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On Algebraic Surfaces Associated with Line Arrangements

Zhenjian Wang

Abstract. For a line arrangement $\mathcal{A}$ in the complex projective plane $\mathbb{P}^2$, we investigate the compactification $\overline{F}$ in $\mathbb{P}^3$ of the affine Milnor fiber F and its minimal resolution $\widetilde{F}$. We compute the Chern numbers of $\widetilde{F}$ in terms of the combinatorics of the line arrangement $\mathcal{A}$. As applications of the computation of the Chern numbers, we show that the minimal resolution is never a quotient of a ball; in addition, we also prove that $\widetilde{F}$ is of general type when the arrangement has only nodes or triple points as singularities. Finally, we compute all the Hodge numbers of some $\widetilde{F}$ by using some knowledge about the Milnor fiber monodromy of the arrangement.

1 Introduction

A line arrangement $\mathcal{A}$ is a finite set of lines in the projective plane $\mathbb{P}^2 = \mathbb{CP}^2$:

$$\mathcal{A} = \{L_1, \dots, L_d\}$$

where for $i = 1, \dots, d$, $L_i : \ell_i(x, y, z) = 0$ is a line in $\mathbb{P}^2$ defined by the linear form ℓ_i. We call $Q = Q(x, y, z) = \prod_{i=1}^{d} \ell_i$ the *defining equation* of $\mathcal{A}$.

In this paper, we consider a construction of surfaces from a line arrangement that is closely related to the Milnor fiber of the arrangement. The *Milnor fiber* $F = F(\mathcal{A})$ of the line arrangement $\mathcal{A}$ is the smooth affine surface in $\mathbb{C}^3$, defined by

$$F : Q(x, y, z) - 1 = 0,$$

where Q is the defining equation of $\mathcal{A}$. As F is not compact, we consider its natural compactification $\overline{F}$ in $\mathbb{P}^3$ defined by

$$\overline{F} : Q(x, y, z) + t^d = 0.$$

Denote by $V(Q)$ the projective curve in $\mathbb{P}^2$ defined by $Q = 0$, or equivalently,

$$V(Q) = \bigcup_{L_i \in \mathcal{A}} L_i;$$

then $\overline{F}$ is a branched cover of $\mathbb{P}^2$ of degree d branched over $V(Q)$. The surface $\overline{F}$ is a singular surface with isolated singularities, and we are mainly interested in the *minimal resolution* $\widetilde{F}$ of $\overline{F}$.

The main aim of this paper is to compute the Chern numbers $c_1^2(\widetilde{F})$ and $c_2(\widetilde{F})$ of the associated surface $\widetilde{F}$, and then to discuss properties of $\widetilde{F}$ using our computations.

We first show that $c_1^2(\widetilde{F})$ and $c_2(\widetilde{F})$ are determined by the combinatorics of the line arrangement $\mathcal{A}$. More precisely, we prove the following theorem.

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Theorem 1.1 *Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$ consisting of $d = |\mathcal{A}|$ lines. For $r \geq 2$, denote by t_r the number of points lying on exactly r lines contained in $\mathcal{A}$.*

Then the following hold.

(i) *The first Chern number of the associated surface $\widetilde{F}$ is given by*

$$c_1^2(\widetilde{F}) = K_{\overline{F}}^2 + \sum_r t_r DCI_{r,d},$$

where $K_{\overline{F}}^2 = d(d-4)^2$. For $d \not\equiv 1 \bmod r$, we have

$$DCI_{r,d} = -d(r-2)^2 - r\sum_{i=1}^{\lambda}(n_i - 2) + 2(r-2)\big(r - \gcd(r,d)\big) + (r-b);$$

for $d \equiv 1 \bmod r$, we have $DCI_{r,d} = -(d-1)(r-2)^2$.

(ii) *The second Chern number of the associated surface $\widetilde{F}$ is given by*

$$c_2(\widetilde{F}) = \chi(\overline{F}) + \sum_r t_r DCII_{r,d},$$

where $\chi(\overline{F})$ is the topological Euler number of $\overline{F}$:

$$\chi(\overline{F}) = d(d^2 - 4d + 6) - (d-1)\sum_r t_r (r-1)^2.$$

In addition, for $d \not\equiv 1 \bmod r$, we have

$$DCII_{r,d} = 1 + r\lambda - (r-2)\big(\gcd(r,d) - 1\big);$$

for $d \equiv 1 \bmod r$, we have $DCII_{r,d} = d - 1$.

In the above formulae, the numbers λ, b, n_i are uniquely determined only by r and d from Theorem 4.1 below.

Theorem 1.1 will not be completely proved until Section 5, before which we will establish step by step all necessary ingredients of the proof. Briefly speaking, we will first show that the computation of the Chern numbers can be localized in Section 3; namely, it suffices to consider minimal resolutions of surface germs. Then we make use of the technical theorem, *i.e.*, Theorem 4.1 on resolutions of weighted homogeneous singularities to complete the calculations. Another technical tool involving continued fractions is outlined in Section 5.

As a first application of our computation of the Chern numbers, we show that the surface $\widetilde{F}$ can never be a ball quotient. Recall that a *ball quotient* is a smooth projective surface that is biholomorphic to $\mathbf{B}/\Gamma$, where

$$\mathbf{B} = \big\{ (z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 < 1 \big\}$$

equipped with the Kähler metric whose Kähler form is given by

$$\omega_{\mathbf{B}} = -\sqrt{-1}\partial\overline{\partial}\log(1 - |z|^2 - |w|^2),$$

and where Γ is a discrete cocompact subgroup of isometries of $\mathbf{B}$. A smooth projective surface X is said to be of *general type* if its canonical divisor K_X is big; and for such surfaces, we have the celebrated Miyaoka-Yau inequality, namely,

$$c_1^2(X) \leq 3c_2(X), \tag{1.1}$$

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where the equality holds if and only if X is a ball quotient; see [22].

We say that $\mathcal{A}$ is a *pencil* if $V(Q)$ has a singularity of multiplicity $d = |\mathcal{A}|$. Using our results on Chern numbers, we prove the following result.

Theorem 1.2 *Assume $d = |\mathcal{A}| \geq 2$.*

(i) *If $d \geq 4$ and $\mathcal{A}$ is a pencil, $c_1^2(\widetilde{F}) > 3c_2(\widetilde{F})$.*

(ii) *If $d \geq 3$ and $\mathcal{A}$ is not a pencil or $d = 2$, $c_1^2(\widetilde{F}) < 3c_2(\widetilde{F})$.*

In particular, if $d \geq 2$ and for $d = 3$, $\mathcal{A}$ is not a pencil, then $\widetilde{F}$ is not a ball quotient.

The non-ball-quotient property of $\widetilde{F}$ is always true for $d \geq 2$; see Remark 6.1. And in fact, it is not very surprising that $\widetilde{F}$ is not a ball quotient; however, our result above is stronger than the non-ball-quotient property of $\widetilde{F}$; see Remark 6.2.

Next, we use the formulae for the Chern numbers to determine whether the surface $\widetilde{F}$ is of general type. We prove the following result in Section 7.

Theorem 1.3 *Assume that $d = |\mathcal{A}| \geq 7$ and $V(Q)$ contains only nodes or triple points as singularities; then $\widetilde{F}$ is of general type.*

Finally, our results can also be applied to study problems in the theory of line arrangements. In fact, one motivation for our work i s to understand whether the Hodge numbers of $\widetilde{F}$ are combinatorially determined, one of the main open questions in the theory of line arrangements; see [16]. We will give some examples in which we compute all the Hodge numbers of $\widetilde{F}$ at the end of this paper.

2 Preliminaries

In this section, we present some basic facts that are stated in the form we will apply in the proof of our main theorem, namely, Theorem 1.1. These facts are well known, and thus we skip the details and only give some references.

2.1 Intersection Theory for Normal Surfaces

Let X be a projective variety of dimension n over $\mathbb{C}$.

When X is smooth, the intersection theory on X is classical and quite well-known. For intersection theory on smooth surfaces, we refer the reader to [1, Chapter 2].

If X is not smooth, there are technical problems about defining the intersection number of $n = \dim X$ divisors on X; see, for instance, [10, Section 2.5]. However, we can always have a well-defined intersection number of n Cartier divisors on the projective variety X and such an intersection theory admits similar properties as in the smooth case; see [5, Section 1.2].

Given n Cartier divisors $D_1, \dots, D_n$ on a (not necessarily smooth) projective variety X, their intersection number will be denoted by $D_1 \cdot D_2 \cdots D_n$; if $D_1 = \cdots = D_n = D$, we use the simpler notation $D^n = D_1 \cdot D_2 \cdots D_n$.

In addition, assume $f: C \to X$ is a morphism from a projective curve to a quasi-projective variety X and D is a Cartier divisor (class) on X. Since f^*D is a Cartier

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divisor on C, we can define

$$D \cdot C = \deg \mathcal{O}_C(f^* D). \tag{2.1}$$

We will mainly be concerned with intersection theory on normal surfaces.

Let $X, \widetilde{X}$ be two normal projective surfaces and let $\pi\colon \widetilde{X} \to X$ be a proper surjective morphism. Then we have the projection formula

$$\pi^* C \cdot \pi^* D = (\deg \pi)\, C \cdot D$$

for any two Cartier divisors on X; see [5, Proposition 1.10]. Together with formula (2.1), this implies the projection formula

$$\pi^* D \cdot E = D \cdot \pi_* E,$$

where D is a Cartier divisor on X, while E is a curve on $\widetilde{X}$. In particular, if E is contracted by π to a point, then $\pi^* D \cdot E = 0$ for any Cartier divisor D on X.

2.2 Canonical Divisors of Normal Surfaces

Let X be a normal surface on a smooth projective threefold Y.

If X is smooth, then we have a well-defined canonical bundle and hence the canonical divisor K_X; moreover, we have the adjunction formula

$$K_X = (K_Y + X)|_X. \tag{2.2}$$

When X is not smooth, we have a canonical bundle on the smooth locus $X \smallsetminus \operatorname{Sing}(X)$ of X, and hence the associated Cartier divisor $K_{X \smallsetminus \operatorname{Sing}(X)}$ on $X \smallsetminus \operatorname{Sing}(X)$. Note that $\operatorname{Sing}(X)$ has codimension 2 in X, since X is normal, so any Weil divisor on X is uniquely determined by its restriction on $X \smallsetminus \operatorname{Sing}(X)$ (see [11, Chapter II, Section 6, Proposition 6.5(b)]); the canonical divisor of X, still denoted by K_X, is the unique Weil divisor whose restriction on $X \smallsetminus \operatorname{Sing}(X)$ is $K_{X \smallsetminus \operatorname{Sing}(X)}$. Furthermore, the adjunction formula (2.2) still holds. Indeed, the equality clearly holds on the smooth locus $X \smallsetminus \operatorname{Sing}(X)$, and thus it is also valid on X, because as we have mentioned, any Weil divisor on X is uniquely determined by its restriction on $X \smallsetminus \operatorname{Sing}(X)$.

