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A REMARK ON COLIMITS

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Let M_R be a right module over the associative ring R (with 1). Assume one has an expression for M as a colimit (direct limit) of a system

$$\left\{F_\alpha \xrightarrow{\pi_\alpha^\beta} F_\beta | \alpha < \beta \in D\right\}$$

over the (directed) poset D. A natural way to get M as a colimit of the family $\{F_\alpha \to F_\beta | \alpha, \beta \in E\}$ for some subset E of D is to take E cofinal in D. However, if one is concerned about the cardinality of the set E, cofinal subsets may be too large. Let us look at a specific example. Lazard [**3**] has shown that any flat M_R is a direct limit of finitely generated free R-modules. The cardinality of his indexing set depends on the cardinality of M. Thus Lazard's indexing set and any cofinal subset thereof may have cardinality much larger than the minimum number of relations required to define M. Thus, knowing that the projective dimension of $\varinjlim_D F_\alpha \leqq \sup \{\text{proj. dim. } (F_\alpha)\} + k + 1$ where D has cardinality $\aleph_k$ does not obviously imply that the projective dimension of an $\aleph_k$-presented flat module $\leqq k + 1$. In this note we show how to get around this kind of problem by looking at (directed) subsets E of D which are not necessarily cofinal but which still have $M = \text{colim}_E\{F_\alpha \to F_\beta\}$.

In this paper, $\aleph$ will denote an infinite cardinal number, and $|D|$ will denote the cardinality of the set D.

Definition. A module M_R is called $\aleph$-*related* if there exists an exact sequence

$$0 \to K \to P \to M \to 0$$

with P free and K $\aleph$-generated. M is $\aleph$-*presented* if it is $\aleph$-generated and $\aleph$-related.

THEOREM. *Let M be an $\aleph$-related module, D a (directed) poset,*

$$\left\{F_\alpha \xrightarrow{\pi_\alpha^\beta} F_\beta | \alpha < \beta \in D\right\}$$

a system of $\aleph$-generated modules such that $M \approx \text{colim}_D F_\alpha$. Then there exists a (directed) subset $D' \subseteq D$ with $|D'| \leqq \aleph$ such that $M \approx \text{colim}_{D'}\{F_\alpha, \pi_\alpha^\beta\} \oplus L$, where L is free. If M is $\aleph$-generated, we may take $L = 0$.

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Proof. We proceed in a series of steps. Let

$$0 \to K \to P \to M \to 0$$

be exact, P free with free basis $\{x_i | i \in \mathscr{I}\}$, K $\aleph$-generated.

(1) (Initial reduction). Since K is $\aleph$-generated, there exists a subset $\mathscr{J} \subseteq \mathscr{I}$ such that $|\mathscr{J}| = \aleph$ and $K \subseteq \sum_{j \in \mathscr{J}} x_j R$. Then

$$M \approx P/K \approx \left(\sum_{j \in \mathscr{J}} x_j R\right)/K \oplus \sum_{i \in \mathscr{I} - \mathscr{J}} x_i R.$$

If $|\mathscr{I}| \leqq \aleph$, take $\mathscr{J} = \mathscr{I}$. If not, any $\aleph$-generated submodule of $\sum_{i \in \mathscr{I} - \mathscr{J}} x_i R$ is contained in a proper summand, so M cannot be $\aleph$-generated. In any case we have $M \approx M' \oplus L'$, where

$$M' = \left(\sum_{j \in \mathscr{J}} x_j R\right)/K$$

is $\aleph$-presented and L' is free on $\{x_i | i \in \mathscr{I} - \mathscr{J}\}$.

(2) (Well-known fact). Let N be any $\aleph$-presented module, $0 \to B \to A \to N \to 0$ exact, A $\aleph$-generated. Then B is $\aleph$-generated. This is a corollary of Schanuel's lemma (see [**2**, p. 167]), for if P' is an $\aleph$-generated free mapping onto A, kernel $(P' \to A \to N)$ is an $\aleph$-generated module mapping onto B.

(3) (Another well-known fact). $\operatorname{Colim}_D\{F_\alpha, \pi_\alpha{}^\beta\} \approx \oplus_{\alpha \in D} F_\alpha / X_D$, where X_D is the submodule of $\oplus F_\alpha$ generated by elements of the form $u(\alpha, \beta)$ for $\alpha < \beta$, where all projections of $u(\alpha, \beta)$ are zero except for $x \in F_\alpha$ and $-\pi_\alpha{}^\beta x \in F_\beta$. This is trivial to verify from the definition of colimit. See [**6**, Chapter VIII, § 4] if a reference is necessary. This motivates the following notation.

