particularly simple type. Furthermore, the general classification may be achieved by an evident reworking of the arguments of this paper.

Thirdly, the quiver underlying the algebra R_2 may be extended by a

second 4-subspace system by the adjunction of idempotents $\varepsilon_7, \varepsilon_8, \varepsilon_9, \varepsilon_0$ and generators $\gamma_{67}, \gamma_{68}, \gamma_{69}, \gamma_{60}$ subject to additional relationships $\alpha_{56}\gamma_{67} = \alpha_{56}\gamma_{68} = \beta_{56}\gamma_{69} = \beta_{56}\gamma_{60} = 0$. The regression schemes defined in Section 2 do not appear to be adequate to determine the representation theory of this quiver. The relevant matrix problem may be obtained by putting $\gamma_{15}, \gamma_{25}, \gamma_{35}, \gamma_{45}$ into the canonical form defined by the filtrations (1), (2), and (3) of Section 4 and simultaneously putting $\gamma_{67}, \gamma_{68}, \gamma_{69}, \gamma_{60}$ into the dual canonical form. The matrix problem so obtained can be described by two regression schemes T_α, T_β in which the last elements of $R_\alpha, C_\alpha, R_\beta, T_\beta$ are not related to any elements. The last block row of the T_α -matrix has the same width as the last block row of the T_β -matrix. If any operation of type 2 (see Section 2) is performed on one of these last block rows, it must also be performed on the other. Similar conditions hold for the last two block columns. The proof of [6] with slight modifications now shows that this extended algebra has tame representation type.

Fourthly, the above generalisations may be carried out simultaneously to yield still more complicated quivers with zero-composite relations. The above methods suffice to prove that they have tame representation type. Hence sundry associated commutative quivers also have tame representation type.

Fifthly, as Brenner has pointed out, the 4-subspace components of the R_2 problem considered here may be replaced by E type extended Dynkin diagrams. For example, the algebra

$$R_3 = K\langle \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, \alpha_{21}, \alpha_{41}, \alpha_{61}, \beta_{32}, \beta_{54}, \beta_{76}, \gamma_{18}, \gamma_{18} \mid \alpha_{21}\gamma_{18} = \alpha_{41}\delta_{18} = \alpha_{61}(\gamma_{18} + \delta_{18}) = 0 \rangle$$

is obtained analogously from an \hat{E}_6 algebra (corresponding to the modules with $\varepsilon_8 M = 0$) and a 'Kronecker' algebra. New methods seem necessary to classify the R_3 -modules.

It would be of some interest to have a method for generating all the

algebras with zero-composite defining relations and with tame representation type.

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