Note that because Y is a smooth manifold, K_Y and X are both Cartier divisors on Y, hence $K_X = (K_Y + X)|_X$ is a Cartier divisor on X, being the restriction to X of the Cartier divisor $K_Y + X$ of Y. Therefore, we have the well-defined intersection number K_X^2. By [5, Proposition 1.8], we obtain

$$K_X^2 = (K_Y + X) \cdot (K_Y + X) \cdot X. \tag{2.3}$$

2.3 Miyaoka–Yau Number

As in the previous subsection, we assume that X is a normal surface on a smooth projective threefold Y.

When X is smooth, it is well known that $c_1^2(X) = K_X^2$ and $c_2(X) = \chi(X)$, where $\chi(X)$ denotes the topological Euler number of X.

If X is not smooth, K_X^2 and $\chi(X)$ are still well-defined numbers for X, and they are good substitutions for the Chern numbers c_1^2 and c_2.

In view of the Miyaoka–Yau inequality (1.1), we give the following definition.

Definition 2.1 Let $X \subseteq Y$ be a normal surface on a smooth projective threefold Y, the *Miyaoka–Yau number* of X is defined by $MY(X) = 3\chi(X) - K_X^2$.

Now assume that the normal surface X has an isolated singularity $0 \in X$, and $\pi: \widetilde{X} \to X$ be a minimal resolution of the singularity 0, given by successive embedded blowups. Let $\pi': \widetilde{Y} \to Y$ be the effect of the successive blowups on Y. Then $\widetilde{Y}$ is a smooth projective threefold, on which $\widetilde{X}$ is a normal hypersurface; hence, we have the canonical divisor $K_{\widetilde{X}} = (K_{\widetilde{Y}} + \widetilde{X})|_{\widetilde{X}}$ and topological Euler number $\chi(\widetilde{X})$ and thus the Miyaoka-Yau number $MY(\widetilde{X})$ of $\widetilde{X}$.

Definition 2.2 Three numerical invariant differences for the minimal resolution $\pi: \widetilde{X} \to X$ around the point 0 are defined as follows:

(i) The difference for the first Chern number is $DCI_0 = K_{\widetilde{X}}^2 - K_X^2$;
(ii) The difference for the second Chern number is $DCII_0 = \chi(\widetilde{X}) - \chi(X)$;
(iii) The difference for the Miyaoka–Yau number is

$$DMY_0 = MY(\widetilde{X}) - MY(X) = 3\,DCII_0 - DCI_0.$$

When X is given by $(X,0) : G_r(u,v) + t^d = 0$ around the local coordinates (u,v,t) centered at 0 on Y where G_r is a product of r distinct linear forms, it turns out that the three differences defined above are determined only by r and d; thus, they will be denoted by $DCI_{r,d}$, $DCII_{r,d}$ and $DMY_{r,d} = 3\,DCII_{r,d} - DCI_{r,d}$, respectively.

3 Surfaces Associated with Line Arrangements and Chern Numbers: From Global to Local

In this section, we consider several surfaces associated with a given line arrangement, and as a first step to prove Theorem 1.1, we show that our computations can indeed be localized; namely, we only need to investigate the resolution of a surface germ.

3.1 Surfaces Constructed from Line Arrangements

Let $\mathcal{A} = \{L_1, \ldots, L_d\}$ with $L_i : \ell_i = 0, i = 1, \ldots, d$, be a line arrangement in $\mathbb{P}^2$ with defining polynomial $Q(x,y,z) = \ell_1\ell_2\cdots\ell_d$.

Given $r \geq 2$. If a point $x \in \mathbb{P}^2$ lies on exactly r lines in $\mathcal{A}$, or equivalently, x is a singular point of multiplicity r of the curve $V(Q) : Q = 0$ in $\mathbb{P}^2$, we say that x is of multiplicity r. The number of points of multiplicity r will be denoted by t_r.

Consider the affine Milnor fiber $F : Q = 1$ in $\mathbb{C}^3$, for which we have a natural compactification $\overline{F} : Q(x,y,z) + t^d = 0$ in $\mathbb{P}^3$. The surface $\overline{F}$ is a singular normal surface in $\mathbb{P}^3$, and a singular point of multiplicity r of $V(Q)$ gives a singular point of multiplicity r of $\overline{F}$ and vice versa. Moreover, since Q is a product of distinct linear forms, around a singular point of $\overline{F}$ of multiplicity r, we have $\overline{F} : G_r(u,v) + t^d = 0$ with $G_r(u,v)$ a product of r distinct linear forms, whose resolution will be investigated in detail in the next section.

For later convenience, we first compute the Chern numbers and the Miyaoka–Yau number of the singular surface $\overline{F}$.

Example 3.1 The adjunction formula (2.2) gives

$$K_{\overline{F}} = (K_{\mathbb{P}^3} + \overline{F})|_{\overline{F}} \sim (d-4)H|_{\overline{F}},$$

where H is a hyperplane section of $\mathbb{P}^3$. Indeed, we have $\overline{F} \sim dH$ and $K_{\mathbb{P}^3} \sim -4H$ (where ~ denotes rational equivalence). It follows from (2.3) that

$$K_{\overline{F}}^2 = d(d-4)^2. \tag{3.1}$$

Moreover, there is a natural projection

$$p\colon \overline{F} \longrightarrow \mathbb{P}^2, \quad (x,y,z,t) \longmapsto (x,y,z),$$

which is a branched covering of degree d with ramification locus $V(Q) \subseteq \mathbb{P}^2$, hence

$$\chi(\overline{F}) = 3d - (d-1)\chi(V(Q)).$$

The Euler number of the singular curve $V(Q)$ is

$$\chi\big(V(Q)\big) = d(3-d) + \sum_r t_r(r-1)^2,$$

which implies that

$$\chi(\overline{F}) = d(d^2-4d+6) - (d-1)\sum_r t_r(r-1)^2 \tag{3.2}$$

Consequently,

$$MY(\overline{F}) = 3\chi(\overline{F}) - K_{\overline{F}}^2 = (d-1)\sum_r t_r(r-1)(3-r). \tag{3.3}$$

Remark 3.2 To deduce (3.3), we have used the well-known equality

$$\frac{d(d-1)}{2} = \sum_r t_r \frac{r(r-1)}{2}.$$

Let $\pi\colon \widetilde{F} \to \overline{F}$ be the minimal resolution of $\overline{F}$; namely, the following three conditions hold:

(i) $\widetilde{F}$ is a smooth surface and π is proper birational morphism;
(ii) $\pi\colon \widetilde{F} \smallsetminus \pi^{-1}(\mathrm{Sing}(\overline{F})) \to \overline{F} \smallsetminus \mathrm{Sing}(\overline{F})$ is an isomorphism;
(iii) there is no exceptional (-1)-curve on $\widetilde{F}$, *i.e.*, a rational curve E on $\widetilde{F}$ such that $E^2 = -1$ and E is contracted to a point by π.

Such a resolution π can be obtained by successive embedded blowups, namely by blowing up along submanifolds of $\mathbb{P}^3$ as well as the resulting manifolds in each step. In particular, we can resolve the singularities of $\overline{F}$ point by point, because one can do blowups in this way. In addition, useful numerical information for the resolution is encoded in its (weighted) dual graph, as explained in [6, p. 50]; for instance, the intersection matrix of the exceptional curves can be read out from the dual graph. For the detailed construction of the dual graph; see loc. cit.

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3.2 Chern Numbers for the Associated Surfaces

Let $p_1, \dots, p_s$ be all the singular points of $\overline{F}$ and let r_i be the multiplicity of p_i. Let $E_{i,1}, \dots, E_{i,\nu_i}$ be the irreducible components of $\pi^{-1}(p_i)$ and $M_{i,j,k} = E_{i,j} \cdot E_{i,k}$ be the intersection product of $E_{i,j}$ and $E_{i,k}$. Moreover, let $\mathbf{M_i} = (M_{i,j,k})$ be the intersection matrix of the $E_{i,j}$ for any fixed i. It is a $\nu_i \times \nu_i$ matrix. Set

$$\mathbf{E_i} = (E_{i,1}, E_{i,2} \dots, E_{i,\nu_i})$$

as a $1 \times \nu_i$ matrix.

The canonical divisor $K_{\widetilde{F}}$ is of the form

$$K_{\widetilde{F}} = \pi^* K_{\overline{F}} + \sum_{i=1}^{s} \sum_{j=1}^{\nu_i} a_{i,j} E_{i,j}.$$

Let $\mathbf{A_i} = (a_{i,1}, a_{i,2}, \dots, a_{i,\nu_i})^T$ be a $\nu_i \times 1$ matrix, where $(\cdot)^T$ denotes the transpose of a matrix; then $K_{\widetilde{F}}$ can be written as

$$K_{\widetilde{F}} = \pi^* K_{\overline{F}} + \sum_{i=1}^{s} \mathbf{E_i} \mathbf{A_i}. \tag{3.4}$$

Let

$$\mathbf{E_i} \cdot K_{\widetilde{F}} = (E_{i,1} \cdot K_{\widetilde{F}}, \dots, E_{i,\nu_i} \cdot K_{\widetilde{F}})$$

be a $1 \times \nu_i$ matrix. By Theorem 4.1, each $E_{i,j}$ is a smooth complete curve, and by [14], each $\mathbf{M_i}$ is a symmetric negative definite $\nu_i \times \nu_i$ matrix. Set $\mathbf{N_i} = -\mathbf{M_i}^{-1}$.

Taking intersection product of $K_{\widetilde{F}}$ with the exceptional divisors $E_{i,j}$, it follows that

$$\mathbf{A_i} = -\mathbf{N_i} (\mathbf{E_i} \cdot K_{\widetilde{F}})^T, \tag{3.5}$$

and hence by (3.4), we have

$$K_{\widetilde{F}} = \pi^* K_{\overline{F}} - \sum_{i=1}^{s} \mathbf{E_i} \mathbf{N_i} (\mathbf{E_i} \cdot K_{\widetilde{F}})^T,$$

so from the projection formula, one obtains

$$c_1^2(\widetilde{F}) = K_{\overline{F}}^2 + \sum_{i=1}^{s} \left(\mathbf{E_i} \mathbf{N_i} (\mathbf{E_i} \cdot K_{\widetilde{F}})^T \right)^2.$$

In addition, since $E_{i,j}$ is a smooth complete curve, we have the adjunction formula

$$E_{i,j} \cdot K_{\widetilde{F}} = 2g(E_{i,j}) - 2 - E_{i,j}^2;$$

thus, it follows from (3.5) that $a_{i,j}$ satisfy

$$\sum_{k=1}^{\nu_i} a_{i,k} (E_{i,j} \cdot E_{i,k}) = 2g(E_{i,j}) - 2 - E_{i,j}^2, \quad j = 1, \dots, \nu_i.$$

So, we have proved the following result.