Notation. For any $E \subseteq D$, set X_E = the submodule of $\oplus_{\alpha \in E} F_\alpha$ generated by

$$\{u(\alpha, \beta) | \alpha, \beta \in E, u(\alpha, \beta) = x - \pi_\alpha{}^\beta x \text{ where } x \in F_\alpha\}.$$

Set $F_E = \oplus_{\alpha \in E} F_\alpha$. We will consider each F_E and X_E as a subset of F_D and X_D by the obvious injections. Let ν_E be the map from F_E to M induced by these identifications, I_E the image of ν_E in M.

(4) Let N be any $\aleph$-generated submodule of M. Then $M' + N$ is $\aleph$-generated. Hence there exists a set $\mathscr{K} \subseteq \mathscr{I} - \mathscr{J}$ such that $|\mathscr{K}| \leqq \aleph$ and $M' + N \subseteq M' \oplus \sum_{i \in \mathscr{K}} x_i R$. Since $F_D \to M$ is onto, for each generator m of $M' \oplus \sum_{i \in \mathscr{K}} x_i R$ there is a finite subset $G(m)$ of D such that $m \in I_{G(m)}$. Then the set $E = \bigcup G(m)$ satisfies

$$M' + N \subseteq M' \oplus \sum_{i \in \mathscr{K}} x_i R \subseteq I_E.$$

We use this plus a snaking argument of Kaplansky [**1**] to get:

(5) Let $N \subseteq M$ be any $\aleph$-generated submodule of M. Then there exists $E(N) \subseteq D$ and $\mathscr{L}(N) \subseteq \mathscr{I} - \mathscr{J}$ such that $N \subseteq I_{E(N)}$, $|E(N)| \leqq \aleph$, and $M \approx I_{E(N)} \oplus \sum_{i \in \mathscr{L}(N)} x_i R$. We show this by finite induction. By (4), we may

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find $\mathscr{K}_0 \subseteq \mathscr{I} - \mathscr{J}$ and $E_0 \subseteq D$ with $|\mathscr{K}_0| \leqq \aleph$ and $|E_0| \leqq \aleph$ such that $N \subseteq M' \oplus \sum_{\iota \in \mathscr{K}_0} x_\iota R \subseteq I_{E_0}$. Assume we have E_n and $\mathscr{K}_n$ for all $n \leqq m$ such that $|E_n|$ and $|\mathscr{K}_n| \leqq \aleph$ and for $0 \leqq n \leqq m - 1$

(a) $E_n \subseteq E_{n+1}$,

(b) $\mathscr{K}_n \subseteq \mathscr{K}_{n+1}$,

(c) $I_{E_n} \subseteq M' \oplus \sum_{\iota \in \mathscr{K}_{n+1}} x_\iota R \subseteq I_{E_{n+1}}$.

Since each F_α is $\aleph$-generated, so is I_{E_m}. Hence by (4) we may find E_{m+1} and $\mathscr{K}_{m+1}$ satisfying (a), (b), and (c) with $|E_{m+1}|$ and $|\mathscr{K}_{m+1}| \leqq \aleph$. Set $E(N) = \bigcup_{n=0}^{\infty} E_n$, $\mathscr{K}(N) = \bigcup_{n=0}^{\infty} \mathscr{K}_n$. Then $|E(N)|$ and $|\mathscr{K}(N)| \leqq \aleph$ and by construction,

$$N \subseteq \sum_{\iota \in \mathscr{K}(N)} x_\iota R \oplus M' \subseteq I_{E(N)} \subseteq \sum_{\iota \in \mathscr{K}(N)} x_\iota R \oplus M',$$

so $M = I_{E(N)} \oplus \sum_{\iota \in \mathscr{L}(N)} x_\iota R$ where $\mathscr{L}(N) = (\mathscr{I} - \mathscr{J}) - \mathscr{K}(N)$. In particular, $I_{E(N)}$ is the $\aleph$-presented module $M' \oplus \sum_{\iota \in \mathscr{K}(N)} x_\iota R$.