Lemma 3.3 *The first Chern number $c_1^2(\widetilde{F})$ of $\widetilde{F}$ is given by*

$$c_1^2(\widetilde{F}) = K_{\overline{F}}^2 + \sum_{i=1}^{s} \Big(\sum_{j=1}^{\nu_i} a_{i,j} E_{i,j} \Big)^2,$$

where $E_{i,j}, j = 1, \dots, \nu_i$ *are the irreducible components of the exceptional divisor* $\pi^{-1}(p_i)$ *and* $a_{i,j}, i = 1, \dots, s; j = 1, \dots, \nu_i$ *satisfy the equation*

$$\sum_{k=1}^{\nu_i} a_{i,k}(E_{i,j} \cdot E_{i,k}) = 2g(E_{i,j}) - 2 - E_{i,j}^2. \tag{3.6}$$

From (3.6), one easily sees that for a fixed i, the $a_{i,j}$ are determined by the genera of the $E_{i,j}$ and the intersection products $E_{i,j} \cdot E_{i,k}, j, k = 1, \dots, \nu_i$; thus, they are determined only by the dual graph of the resolution of the surface germ $(\overline{F}, p_i)$.

Now we consider the more general setting. Let X be a normal surface in a smooth projective threefold Y such that X has a singularity 0. Let $\widetilde{X} \to X$ be the minimal embedded resolution of X about the singularity 0.

Assume that $(X, 0) \simeq (\overline{F}, p_i)$ for some i and $q_1 = 0, q_2, \dots, q_m$ are all the singular points of X. Note that $\widetilde{X}$ may not be smooth, so let $\widetilde{X}^*$ be the minimal resolution of $\widetilde{X}$, then $\widetilde{X}^*$ is also the minimal resolution of X.

By our discussion above, we have

$$c_1^2(\widetilde{X}^*) = K_{\widetilde{X}}^2 + R,$$

where R depends only on the resolution of the singularities of $(X, q_j), j = 2, \dots, m$. Similarly, considering $\widetilde{X}^*$ as a minimal resolution of X, we obtain

$$c_1^2(\widetilde{X}^*) = K_X^2 + R + \Big(\sum_{j=1}^{\nu_i} a_{i,j}E_{i,j}\Big)^2$$

with $a_{i,j}$ satisfying (3.6). Therefore, we obtain that

$$K_{\widetilde{X}}^2 - K_X^2 = \Big(\sum_{j=1}^{\nu_i} a_{i,j}E_{i,j}\Big)^2.$$

Finally, the surface germ $(\overline{F}, p_i)$ is defined by $G_{r_i}(u, v) + t^d = 0$ around the local coordinates (u, v, t) centered at p_i on $\mathbb{P}^3$, where G_{r_i} is a product of r_i distinct linear forms. So, by our notation, we have

$$DCI_{r_i,d} = \Big(\sum_{j=1}^{\nu_i} a_{i,j}E_{i,j}\Big)^2,$$

where the $a_{i,j}$ satisfy (3.6).

Consequently, we have the following theorem, which essentially says that our computation of first Chern number can be localized.

Theorem 3.4 *For the first Chern number* c_1^2*, the following hold:*

(i) *Assume that* $p_1, \dots, p_s$ *are all the singular points of* $\overline{F}$ *and* r_i *is the multiplicity of* p_i*. Then*

$$c_1(\widetilde{F}) = K_{\overline{F}}^2 + \sum_{i=1}^{s} DCI_{r_i,d} = K_{\overline{F}}^2 + \sum_{r} t_r DCI_{r,d}.$$

(ii) *Let* $(X, 0)$ *be an isolated surface singularity germ defined by* $G_r(u, v) + t^d = 0$*, where* G_r *is a product of* $r \le d$ *distinct linear forms and* $\pi_X\colon \widetilde{X} \to X$ *is the minimal resolution. Let* $E_1, \dots, E_M$ *are the irreducible components of* $\pi_X^{-1}(0)$ *and*

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$E_j \cdot E_k, j,k = 1,\dots,M$ are the intersection products. Then the E_j are smooth complete curves and the matrix $(E_j \cdot E_k)$ is negative definite. Moreover,

$$DCI_{r,d} = \Big(\sum_{j=1}^{M} a_i E_i\Big)^2$$

with a_i satisfying the following equations

$$\sum_{k=1}^{M} a_k(E_j \cdot E_k) = 2g(E_j) - 2 - E_j^2, \qquad j = 1,\dots,M.$$

In particular, $DCI_{r,d}$ is determined only by the dual graph of $(X,0)$.

Computation of the second Chern number is relatively easier. Just notice that resolution of a singular point has the topological effect of replacing the point by its exceptional divisor. Thus, by a similar argument as above and using the additivity of the Euler number, we obtain the following theorem.

Theorem 3.5 *For the second Chern number c_2, the following hold:*

(i) *Assume that $p_1,\dots,p_s$ are all the singular points of $\overline{F}$ and r_i is the multiplicity of p_i. Then*

$$c_2(\widetilde{F}) = \chi(\overline{F}) + \sum_{i=1}^{s} DCII_{r_i,d} = \chi(\overline{F}) + \sum_r t_r DCII_{r,d}.$$

(ii) *Assume that $(X,0)$ is an isolated surface singularity germ defined by $G_r(u,v) + t^d = 0$, where G_r is a product of $r \le d$ distinct linear forms and $\pi_X\colon \widetilde{X} \to X$ is the minimal resolution. Then*

$$DCII_{r,d} = \chi(\pi_X^{-1}(0)) - 1.$$

In particular, $DCII_{r,d}$ is determined only by the dual graph of $(X,0)$.

From Theorems 3.4 and 3.5, it follows that the computation of $MY(\widetilde{F})$ can also be localized, and in fact, by (3.3),

$$\begin{aligned} MY(\widetilde{F}) &= MY(\overline{F}) + \sum_r t_r DMY_{r,d} \\ &= \sum_r t_r\big((d-1)(r-1)(3-r) + DMY_{r,d}\big). \end{aligned} \tag{3.7}$$

For later convenience, we set

$$\mathcal{E}_{r,d} = (d-1)(r-1)(3-r) + DMY_{r,d}.$$

4 Resolution of Singularities

In this section, we consider singularities of the type $(X,0) : f(u,v,t) = 0$ with $f(u,v,t) = G_r(u,v) + t^d$, where $G_r(u,v)$ is a product of r distinct linear forms in u,v. Such a type of singularity in fact belongs to a special class of singularities, namely weighted homogeneous singularities, whose resolutions are explicitly known.

Using the recalled Theorem 4.1 about the structure of the dual graph, we compute the main numerical invariants $DCI_{r,d}$ and $DCII_{r,d}$ for some special cases, while such computation for the general case will be completed in the next section.

4.1 Weighted Homogeneous Singularities

Consider the $\mathbb{C}^*$ action on $\mathbb{C}^3$ given by

$$a \cdot (z_1, z_2, z_3) = (a^{w_1} z_1, a^{w_2} z_2, a^{w_3} z_3),$$

where the weights $w_i = \text{weight}(z_i)$ are strictly positive integers satisfying

$$\gcd(w_1, w_2, w_3) = 1.$$

An isolated surface singularity $(X', 0) : f'(z_1, z_2, z_3) = 0$ is called *weighted homogeneous* of degree N for the weights w_i if

$$a \cdot f'(z_1, z_2, z_3) = f'(a^{w_1} z_1, a^{w_2} z_2, a^{w_3} z_3) = a^N f'(z_1, z_2, z_3), \qquad \forall a \in \mathbb{C}^*.$$

Theorem 4.1 (see [15] and [6, Section 4.10]) *Let $(X, 0) : f(u, v, t) = G_r(u, v) + t^d = 0, r \le d$ be an isolated weighted homogeneous singularity of degree $N = rd/\gcd(r, d)$, where $G_r(u, v)$ is a product of r distinct linear forms in u, v, for the weights*

$$\begin{aligned} w_1 &= \text{weight}(u) = d/\gcd(r, d),\\ w_2 &= \text{weight}(v) = d/\gcd(r, d),\\ w_3 &= \text{weight}(t) = r/\gcd(r, d). \end{aligned}$$

Then there is a resolution $\pi\colon \widetilde{X} \to X$ such that the following hold.

(i) *There is a $\mathbb{C}^*$ action on $\widetilde{X}$ under which the morphism π is equivariant.*

(ii) *The exceptional divisor $\pi^{-1}(0)$ has exactly one component, denoted by E_0, that is fixed pointwise by the $\mathbb{C}^*$ action on $\widetilde{X}$.*

(iii) *$\pi^{-1}(0)$ has the form*

$$\pi^{-1}(0) = E_0 \cup E_1 \cup \cdots \cup E_\lambda,$$

where for $k = 1, \ldots, \lambda$, $E_k = E_k^1 \cup \cdots \cup E_k^r$ is a disjoint union of r curves, corresponding to vertices at distance k from the center in the dual graph below.

(iv) *For each $k = 1, \ldots, \lambda$ and $j = 1, \ldots, r$, the curve E_k^j is a smooth rational irreducible curve and has self-intersection $(E_k^j)^2 = -n_k \le -2$ (independent of j).*

(v) *E_0 is a smooth complete curve of genus*

$$\begin{aligned} g(E_0) &= \frac{1}{2}\left[\frac{N^2}{w_1 w_2 w_3} - \sum_{i<j} \frac{N \gcd(w_i, w_j)}{w_i w_j} + \sum_i \frac{\gcd(N, w_i)}{w_i} - 1\right]\\ &= \frac{1}{2}(r-2)(\gcd(r, d) - 1). \end{aligned}$$

(vi) *The components E_0, E_k^j meet transversally according to the following star-shaped graph*

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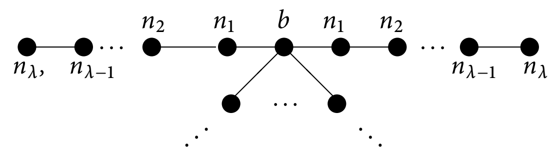


where the central vertex corresponds to E_0 *and there are exactly* r *arms, having the same length* λ *and the same weight sequences* $n_1, \dots, n_\lambda$.

(vii) *Moreover, the above dual graph satisfies the following: if we index the arms* $1, 2, \dots, r$ *from leftmost to right by the anticlockwise order and go along the arm indexed by* j *from the end closest to* E_0 *to the one farthest to* E_0*, we get, in order, the vertices corresponding to the curves* $E_1^j, E_2^j, \dots, E_\lambda^j$.

(viii) *Let* $\alpha = w_1 = d/\gcd(r,d)$ *and* $b' = w_3 = r/\gcd(r,d)$*. When* $\alpha = 1$*, there are in fact no arms,* i.e., $\lambda = 0$*. In this case, let* $\beta = 0$*. When* $\alpha > 1$*, choose* $0 < \beta < \alpha$ *such that* $\beta b' \equiv -1 \bmod \alpha$*. Then the weights of the vertices of the dual graph are determined as follows:*

- *The weight of the central vertex is*

$$b = \frac{N}{w_1 w_2 w_3} + r\beta/\alpha = \frac{\gcd(r,d)(1 + b'\beta)}{\alpha}.$$

- *The weight sequence* $(n_1, \dots, n_\lambda)$ *along each arm is given by the following continued fraction decomposition*

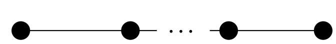
$$\frac{\alpha}{\beta} = n_1 - \cfrac{1}{n_2 - \cfrac{1}{\cdots - \cfrac{1}{n_\lambda}}}.$$

4.2 Examples of Minimal Resolutions

Assume that X is a surface on a smooth projective threefold Y with only on singular point 0 and that $(X, 0) : f(u, v, t) = G_r(u, v) + t^d = 0$, where (u, v, t) is the local coordinate system on Y around 0 and G_r is a product of r distinct linear binary forms.

Let $\pi\colon \widetilde{X} \to X$ be the resolution obtained by Theorem 4.1 and let $\pi_m\colon \widetilde{X}^m \to X$ be the minimal resolution of X.

Example 4.2 When $r = 2$, $(X, 0)$ is a singularity of type A_{d-1}, and its minimal resolution $\pi_m\colon \widetilde{X}^m \to X$ is well-known: the dual graph is



where there are $(d-1)$ vertices and each vertex has weight 2. Moreover, $K_{\widetilde{X}^m} = \pi_m^* K_X$ (see [17]), so by Theorems 3.4 and 3.5, we obtain

$$DCI_{r,d} = 0 \quad \text{and} \quad DCII_{r,d} = d - 1,$$

and hence

$$DMY_{r,d} = 3(d-1), \qquad \mathcal{E}_{r,d} = (d-1)(r-1)(3-r) + DMY_{r,d} = 4(d-1).$$

Note that when $r = 2$ and $d = rp+1$ for $p \geq 1$, the resolution $\widetilde{X} \to X$ is not minimal. Indeed, the central curve E_0 is a (-1)-curve, *i.e.*, $g(E_0) = 0$ and $b = 1$ in Theorem 4.1. More generally, we have the following proposition.

Proposition 4.3 *The resolution $\widetilde{X} \to X$ is not minimal if and only if $d \equiv 1 \bmod r$.*

Proof The resolution is not minimal only if E_0 is a (-1)-curve, since $(E_k^j)^2 = -n_k \leq -2$ for $k \geq 1$. This is the case if and only if that $g(E_0) = 0$ and $b = 1$, namely,

$$0 = g(E_0) = \tfrac{1}{2}(r-2)\big(\gcd(r,d) - 1\big),$$
$$1 = b = \gcd(r,d)(b'\beta + 1)/\alpha.$$

From the second equality, it follows that $\gcd(r,d) = 1$ and $b'\beta + 1 = \alpha$. Now from $\gcd(r,d) = 1$, we have, by definition, $\alpha = d/\gcd(r,d) = d$ and $b' = r/\gcd(r,d) = r$, so $d = r\beta + 1$. ∎

Thus, if d cannot be written as $d = rp + 1$ for some $p \geq 1$, the resolution $\widetilde{X} \to X$ given in Theorem 4.1 is already minimal. The canonical divisor $K_{\widetilde{X}}$ has the form

$$K_{\widetilde{X}} = \pi^* K_X + a_0 E_0 + \sum_{k,j} a_k^j E_k^j.$$

By considering the adjunction formula, we have

$$E_0 \cdot K_{\widetilde{X}} = 2g(E_0) - 2 - E_0^2 = 2g(E_0) - 2 + b,$$

and for all k, j,

$$E_k^j \cdot K_{\widetilde{X}} = 2g(E_k^j) - 2 - (E_k^j)^2 = -2 + n_k,$$

hence, by the projection formula and Theorem 4.1, we get a system of equations

$$\begin{aligned} &-ba_0 + (a_1^1 + \cdots + a_1^r) = (r-2)(\gcd(r,d) - 1) - 2 + b, \\ &-n_k a_k^j + (a_{k-1}^j + a_{k+1}^j) = -2 + n_k, \quad \forall k, j, \end{aligned} \tag{4.1}$$

where we have set $a_0^j = a_0$ and $a_{\lambda+1}^j = 0$ for all j.

For $j = 1, \ldots, r$, set

$$\mathbf{E^j} = (E_1^j, \ldots, E_\lambda^j) \quad \text{and} \quad \mathbf{a^j} = (a_1^j, \ldots, a_\lambda^j);$$

then

$$K_{\widetilde{X}} = \pi^* K_X + a_0 E_0 + \sum_{j=1}^{r} \mathbf{E^j} (\mathbf{a^j})^T.$$

The intersection matrix of E_0, E_k^l is negative definite (see [14]), so from (4.1) we can uniquely solve for the a_0 and $\mathbf{a^j}$. Moreover, we can see that if $(a_0, \mathbf{a^1}, \ldots, \mathbf{a^r})$ is a solution of the system (4.1), then $(a_0, \mathbf{a^j}, \mathbf{a^2}, \ldots, \mathbf{a^{j-1}}, \mathbf{a^1}, \mathbf{a^{j+1}}, \ldots, \mathbf{a^r})$ is also a solution for any $j > 1$; hence, from the uniqueness of the solution, it follows that $a_k^1 = a_k^2 = \cdots = a_k^r$ for all k.

Write

$$E_k = E_k^1 + E_k^2 + \cdots + E_k^r, \qquad k = 1, \ldots, \lambda.$$

Then

$$K_{\widetilde{X}} = \pi^* K_X + \sum_{k=0}^{\lambda} a_k E_k,$$

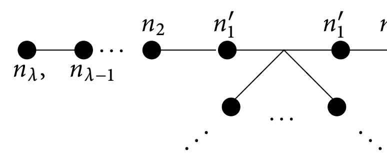
with the a_k satisfying

$$(4.2) \qquad -ba_0 + ra_1 = (r-2)(\gcd(r,d) - 1) - 2 + b$$
$$-n_k a_k + (a_{k-1} + a_{k+1}) = -2 + n_k, \quad k = 1, \ldots, \lambda,$$

where $a_{\lambda+1} = 0$. In particular,

$$DCI_{r,d} = \Big(\sum_{k=0}^{\lambda} a_k E_k\Big)^2.$$

Now assume that $d = rp + 1$ for some $p \geq 1$; then the resolution $\widetilde{X} \to X$ is not minimal and E_0 is a (-1)-curve. By blowing down E_0, we get another resolution $\widetilde{X}'$ of X, and moreover, since in this case $\alpha = b'\beta + 1 = r\beta + 1$, by performing the continued fraction decomposition of $\alpha/\beta = (r\beta + 1)/\beta$, we have $n_1 = r + 1 \geq 3$; hence, $\widetilde{X}'$ is the minimal resolution $\widetilde{X}^m$ of X, with the dual graph being



where $n_1' = n_1 - 1$ and there is no central vertex, meaning that for the r exceptional curves $E_1^1, \ldots, E_1^r$ corresponding to the vertices of weight n_1', we have $E_1^j \cdot E_1^{j'} = 1$ for $j \neq j'$. In particular, the exceptional divisor does not have normal crossings. Moreover, a similar argument gives that

$$DCI_{r,d} = \Big(\sum_{k=1}^{\lambda} a_k E_k\Big)^2,$$

with the a_k satisfying

$$-n_1' a_1 + (r-1)a_1 + a_2 = -2 + n_1',$$
$$-n_k a_k + (a_{k-1} + a_{k+1}) = -2 + n_k, \quad k = 2, \ldots, \lambda,$$

where $a_{\lambda+1} = 0$.

For later convenience, we first give the formulae $DCI_{r,d}$ and $DCII_{r,d}$ for the case $d = rp + 1$.

Example 4.4 Let $r \geq 3$ and $d = rp + 1$, $p \geq 1$. Then according to Theorem 4.1, we have

(i) $\alpha = d/\gcd(r,d) = d$ and $b' = r/\gcd(r,d) = r$, so $\alpha = b'p + 1$; since $0 < \beta < \alpha$ is chosen so that $b'\beta \equiv -1 \bmod \alpha$, we have $\beta = p$;

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(ii) we have $\alpha/\beta = (rp+1)/p$, so considering the continued fraction decomposition, we have $\lambda = p$, and $n_1 = r+1$, $n_2 = n_3 = \cdots = n_\lambda = 2$.

By the discussions above,

$$DCI_{r,d} = \Big(\sum_{k=1}^{\lambda} a_k E_k\Big)^2$$

with

$$\begin{aligned} -ra_1 + a_2 + (r-1)a_1 &= -2 + r \\ -2a_2 + (a_1 + a_3) &= 0 \\ -2a_2 + (a_2 + a_4) &= 0 \\ &\vdots \\ -2a_{\lambda-1} + (a_{\lambda-2} + a_\lambda) &= 0 \\ -2a_\lambda + a_{\lambda-1} &= 0. \end{aligned}$$

By considering the above equations from the bottom to the second top one, we have

$$a_k = (\lambda + 1 - k)a_\lambda, \quad k = 1, \dots, \lambda - 1;$$

hence, from the first equation, we get $a_\lambda = -(r-2)$. It follows that

$$DCI_{r,d} = (r-2)^2(E_p + 2E_{p-1} + \cdots + pE_1)^2.$$

Note that $E_k^2 = -rn_k = -2r$ for $k > 1$ and

$$E_1^2 = (E_1^1 + \cdots + E_1^r)^2 = -r;$$

moreover, for $k \neq k'$,

$$E_k \cdot E_{k'} = \begin{cases} r & \text{for } k' = k \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we have

$$DCI_{r,d} = (r-2)^2(E_p + 2E_{p-1} + \cdots + pE_1)^2 = -(r-2)^2 rp = -(d-1)(r-2)^2.$$

In addition, we have $DCII_{r,d} = r\lambda = rp = d - 1$; thus,

$$DMY_{r,d} = 3\,DCII_{r,d} - DCI_{r,d} = 3(d-1) + (d-1)(r-2)^2.$$

Consequently,

$$\mathcal{E}_{r,d} = DMY_{r,d} + (d-1)(r-1)(3-r) = 4(d-1).$$

5 Proof of Theorem 1.1

In this section, we consider the general case of Theorem 4.1. Although our method applies for more general situations, we assume that $r \geq 3$ and $d \not\equiv 1 \bmod r$, since otherwise we are done by Examples 4.2 and 4.4. In particular, the resolution $\pi\colon \widetilde{X} \to X$ given in Theorem 4.1 is minimal.

By the discussions before Example 4.4, we have

$$DCI_{r,d} = \Big(\sum_{k=0}^{\lambda} a_k E_k\Big)^2$$

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with a_k satisfying

$$-ba_0 + ra_1 = (r-2)(\gcd(r,d)-1) - 2 + b$$
$$-n_k a_k + (a_{k-1} + a_{k+1}) = -2 + n_k, \quad k = 1, \dots, \lambda,$$

where $a_{\lambda+1} = 0$.