(6) Step (5) says that any $\aleph$-generated submodule $N \subseteq M$ can be embedded in a direct summand $I_{E(N)}$ of M which is the image of $\oplus_{\alpha \in E(N)} F_\alpha$. The E we are looking for in our theorem has the additional property that the kernel $X_D \cap F_E$ of $\nu_E : F_E \to M$ is the colim_E kernel X_E; i.e., we must construct $E \subseteq D$ such that $X_E = X_D \cap F_E$ and $M = I_E \oplus \sum_{\iota \in \mathscr{L}} x_\iota R$. We again use finite induction to union up to such an E.

Set $E_0 = E(M')$, $\mathscr{L}_0 = \mathscr{L}(M')$. Assume for all $n \leqq m$ we have E_n and $\mathscr{L}_n$ such that

(a) $|E_n| \leqq \aleph$;

(b) $M = I_{E_n} \oplus \sum_{\iota \in \mathscr{L}_n} x_\iota R$, where $I_{E_n} = M' \oplus \sum_{\iota \in (\mathscr{I} - \mathscr{J}) - \mathscr{L}_n} x_\iota R$ and for $0 \leqq n \leqq m - 1$;

(c) $E_n \subseteq E_{n+1}$ (so $\mathscr{L}_n \supseteq \mathscr{L}_{n+1}$);

(d) $X_D \cap F_{E_n} = X_{E_{n+1}} \cap F_{E_n}$;

(e) if D is directed, then every finite subset of E_n has an upper bound in E_{n+1}.

Since I_{E_m} is $\aleph$-presented, kernel $\nu_{E_m} = X_D \cap F_{E_m}$ is $\aleph$-generated, say by $\{J_\beta | \beta \in \mathscr{I}'\}$. For each $\beta \in \mathscr{I}'$ there exists a finite set $G(\beta) \subseteq D$ such that $y_\beta = \sum u(\alpha, \alpha') r(\alpha, \alpha')$ where $\alpha < \alpha'$ are elements of $G(\beta)$ and $r(\alpha, \alpha') \in R$. Set $G = \bigcup G(\beta)$. Then $|G| \leqq \aleph$. If D is directed, for each finite subset S of of E_m, let $b(S)$ be an upper bound of S in D, and set

$$G' = \{b(S) | S \text{ a finite subset of } E_m\}.$$

Since there are at most $\aleph$ finite subsets of E_m, $|G'| \leqq \aleph$. Now let

$$E_{m+1} = E(I_{E_m \cup G \cup G'}) \cup E_m \cup G \cup G'$$
$$\mathscr{L}_{m+1} = \mathscr{L}(I_{E_m \cup G \cup G'}).$$

Then E_{m+1} and $\mathscr{L}_{m+1}$ satisfy (a) through (e). E_{m+1} was obtained by looking at $E' = E_m \cup G \cup G'$, taking the image $I_{E'}$, and then applying (5) to get a direct summand $I_{E(I_{E'})}$ required in (b). To insure (c), it is not sufficient to

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take $E_{m+1} = E(I_{E'})$. We must also throw in E'. However, since $\oplus_{\alpha \in E'} F_\alpha$ maps into $I_{E(I_{E'})}$, $I_{E_{m+1}} = I_{E(I_{E'})}$. We get (a) since $|E'| \subseteq \aleph$, and (d) and (e) are insured by including G and G' in E_{m+1}. Then if

$$E = \bigcup_{n=0}^{\infty} E_n \quad \text{and} \quad \mathscr{L} = \bigcap_{n=0}^{\infty} \mathscr{L}_n,$$

(b) and (c) insure that $M = I_E \oplus \sum_{t \in \mathscr{L}} x_t R$ and (d) insures that $X_E = F_E \cap X_D$ so $I_E = \operatorname{colim}_E F$.

COROLLARY. *Let M be an $\aleph_k$-related flat R-module. Then* proj. dim $(M) \leqq k + 1$.

Proof. If $k = -1$, it is well-known that M is projective. Otherwise, by Lazard [**3**], M is a direct limit of finitely generated frees. By the theorem,

$$M = \varinjlim_E F_\alpha \oplus L$$

where L is free, each F_α is finitely generated free, and $|E| \leqq \aleph_k$. By Osofsky [**5**], proj. dim. $(\varinjlim_E F_\alpha) \leqq k + 1$, so proj. dim. $(M) \leqq k + 1$.

We remark that step (1) of the proof of the theorem plus the standard argument (as in [**6**] for example) shows that any $\aleph$-related module is a direct union (direct limit) of $\aleph$ finitely generated (finitely presented) modules plus a free.

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