The hard part, which will be accomplished in this section, is to find a compact formula for the invariant $DCI_{r,d}$ without the implicit use of the a_i. Essentially, we achieve this goal by applying an effective method to deal with the continued fraction decomposition coming from Theorem 4.1.

The main results are formulae (5.7) and (5.8), which we summarized in Theorem 1.1.

5.1 Continued Fraction Decomposition

In order to apply Theorem 4.1, we first deal with the continued fraction decomposition

$$\frac{\alpha}{\beta} = n_1 - \cfrac{1}{n_2 - \cfrac{1}{\cdots - \cfrac{1}{n_\lambda}}}.$$

Recall that β is chosen such that $b'\beta \equiv -1 \bmod \alpha$, hence $\gcd(\alpha,\beta) = 1$. Let

$$\alpha_0, \alpha_1, \dots, \alpha_{\lambda-1}, \alpha_\lambda = 1, \alpha_{\lambda+1} = 0$$

be a sequence of natural numbers such that $\gcd(\alpha_i, \alpha_{i+1}) = 1$ for $i = 0, 1, \dots, \lambda$ and

$$\frac{\alpha_i}{\alpha_{i+1}} = n_{i+1} - \cfrac{1}{n_{i+2} - \cfrac{1}{\cdots - \cfrac{1}{n_\lambda}}}, \qquad i = 0, 1, \dots, \lambda - 1. \tag{5.1}$$

Clearly, the numbers α_i are uniquely determined by the continued fraction decomposition above, and $\alpha_i > 0$ for $i < \lambda + 1$.

Moreover, by definition (5.1), we have

$$\frac{\alpha_{i-1}}{\alpha_i} = n_i - \frac{1}{\alpha_i/\alpha_{i+1}} = \frac{n_i\alpha_i - \alpha_{i+1}}{\alpha_i};$$

hence,

$$\alpha_{i-1} = n_i\alpha_i - \alpha_{i+1},$$

or, in another more convenient formulation,

$$\begin{pmatrix} \alpha_{i-1} \\ \alpha_i \end{pmatrix} = \begin{pmatrix} n_i & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_i \\ \alpha_{i+1} \end{pmatrix}. \tag{5.2}$$

For $i = 1, \dots, \lambda$, set

$$\mathbf{G_i} = \begin{pmatrix} n_i & -1 \\ 1 & 0 \end{pmatrix} \tag{5.3}$$

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be a 2×2 matrix. Then the relation (5.2) can be formulated as

$$\begin{pmatrix} \alpha_{i-1} \\ \alpha_i \end{pmatrix} = \mathbf{G_i} \begin{pmatrix} \alpha_i \\ \alpha_{i+1} \end{pmatrix}.$$

Thus, we have

$$\begin{pmatrix} \alpha_{i-1} \\ \alpha_i \end{pmatrix} = \mathbf{G_i G_{i+1}} \cdots \mathbf{G}_\lambda \begin{pmatrix} \alpha_\lambda \\ \alpha_{\lambda+1} \end{pmatrix} = \mathbf{G_i G_{i+1}} \cdots \mathbf{G}_\lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5.4}$$

for all $i \geq 1$.

Note also that by definition (5.1) and our conventions, $\alpha_0 = \alpha$ and $\alpha_1 = \beta$.

Let $\mathbf{G} = \mathbf{G_1 G_2} \cdots \mathbf{G}_\lambda$; then by (5.4), we have

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \mathbf{G} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

So $\mathbf{G}$ is of the form

$$\mathbf{G} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$$

for some integers γ, δ. In fact, we have the following more precise result.

Proposition 5.1 *With the notation as above and in Theorem 4.1, we have*

$$\mathbf{G} = \begin{pmatrix} \alpha & b' - \alpha \\ \beta & \frac{1+b'\beta}{\alpha} - \beta \end{pmatrix},$$

namely, $\gamma = b' - \alpha$ *and* $\delta = -\beta + (1 + b'\beta)/\alpha$.

Proof First, we establish the following claim.

Claim 5.2 $-\alpha < \gamma \leq 0$ *and* $-\beta < \delta \leq 0$.

Assuming the claim, note that by definition,

$$\det \mathbf{G} = \alpha\delta - \beta\gamma = 1;$$

hence, $\beta\gamma \equiv -1 \bmod \alpha$. Recall also that $b'\beta \equiv -1 \bmod \alpha$, so we have $\gamma = b' - \alpha$, since $\gamma, b' - \alpha \in (-\alpha, 0]$, and the equation $\beta x \equiv -1 \bmod \alpha$ admits a unique solution satisfying $x \in (-\alpha, 0]$. In addition,

$$\delta = \frac{1 + \beta\gamma}{\alpha} = \frac{1 + \beta(b' - \alpha)}{\alpha} = \frac{1 + b'\beta}{\alpha} - \beta.$$

Proof of Claim 5.2 For $i \geq 1$, let

$$\begin{pmatrix} \xi_i & \gamma_i \\ \eta_i & \delta_i \end{pmatrix} = \mathbf{G_1 G_2} \cdots \mathbf{G_i};$$

then $\xi_i, \eta_i, \gamma_i, \delta_i$ are all integers. It suffices to show the following:

(i) $\xi_i, \eta_i > 0$ for all i;

(ii) $\gamma_i \in (-\xi_i, 0]$ and $\delta_i \in (-\eta_i, 0]$ for all i.

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We prove this by induction on i. When $i = 1$, we have

$$\begin{pmatrix} \xi_1 & \gamma_1 \\ \eta_1 & \delta_1 \end{pmatrix} = \mathbf{G_1} = \begin{pmatrix} n_1 & -1 \\ 1 & 0 \end{pmatrix},$$

and the conclusion obviously holds. Now assuming the validity of the result for i, we have

$$\begin{pmatrix} \xi_{i+1} & \gamma_{i+1} \\ \eta_{i+1} & \delta_{i+1} \end{pmatrix} = \begin{pmatrix} \xi_i & \gamma_i \\ \eta_i & \delta_i \end{pmatrix} \mathbf{G_{i+1}} = \begin{pmatrix} \xi_i & \gamma_i \\ \eta_i & \delta_i \end{pmatrix} \begin{pmatrix} n_{i+1} & -1 \\ 1 & 0 \end{pmatrix}.$$

Therefore,

(i) $\xi_{i+1} = n_{i+1}\xi_i + \gamma_i > 2\xi_i - \xi_i > 0$, since $n_{i+1} \geq 2$ and by inductive hypothesis, $\xi_i > 0$ and $\gamma_i \in (-\xi_i, 0]$. Similarly, $\eta_{i+1} = n_{i+1}\eta_i + \gamma_i > 0$ by the inductive hypothesis $\eta_i > 0$ and $\gamma_i \in (-\eta_i, 0]$;

(ii) $\gamma_{i+1} = -\xi_i < 0$ since $\xi_i > 0$; in addition,

$$\gamma_{i+1} + \xi_{i+1} = -\xi_i + (n_{i+1}\xi_i + \gamma_i) > (n_{i+1} - 2)\xi_i \geq 0,$$

since $n_{i+1} \geq 2$ and $\gamma_i > -\xi_i$ by the inductive hypothesis; similarly, $\delta_{i+1} = -\eta_i < 0$ and

$$\delta_{i+1} + \eta_{i+1} = -\eta_i + (n_{i+1}\eta_i + \delta_i) > (n_{i+1} - 2)\eta_i \geq 0.$$

We are done. ∎

5.2 Compact Formulae for $DCI_{r,d}$ and $DCII_{r,d}$

As stated in the beginning of this section,

$$DCI_{r,d} = \sum_{i=0}^{\lambda} a_i^2 E_i^2 + 2\sum_{i=0}^{\lambda-1} a_i a_{i+1} E_i \cdot E_{i+1}.$$

Recall that $E_0^2 = -b$ and $E_i^2 = -rn_i$ for $i > 0$. In addition, for $i \neq i'$,

$$E_i \cdot E_{i'} = \begin{cases} r & \text{for } i' = i \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we have

$$DCI_{r,d} = -ba_0^2 + r\sum_{i=1}^{\lambda} a_i(2a_{i-1} - n_i a_i).$$

Since $-n_i a_i + a_{i-1} + a_{i+1} = -2 + n_i$, it follows that

$$DCI_{r,d} = a_0(-ba_0 + ra_1) + r\Big(\sum_{i=1}^{\lambda} n_i a_i - 2\sum_{i=1}^{\lambda} a_i\Big). \tag{5.5}$$

Using the equality $-n_i a_i + a_{i-1} + a_{i+1} = -2 + n_i$ again, we get

$$\begin{aligned}\sum_{i=1}^{\lambda} n_i a_i - 2\sum_{i=1}^{\lambda} a_i &= \sum_{i=1}^{\lambda}\big(a_{i-1} + a_{i+1} - (n_i - 2)\big) - 2\sum_{i=1}^{\lambda} a_i \\ &= -\sum_{i=1}^{\lambda}(n_i - 2) + (a_0 - a_1 - a_\lambda);\end{aligned}$$

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consequently, by (5.5), we obtain

$$DCI_{r,d} = a_0(-ba_0 + ra_1) - r\sum_{i=1}^{\lambda}(n_i - 2) + r(a_0 - a_1 - a_\lambda). \tag{5.6}$$

To compute a_0, a_1 and a_λ, we use the whole system of equations in (4.2). Let $a_i^* = a_i + 1$ for $i = 0, 1, \dots, \lambda + 1$. Recall also that $a_{\lambda+1} = 0$. Then the equations in (4.2) can be reformulated into a more convenient form:

$$\begin{aligned} -ba_0^* + ra_1^* &= \gcd(r,d)(r-2) \\ -n_1 a_1^* + (a_0^* + a_2^*) &= 0 \\ -n_2 a_2^* + (a_1^* + a_3^*) &= 0 \\ &\vdots \\ -n_{\lambda-1} a_{\lambda-1}^* + (a_{\lambda-2}^* + a_\lambda^*) &= 0 \\ -n_\lambda a_\lambda^* + (a_{\lambda-1}^* + a_{\lambda+1}^*) &= 0. \end{aligned}$$

With the help of the matrices $\mathbf{G_i}$ defined in (5.3), we have

$$\begin{pmatrix} a_{i-1}^* \\ a_i^* \end{pmatrix} = \mathbf{G_i} \begin{pmatrix} a_i^* \\ a_{i+1}^* \end{pmatrix};$$

hence,

$$\begin{pmatrix} a_0^* \\ a_1^* \end{pmatrix} = \mathbf{G_1} \cdots \mathbf{G}_\lambda \begin{pmatrix} a_\lambda^* \\ a_{\lambda+1}^* \end{pmatrix} = \mathbf{G} \begin{pmatrix} a_\lambda^* \\ 1 \end{pmatrix}.$$

By Proposition 5.1, we thus have

$$\begin{cases} a_0^* = \alpha a_\lambda^* + (b' - \alpha) = \alpha a_\lambda + b' \\ a_1^* = \beta a_\lambda^* + (\frac{1+b'\beta}{\alpha} - \beta) = \beta a_\lambda + (1 + b'\beta)/\alpha. \end{cases}$$

Furthermore, it also holds $-ba_0^* + ra_1^* = \gcd(r,d)(r-2)$; thus, we obtain three equations in a_0, a_1, a_λ. The solution is as follows; the proof involves only direct computations and is left to the reader.

Lemma 5.3

$$\begin{cases} a_0 = (2-r)\alpha + b' - 1, \\ a_1 = (2-r)\beta + \frac{1+b'\beta}{\alpha} - 1, \\ a_\lambda = -(r-2). \end{cases}$$

Therefore, it follows from (5.6) that

$$DCI_{r,d} = -d(r-2)^2 - r\sum_{i=1}^{\lambda}(n_i - 2) + 2(r-2)\big(r - \gcd(r,d)\big) + (r - b). \tag{5.7}$$

Furthermore, we have

$$DCII_{r,d} = -1 + \chi(E_0) + r\lambda = 1 + r\lambda - (r-2)\big(\gcd(r,d) - 1\big). \tag{5.8}$$

Indeed, $\widetilde{X}$ is essentially obtained from X by replacing 0 by $(1+r\lambda)$ curves intersecting according to the dual graph; E_0 contributes to $\chi(E_0)$ for $\chi(\widetilde{X})$; each arm in the dual

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graph gives rise to a disjoint union of λ copies of $\mathbb{P}^1 \setminus \{\text{one point}\} \cong \mathbb{C}$ and hence contributes λ for $\chi(\widetilde{X})$.

5.3 Estimates of the Miyaoka–Yau Number

Now we continue to consider the Miyaoka–Yau number. By definition, $DMY_{r,d} = 3\,DCII_{r,d} - DCI_{r,d}$. Hence, in view of (5.7) and (5.8), we get

$$DMY_{r,d} = \Big(3(1+r\lambda) + r\sum_{i=1}^{\lambda}(n_i - 2)\Big) + (d-1)(r-2)^2 - \Big((r-2)\big(\gcd(r,d) + r - 1\big) + (r-b)\Big).$$

By definition, $\mathcal{E}_{r,d} = DMY_{r,d} + (d-1)(r-1)(3-r)$; it follows that

$$\mathcal{E}_{r,d} = \Big(r\sum_{i=1}^{\lambda}(n_i + 1)\Big) + (d+2) - \Big((r-2)\big(\gcd(r,d) + r - 1\big) + (r-b)\Big). \tag{5.9}$$

We need an estimate of $\mathcal{E}_{r,d}$. First, we have

$$(r-2)\big(\gcd(r,d) + r - 1\big) + (r-b) \le (r-2)(2r-1) + r = 2(r-1)^2,$$

so

$$\mathcal{E}_{r,d} \ge \Big(r\sum_{i=1}^{\lambda}(n_i + 1)\Big) + (d+2) - 2(r-1)^2 > -2r(r-1). \tag{5.10}$$

Remark 5.4 The above estimate is also true when $d \equiv 1 \bmod r$ by Example 4.4 and when $r = 2$ by Example 4.2.

As an application of the above calculations, we give new examples of computing Chern numbers and $\mathcal{E}_{r,d}$ by directly using formulae (5.7)–(5.9).

Example 5.5 Let $r \ge 3$ and $d = rp$, $p \ge 1$. Then the resolution $\pi\colon \widetilde{X} \to X$ in Theorem 4.1 is minimal.

(i) We have $\gcd(r,d) = r$, so $\alpha = d/\gcd(r,d) = p$ and $b' = r/\gcd(r,d) = 1$. Since $\alpha = b'p$ and by assumption β is chosen so that $0 \le \beta < \alpha$ satisfying $b'\beta \equiv -1 \bmod \alpha$, we have $\beta = p - 1$.

(ii) We get

$$b = \frac{\gcd(r,d)(1 + b'\beta)}{\alpha} = r.$$

(iii) We have

$$\frac{\alpha}{\beta} = \frac{p}{p-1};$$

doing the continued fraction decomposition, we see that

$$\lambda = p - 1, \qquad n_1 = \cdots = n_\lambda = 2.$$

Therefore,

$$r\sum_{i=1}^{\lambda}(n_i + 1) = 3r\lambda = 3r(p-1) = 3d - 3r.$$

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(iv) Eventually, by (5.7), we have

$$DCI_{r,d} = -d(r-2)^2;$$

by (5.8), we have

$$DCII_{r,d} = 1 + r(p-1) - (r-2)(r-1) = d - (r-1)^2;$$

by (5.9), we obtain

$$\begin{aligned}\mathcal{E}_{r,d} &= (3d-3r) + (d+2) - \big((r-2)(2r-1)+0\big)\\ &= 4d - 2r(r-1).\end{aligned}$$

Example 5.6 Let $r \geq 3$ and $d = r(p-1) + (r-1) = rp - 1$ for $p \geq 2$.

(i) We have $\gcd(r,d) = 1$, so $\alpha = d/\gcd(r,d) = d$ and $b' = r/\gcd(r,d) = r$. Since $\alpha = b'p - 1$ and by assumption β is chosen so that $0 \leq \beta < \alpha$ satisfying $b'\beta \equiv -1 \bmod \alpha$, we have $\beta = \alpha - p = p(r-1) - 1$.

(ii) We get

$$b = \frac{\gcd(r,d)(1+b'\beta)}{\alpha} = r - 1.$$

(iii) We have

$$\frac{\alpha}{\beta} = \frac{d}{r(p-1)-1} = \frac{rp-1}{r(p-1)-1},$$

doing the continued fraction decomposition, we see that $\lambda = p + r - 3$, and

$$n_1 = \cdots = n_{p-2} = 2, \qquad n_{p-1} = 3, \qquad n_p = n_{p+1} = \cdots = n_{p+r-3} = 2.$$

Therefore,

$$r\sum_{i=1}^{\lambda}(n_i+1) = r(3\lambda+1) = 3r(p+r) - 8r = 3(d+1) + 3r^2 - 8r.$$

(iv) Eventually, by (5.7), we have

$$DCI_{r,d} = -d(r-2)^2 + (2r-5)(r-1);$$

by (5.8), we have

$$DCII_{r,d} = 1 + r(p+r-3) = d + (r-1)(r-2);$$

by (5.9), we obtain

$$\begin{aligned}\mathcal{E}_{r,d} &= \big(3(d+1) + 3r^2 - 8r\big) + (d+2) - \big(r(r-2)+1\big)\\ &= 4(d+1) + 2r(r-3).\end{aligned}$$

Consequently, when $r = 3$, we have the following:

(a) when $3|d$, we have $\mathcal{E}_{3,d} = 4d - 12$ by Example 5.5;
(b) when $d \equiv 1 \bmod 3$, we have $\mathcal{E}_{3,d} = 4(d-1)$ by Example 4.4;
(c) when $d \equiv 2 \bmod 3$, we have $\mathcal{E}_{3,d} = 4(d+1)$ by the results above.

In particular, when $d \geq 4$, it is always true that $\mathcal{E}_{3,d} \geq 4(d-3)$.

6 Proof of Theorem 1.2

Let $\pi: \widetilde{F} \to \overline{F}$ be the minimal resolution obtained in previous sections. We prove that $MY(\widetilde{F}) \neq 0$ under the assumption of Theorem 1.2.

The proof will be divided into three cases with respect to the values of t_d and t_{d-1}.

Case 1 When the lines in $\mathcal{A}$ form a pencil, namely, $t_d = 1$, we have, $d \neq 3$ by the assumption of Theorem 1.2; moreover, by Examples 4.2 and 5.5 and formula (3.7),

$$MY(\widetilde{F}) = \mathcal{E}_{d,d} = 4d - 2d(d-1) = 2d(3-d).$$

Case 2 If $t_d = 0$ while $t_{d-1} \neq 0$, then we have $t_{d-1} = 1$ and $t_2 = d-1$ (if $d = 3$, $t_2 = d = 3$). Moreover, by Examples 4.2 and 4.4, in view of (3.7), we have

$$\begin{aligned} MY(\widetilde{F}) &= t_2\mathcal{E}_{2,d} + t_{d-1}\mathcal{E}_{r,d} = (d-1)\big(4(d-1)\big) + 4(d-1) \\ &= 4d(d-1) > 0. \end{aligned}$$

Case 3 Now we consider the case $t_d = 0$, $t_{d-1} = 0$. Then by estimate (5.10), we have

$$\begin{aligned} MY(\widetilde{F}) &= t_2\mathcal{E}_{2,d} + t_3\mathcal{E}_{3,d} + \sum_{r\geq 4} t_r\mathcal{E}_{r,d} \geq t_2\mathcal{E}_{2,d} + t_3\mathcal{E}_{3,d} - 2\sum_{r\geq 4} t_r r(r-1) \\ &= \Big(t_2(\mathcal{E}_{2,d}+4) + t_3(\mathcal{E}_{3,d}+12)\Big) - 2\sum_r t_r r(r-1). \end{aligned}$$

From Remark 3.2, we have $\sum_r t_r r(r-1) = d(d-1)$; moreover, from Example 4.2 and the end of Example 5.6, we deduce that

$$MY(\widetilde{F}) \geq 4d(t_2+t_3) - 2d(d-1) = 2d\big(2(t_2+t_3) - (d-1)\big). \tag{6.1}$$

Now we use the celebrated inequality in the second remark added in proof of [12], which states that

$$t_2 + \frac{3}{4}t_3 \geq d + \sum_{r\geq 5}(r-4)t_r;$$

see also [18] or [21, Appendix A]. In particular, $t_2 + t_3 \geq d$. It follows immediately by (6.1), that $MY(\widetilde{F}) > 0$.

The proof now is complete. ∎

Remark 6.1 When $d = |\mathcal{A}| = 3$ and $\mathcal{A}$ is a pencil, *i.e.*, $t_3 = 1$, $MY(\widetilde{F}) = 0$. Moreover, from Example 5.5, we obtain $DCI_{3,3} = -3$. Hence, by Example 3.1, we have

$$c_1^2(\widetilde{F}) = K_{\widetilde{F}}^2 = K_{\overline{F}}^2 + DCI_{3,3} = 3\times(3-4)^2 - 3 = 0.$$

Moreover, $c_2(\widetilde{F}) = 0$, since $MY(\widetilde{F}) = 3c_2(\widetilde{F}) - c_1^2(\widetilde{F}) = 0$.

Since $c_2 > 0$ for a smooth projective surface of general type (see [1, Chapter VII]), it follows that $\widetilde{F}$ is not of general type, a fortiori, $\widetilde{F}$ is not a ball quotient.

Remark 6.2 Note that a ball quotient cannot admit any rational curves; see [3, Proposition 19]. In fact, for any smooth projective surface X, any given morphism $f: \mathbb{P}^1 \to X$ lifts to a morphism $\widetilde{f}: \mathbb{P}^1 \to \widetilde{X}$, since $\mathbb{P}^1$ is simply connected, where $\widetilde{X}$ is the universal cover of X. If X is a ball quotient, or, equivalently, $\widetilde{X}$ is biholomorphic to a ball, then by the maximum principle, $\widetilde{f}$ and hence f must be constant. When the line arrangement $\mathcal{A}$ is not a pencil, from Theorem 4.1, $\widetilde{F}$ clearly contains rational curves in the exceptional divisors; it follows immediately that $\widetilde{F}$ is not a ball quotient.

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However, we have showed above that $MY(\widetilde{F}) > 0$ when $\mathcal{A}$ is not a pencil. This is stronger than the non-ball-quotient property in the following sense: a non-ball quotient surface X of general type can have $MY(X) > 0$ or $MY(X) < 0$, while our results assert that $MY(\widetilde{F}) < 0$ can never happen even if $\widetilde{F}$ is of general type.

7 Surfaces of General Type Associated with Line Arrangements

Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$. By Theorem 1.2, $MY(\widetilde{F}) > 0$ when $\mathcal{A}$ is not a pencil. It is natural to ask whether $\widetilde{F}$ is a surface of general type.

7.1 A General Type Criterion

We first provide a criterion for a surface to be of general type.

Proposition 7.1 *Let X be a smooth projective surface. If $c_1^2(X) > 9$, then X is of general type.*

Proof Let X' be a minimal model of X. Then X' is obtained by successively blowing down (-1)-curves. Note that once we blow down a (-1)-curve, c_1^2 increases by 1, so $c_1^2(X') \ge c_1^2(X) > 9$; hence, by the Enriques–Kodaira classification of surfaces (see [1, Chapter VI], X' is of general type, and thus, so is X. ∎

7.2 Surfaces Associated with Line Arrangements with only Nodes and Triple Points

In the sequel, we consider surfaces associated with line arrangements such that $t_r = 0$ whenever $r \ge 4$, and we prove Theorem 1.3.

For $r = 2$, by Example 4.2, we have $DCI_{2,d} = 0$ and $DCII_{2,d} = d - 1$.

When $r = 3$, the following hold.

(i) If $3|d$, $DCI_{3,d} = -d$, $DCII_{3,d} = d - 4$ by Example 5.5.
(ii) If $d \equiv 1 \bmod 3$, $DCI_{3,d} = -(d-1)$, $DCII_{3,d} = d - 1$ by Example 4.4.
(iii) If $d \equiv 2 \bmod 3$, we have $DCI_{3,d} = -(d-2)$, $DCII_{r,d} = d + 2$ by Example 5.6.

The Case $3|d$

When $d = 3p$, we have by (3.1) that

$$c_1^2(\widetilde{F}) = K_{\widetilde{F}}^2 + \sum_r t_r DCI_{r,d} = d(d-4)^2 - dt_3 = d\big((d-4)^2 - t_3\big),$$

and by (3.2),

$$c_2(\widetilde{F}) = \chi(\overline{F}) + \sum_r t_r DCII_{r,d} = d(d^2 - 4d + 6) - 3t_3 d.$$

By Remark 3.2, we have $2t_2 + 6t_3 = d(d-1)$; hence, $t_3 \le d(d-1)/6$, and thus when $d = 3p \ge 9$,

$$c_1^2(\widetilde{F}) \ge d\Big((d-4)^2 - \frac{d(d-1)}{6}\Big) > 9.$$

Therefore, $\widetilde{F}$ is of general type by Proposition 7.1. In addition,

$$\frac{c_1^2(\widetilde{F})}{c_2(\widetilde{F})} = \frac{(d-4)^2 - t_3}{d^2 - 4d + 6 - 3t_3} = \frac{1}{3}\Big(1 + \frac{2(d-3)(d-7)}{d^2 - 4d + 6 - 3t_3}\Big).$$

The Case $d \equiv 1 \bmod 3$

When $d = 3p + 1$, we have

$$c_1^2(\widetilde{F}) = K_{\overline{F}}^2 + \sum_r t_r DCI_{r,d} = d(d-4)^2 - (d-1)t_3,$$

$$c_2(\widetilde{F}) = \chi(\overline{F}) + \sum_r t_r DCII_{r,d} = d(d^2 - 4d + 6) - 3(d-1)t_3.$$

Since $2t_2 + 6t_3 = d(d-1)$, we have $t_3 \le d(d-1)/6$, so when $p \ge 2$; or equivalently, $d \ge 7$,

$$c_1^2(\widetilde{F}) \ge d(d-4)^2 - \frac{1}{6}d(d-1)^2 > 9;$$

hence, $\widetilde{F}$ is of general type by Proposition 7.1. In addition,

$$\frac{c_1^2(\widetilde{F})}{c_2(\widetilde{F})} = \frac{d(d-4)^2 - (d-1)t_3}{d(d^2 - 4d + 6) - 3(d-1)t_3} = \frac{1}{3}\Big(1 + \frac{2d(d-3)(d-7)}{d(d^2 - 4d + 6) - 3(d-1)t_3}\Big).$$

The Case $d \equiv 2 \bmod 3$

When $d = 3p + 2$, we have

$$c_1^2(\widetilde{F}) = K_{\overline{F}}^2 + \sum_r t_r DCI_{r,d} = d(d-4)^2 - (d-2)t_3,$$

$$c_2(\widetilde{F}) = \chi(\overline{F}) + \sum_r t_r DCII_{r,d} = d(d^2 - 4d + 6) - 3(d-2)t_3.$$

Since $2t_2 + 6t_3 = d(d-1)$, we have $t_3 \le d(d-1)/6$, so when $p \ge 2$, or, equivalently, $d \ge 8$,

$$c_1^2(\widetilde{F}) \ge d(d-4)^2 - \frac{1}{6}d(d-1)(d-2) > 9;$$

hence, $\widetilde{F}$ is of general type by Proposition 7.1. In addition,

$$\frac{c_1^2(\widetilde{F})}{c_2(\widetilde{F})} = \frac{d(d-4)^2 - (d-2)t_3}{d(d^2 - 4d + 6) - 3(d-2)t_3} = \frac{1}{3}\Big(1 + \frac{2d(d-3)(d-7)}{d(d^2 - 4d + 6) - 3(d-2)t_3}\Big).$$

Conclusion

In any case, $c_1^2(\widetilde{F})/c_2(\widetilde{F})$ is an increasing function in t_3 with fixed $d \ge 7$. As $t_3 \le d(d-1)/6$, it follows that

$$1 \le \liminf_{d\to\infty} \frac{c_1^2(\widetilde{F})}{c_2(\widetilde{F})} \le \limsup_{d\to\infty} \frac{c_1^2(\widetilde{F})}{c_2(\widetilde{F})} \le \frac{5}{3}.$$

Theorem 1.3 follows from the above discussion.

8 Chern Numbers and Hodge Numbers

In this section, we compute the Hodge numbers of the associated surfaces in some examples.

8.1 Relations Between Hodge Numbers and Chern Numbers

Fix a smooth projective surface X. Denote by $q = h^{0,1}(X)$ its irregularity and by $p = h^{0,2}(X)$ its geometric genus. Denote also by b_i, $i = 1, 2, 3, 4$ the Betti numbers of X and by c_1^2, c_2 the Chern numbers, as well as by $h^{s,t}$ the Hodge numbers.

Then by Noether's formula (see [1, Chapter I, the examples after Theorem 5.5]), we first have

$$1 - q + p = \frac{1}{12}(c_1^2 + c_2); \tag{8.1}$$

secondly, from the formula for Euler characteristic, we have

$$2 - 2b_1 + b_2 = c_2. \tag{8.2}$$

Moreover, from Hodge decomposition and Serre duality, we have

$$b_1 = 2q, \quad b_2 = h^{0,2} + h^{2,0} + h^{1,1}, \quad h^{s,t} = h^{t,s} = h^{2-s,2-t}, \quad s, t = 0, 1, 2. \tag{8.3}$$

We can view the equalities (8.1)–(8.3) as equations for the Hodge numbers $h^{s,t}$, assuming known c_1, c_2, q, and we have the solution

$$\begin{cases} h^{0,0} = h^{2,2} = 1, \\ h^{0,1} = h^{1,0} = h^{1,2} = h^{2,1} = q, \\ h^{0,2} = h^{2,0} = \frac{1}{12}(c_1^2 + c_2) - (1 - q), \\ h^{1,1} = -\frac{1}{6}c_1^2 + \frac{5}{6}c_2 + 2q. \end{cases} \tag{8.4}$$

8.2 Computing Hodge Numbers via Chern Numbers

In the sequel, $\mathcal{A}$ is a line arrangement in $\mathbb{P}^2$ and $\widetilde{F}$ is the associated surface. All the Hodge numbers $h^{s,t}$ and Chern numbers c_1^2, c_2 are those of $\widetilde{F}$; namely, we abbreviate the notations $h^{s,t}(\widetilde{F})$ by $h^{s,t}$, etc.

One of motivations of our work is to understand whether the Hodge numbers $h^{s,t}$ are combinatorially determined, one of the main open questions in the theory of line arrangements; see [16]. By Theorem 1.1, the Chern numbers c_1^2, c_2 of $\widetilde{F}$ are determined by the combinatorics of $\mathcal{A}$. On the other hand, the irregularity q is closely related to the monodromy of $h^*\colon H^1(F) \to H^1(F)$, where $h\colon F \to F$ is defined by

$$h(x, y, z) = \big(\exp(2\pi\sqrt{-1}/d)x, \exp(2\pi\sqrt{-1}/d)y, \exp(2\pi\sqrt{-1}/d)z\big).$$

Indeed, as $\widetilde{F}$ can be seen as a smooth compactification of F, we have $H^1(\widetilde{F}) \simeq W_1H^1(F)$, where $W_\bullet$ denotes the weight filtration on the cohomology $H^\bullet(F)$; see, for instance, [6, Appendix C]. In addition, by [8, Theorem 4.1], we have

$$W_1H^1(F) = H^1(F)_{\neq 1} = \ker\big(h^{*d-1} + \cdots + h^* + \mathrm{Id}\colon H^1(F) \to H^1(F)\big)$$

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So

$$2q = b_1(\widetilde{F}) = \dim H^1(F)_{\neq 1}; \tag{8.5}$$

this is known for many line arrangements; see [4]. In fact, in [16], a combinatorial formula for q is given when $\mathcal{A}$ has only double or triple points; more examples are given in [19] where q is computed. A good, and recent, survey of the monodromy computations is [20] and in a recent preprint [9], an effective algorithm to compute q is provided.

We give now some examples in which we compute all the Hodge numbers of the associated surfaces. In the first two examples below, $t_r \neq 0$ only if $r|d$. By Example 5.5, we have

$$DCI_{r,d} = -d(r-2)^2 \quad \text{and} \quad DCII_{r,d} = d - (r-1)^2.$$

Example 8.1 (Hesse arrangement) The Hesse arrangement is defined by

$$Q = xyz\big((x^3+y^3+z^3)^3 - 27x^3y^3z^3\big)$$

with $d = 12$ with $t_2 = 12$, $t_4 = 9$. Moreover, we have $\dim H^1(F)_{\neq 1} = 6$; see [2, Remark 3.3(iii)], thus $q = 3$ by (8.5).

For the Chern numbers, we first have by (3.1),

$$K_{\overline{F}}^2 = d(d-4)^2 = 768.$$

Since $DCI_{2,12} = 0$ and $DCI_{4,12} = -48$, we obtain

$$c_1^2 = K_{\overline{F}}^2 + \sum_r t_r DCI_{r,d} = 336 > 9.$$

So by Proposition 7.1, $\widetilde{F}$ is of general type. Moreover, by (3.2),

$$\chi(\overline{F}) = d(d^2-4d+6) - (d-1)\sum_r t_r(r-1)^2 = 201.$$

Since $DCII_{2,12} = 11$ and $DCII_{4,12} = 3$, we have

$$c_2 = \chi(\overline{F}) + \sum_r t_r DCII_{r,d} = 360.$$

Finally, by formula (8.4), we obtain

$$\begin{aligned}
h^{0,0} &= h^{2,2} = 1,\\
h^{0,1} &= h^{1,0} = h^{1,2} = h^{2,1} = q = 3,\\
h^{0,2} &= h^{2,0} = \frac{1}{12}(c_1^2 + c_2) - (1-q) = 60,\\
h^{1,1} &= -\frac{1}{6}c_1^2 + \frac{5}{6}c_2 + 2q = 250.
\end{aligned}$$

Example 8.2 Consider the arrangement $\mathcal{A}(m,m,3)$ defined by

$$Q = (x^m - y^m)(y^m - z^m)(z^m - x^m) = 0.$$

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Then if $m = 3$, we have $t_3 = 12$ and if $m \neq 3$, we have $t_3 = m^2, t_m = 3$. In addition, $\dim H^1(F)_{\neq 1}$ can be computed by [7, Theorem 1.4]; it follows from (8.5) that

$$q = \begin{cases} 2 & \text{if } m \equiv 0 \bmod 3, \\ 1 & \text{otherwise.} \end{cases}$$

Moreover, by Example 5.5, the following hold:

$$\begin{aligned} DCI_{3,d} &= -d = -3m, & DCII_{3,d} &= d - 4 = 3m - 4 \\ DCI_{m,d} &= -d(m-2)^2 = -3m(m-2)^2, & DCII_{m,d} &= d - (m-1)^2 = 3m - (m-1)^2. \end{aligned}$$

Therefore, by (3.1),

$$c_1^2 = K_{\overline{F}}^2 + \sum_r t_r DCI_{r,d} = 3m(m-2)(5m-2),$$

and by (3.2),

$$c_2 = \chi(\overline{F}) + \sum_r t_r DCII_{r,d} = 9m(m^2 - 2m + 2).$$

(i) First consider the case where $m = 2$. Then $q = 1$, $c_1^2 = 0$, and $c_2 = 36$. Therefore, by formula (8.4), we have

$$\begin{aligned} h^{0,0} &= h^{2,2} = 1, \\ h^{0,1} &= h^{1,0} = h^{1,2} = h^{2,1} = q = 1, \\ h^{0,2} &= h^{2,0} = \frac{1}{12}(c_1^2 + c_2) - (1 - q) = 3, \\ h^{1,1} &= -\frac{1}{6}c_1^2 + \frac{5}{6}c_2 + 2q = 32. \end{aligned}$$

(ii) Second, consider the case where $m = 3$; then $q = 2$. Moreover, $c_1^2 = 117 > 9$, so $\widetilde{F}$ is of general type by Proposition 7.1. In addition, $c_2 = 135$. Therefore, by formula (8.4), we have

$$\begin{aligned} h^{0,0} &= h^{2,2} = 1, \\ h^{0,1} &= h^{1,0} = h^{1,2} = h^{2,1} = q = 2, \\ h^{0,2} &= h^{2,0} = \frac{1}{12}(c_1^2 + c_2) - (1 - q) = 22, \\ h^{1,1} &= -\frac{1}{6}c_1^2 + \frac{5}{6}c_2 + 2q = 97. \end{aligned}$$

(iii) Consider the case where $m > 3$ and $m \not\equiv 0 \bmod 3$. Then $q = 1$ and $c_1^2 = 3m(m-2)(5m-2)$. Note that in our situation $m \geq 4$, hence $c_1^2 \geq 3 \cdot 4 \cdot 2 \cdot 18 = 432 > 9$; hence, $\widetilde{F}$ is of general type by Proposition 7.1.

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In addition, $c_2 = 9m(m^2 - 2m + 2)$. Therefore, by formula (8.4), we have

$$\begin{aligned} h^{0,0} &= h^{2,2} = 1, \\ h^{0,1} &= h^{1,0} = h^{1,2} = h^{2,1} = q = 1, \\ h^{0,2} &= h^{2,0} = \frac{1}{12}(c_1^2 + c_2) - (1-q) = \frac{1}{2}m(m-1)(4m-5), \\ h^{1,1} &= -\frac{1}{6}c_1^2 + \frac{5}{6}c_2 + 2q = 5m^3 - 9m^2 + 13m + 2. \end{aligned}$$

(iv) Finally, we consider the case where $m > 3$ and $m \equiv 0 \bmod 3$. Then $q = 2$, and

$$c_1^2 = 3m(m-2)(5m-2) \geq 3 \cdot 6 \cdot (6-2) \cdot (5 \cdot 6 - 2) > 9,$$

hence $\widetilde{F}$ is of general type by Proposition 7.1. Moreover,

$$c_2 = 9m(m^2 - 2m + 2);$$

hence, by formula (8.4), we have

$$\begin{aligned} h^{0,0} &= h^{2,2} = 1, \\ h^{0,1} &= h^{1,0} = h^{1,2} = h^{2,1} = q = 2, \\ h^{0,2} &= h^{2,0} = \frac{1}{12}(c_1^2 + c_2) - (1-q) = \frac{1}{2}m(m-1)(4m-5) + 1, \\ h^{1,1} &= -\frac{1}{6}c_1^2 + \frac{5}{6}c_2 + 2q = 5m^3 - 9m^2 + 13m + 4. \end{aligned}$$

Conclusion For $m \geq 3$, the surface $\widetilde{F}$ is of general type. Furthermore, as $m \to \infty$, we have $h^{0,2} = h^{2,0}, h^{1,1} \to \infty$, while other Hodge numbers remain 1 or 2. In addition, the Chern ratio

$$\frac{c_1^2}{c_2} = \frac{3m(m-2)(5m-2)}{9m(m^2-2m+2)} \longrightarrow \frac{5}{3} \qquad \text{as } m \longrightarrow \infty.$$

Example 8.3 Now we consider line arrangements that arise from restriction of higher dimensional hyperplane arrangements. The braid arrangement in $\mathbb{P}^n$ is given by

$$\mathfrak{B}_n : \prod_{0 \leq i < j \leq n} (x_i - x_j) = 0,$$

consisting of $\binom{n+1}{2}$ hyperplanes. Let $E \subseteq \mathbb{P}^n$ be a generic projective plane and let $\mathcal{A}_n = \mathfrak{B}_n|_E$ the restriction of $\mathfrak{B}_n$ to E. Then $\mathcal{A}_n$ is a line arrangement in the projective plane with only nodes and triple points such that

$$d = \binom{n+1}{2} = \frac{n(n+1)}{2} \quad \text{and} \quad t_3 = \binom{n+1}{3}.$$

Indeed, any triple point of $\mathcal{A}_n$ corresponds to the intersection of exactly three hyperplanes in $\mathfrak{B}_n$, and is then of the form $\{x_{i_1} = x_{i_2} = x_{i_3}\}$ for some $i_1 < i_2 < i_3$. Hence,

$$t_3 = \#\{(i_1, i_2, i_3) : i_1, i_2, i_3 \in [0, n], i_1 < i_2 < i_3\} = \binom{n+1}{3}.$$

From Remark 3.2, we have

$$2t_2 + 6t_3 = d(d-1) = \binom{n+1}{2}\left(\binom{n+1}{2} - 1\right),$$

hence

$$t_2 = \frac{d(d-1)}{2} - 3t_3 = \frac{n^2(n^2-1)}{4}.$$

Note that if $n \equiv 1 \bmod 3$, then $d \equiv 1 \bmod 3$; otherwise, $3|d$. So we consider the following two cases:

(i) If $n \not\equiv 1 \bmod 3$, we have $3|d$. Moreover, $\dim H^1(F)_{\neq 1}$ can be computed by [13, Theorem A, Lemma 4.1, Proposition 4.14 and Proposition 5.1]; we have by (8.5) that

$$q = \begin{cases} 1 & \text{if } n = 2, 3, \\ 0 & \text{otherwise.} \end{cases}$$

In addition,

$$DCI_{2,d} = 0, \qquad DCII_{2,d} = d - 1$$

and

$$DCI_{3,d} = -d, \qquad DCII_{3,d} = d - 4.$$

Hence,

$$c_1^2 = K_{\overline{F}}^2 + \sum_r t_r DCI_{r,d} = \frac{1}{24} n(n+1)(n-2)(n-3)(3n^2 + 19n + 32),$$

so if $n \geq 4$, then $c_1^2 > 9$, and thus $\widetilde{F}$ is of general type by Proposition 7.1. Moreover,

$$c_2 = \chi(\overline{F}) + \sum_r t_r DCII_{r,d} = \frac{1}{8} n(n+1)(n-2)(n^3 + 2n^2 - 3n - 12).$$

(ii) If $n \equiv 1 \bmod 3$, then $d \equiv 1 \bmod 3$ and $\dim H^1(F)_{\neq 1} = 0$ by [13, Theorem A], and thus $q = 0$. In addition,

$$DCI_{2,d} = 0, \qquad DCII_{2,d} = d - 1,$$

and by Example 4.4, we have

$$DCI_{3,d} = -(d-1), \qquad DCII_{3,d} = d - 1.$$

Thus,

$$\begin{aligned} c_1^2 &= K_{\overline{F}}^2 + \sum_r t_r DCI_{r,d} \\ &= \frac{1}{24} n(n+1)\big(3n^2(n^2 - 15) + 2n(2n^2 - 21) + 188\big), \end{aligned}$$

so if $n \geq 4$, we have $c_1^2 > 9$ and thus $\widetilde{F}$ is of general type by Proposition 7.1. Moreover,

$$c_2 = \chi(\overline{F}) + \sum_r t_r DCII_{r,d} = \frac{1}{8} n(n+1)(n^4 - 7n^2 - 2n + 20).$$

Finding the concrete formulae for the Hodge numbers by applying (8.4) is left to the reader. For the Chern ratio, we have $\lim_{n\to\infty} c_1^2/c_2 = 1$.

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CNRS, LJAD, UMR 7351, Univ. Nice Sophia Antipolis, 06100 Nice, France
e-mail: wzhj01@gmail.com