EXIT TIMES FOR ARMA PROCESSES

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Abstract

We study the asymptotic behaviour of the expected exit time from an interval for the ARMA process, when the noise level approaches 0.

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1. Introduction

Autoregressive moving average (ARMA) processes are amongst the most widespread and important tools in time series analysis. In this paper we study exit or first passage problems for these processes and give explicit asymptotic formulae for the expected exit time from an interval for a standard ARMA(n, m) process with normal noise when the noise level approaches 0. The formula is an immediate consequence of results for linear autoregressive (AR) processes in [6]. The essential ingredient in the formula is the invariant probability distribution of an associated multidimensional AR process. This is a centered normal distribution, and hence determined entirely by its covariance matrix.

Our work was inspired by a paper on large deviations by Klebaner and Liptser [7]. In an example in that paper, they derived the upper bound of the asymptotic expected exit time of an AR process with normally distributed noise. The corresponding lower bound was then derived in [10] by using Novikov's martingale method (see [8] and [9]). The result for the asymptotic expected exit time for a multivariate AR process was derived later in [6]. A more general stochastic difference equation of AR type has also been studied, in [4], where an upper bound of the expected exit time was expressed in terms of the invariant probability measure of the process. In this paper we show how the result in [6] can be applied to the ARMA process.

Other recent work on related topics can be found for example in [3] and [5], where the probability distribution of the first passage time of an AR(n) process was studied, in [2], which focused on the probability that the AR(n) process does not exceed a barrier before a certain time, and in [1], where an extension of Novikov's method was used to get a representation of a mean first passage time for the ARMA process, as the mean of an integral containing the process.

2. Exit times for AR processes

Jung [6] studied exit times for multivariate AR processes. The methods used also give a result for the expected exit time from an interval of the univariate AR process of order n (the AR(n) process) $\{X_t^\varepsilon\}_{t\geq 0}$ defined by

$$X_t^\varepsilon = b_1 X_{t-1}^\varepsilon + \cdots + b_n X_{t-n}^\varepsilon + \varepsilon\xi_t \quad \text{for } t \geq n, \qquad X_0^\varepsilon = \cdots = X_{n-1}^\varepsilon = 0. \tag{1}$$

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Here $X_t^\varepsilon \in \mathbb{R}$ for all $t \geq 0, b_1, \dots, b_n$ are real parameters, ε is a positive parameter, and $\{\xi_t\}_{t\geq n}$ is an independent and identically distributed (i.i.d.) sequence of standard normal random variables. The exit time from the interval $(-1, 1)$ is defined by

$$\tau^\varepsilon_{(-1,1)} = \min\{t \geq n \colon |X_t^\varepsilon| \geq 1\}.$$

Assuming that $b_1, \dots, b_n$ are such that the roots of the characteristic polynomial

$$b(x) := 1 - \sum_{i=1}^{n} b_i x^i$$

lie outside the unit disk (hence, $|b(x)| > 0$ for $|x| \leq 1$), and that $X_0^\varepsilon = \cdots = X_{n-1}^\varepsilon = 0$ as in (1), Jung [6, Thorem 4.1] established the following result.

Theorem 1. *The mean exit time* $\mathbb{E}[\tau^\varepsilon_{(-1,1)}]$ *of the AR*(n) *process* $\{X_t^\varepsilon\}_{t\geq 0}$ *at (1) satisfies*

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log(\mathbb{E}[\tau^\varepsilon_{(-1,1)}]) = \frac{1}{2\sigma^2},$$

where $\varepsilon^2\sigma^2 = \lim_{t\to\infty} \operatorname{var}(X_t^\varepsilon)$.

The proof of Theorem 1 given in [6] uses a multivariate representation $\{\boldsymbol{Y}_t^\varepsilon\}_{t\geq n-1}$ of the AR(n) process, where $\boldsymbol{Y}_t^\varepsilon \in \mathbb{R}^n$ and

$$\boldsymbol{Y}_t^\varepsilon = (X_t^\varepsilon, \dots, X_{t-n+1}^\varepsilon)^\top.$$

The process $\{\boldsymbol{Y}_t^\varepsilon\}_{t\geq n-1}$ satisfies

$$\boldsymbol{Y}_t^\varepsilon = \boldsymbol{B}\boldsymbol{Y}_{t-1}^\varepsilon + \varepsilon(\xi_t, 0, \dots, 0)^\top \quad \text{for } t \geq n, \qquad \boldsymbol{Y}_{n-1}^\varepsilon = (0, \dots, 0)^\top,$$

for the $n \times n$ matrix $\boldsymbol{B}$, where

$$\boldsymbol{B} = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ 1 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

For any $t \geq n$, $\boldsymbol{Y}_t^\varepsilon$ is normally distributed with zero-mean vector and covariance matrix $\varepsilon^2\boldsymbol{\Sigma}_t$, where

$$\boldsymbol{\Sigma}_t = \boldsymbol{B}\boldsymbol{\Sigma}_{t-1}\boldsymbol{B}^\top + (1, 0, \dots, 0)^\top(1, 0, \dots, 0), \qquad t \geq n+1,$$
$$\boldsymbol{\Sigma}_n = (1, 0, \dots, 0)^\top(1, 0, \dots, 0).$$

Under our assumptions on $b_1, \dots, b_n$, all eigenvalues of the matrix $\boldsymbol{B}$ have absolute values smaller than 1. Then $\boldsymbol{\Sigma}_t$ converges to $\boldsymbol{\Sigma}_\infty$, where the $n \times n$ matrix $\boldsymbol{\Sigma}_\infty$ is the solution of the equation

$$\boldsymbol{\Sigma}_\infty = \boldsymbol{B}\boldsymbol{\Sigma}_\infty\boldsymbol{B}^\top + \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix},$$

and the process $\{\boldsymbol{Y}_t^\varepsilon\}_{t\geq n-1}$ has an invariant distribution which is multivariate normal with zero-mean vector and covariance matrix $\varepsilon^2\boldsymbol{\Sigma}_\infty$.

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What is studied in [6] is the exit time from the set $\{\boldsymbol{x} \in \mathbb{R}^n : |\boldsymbol{v}^\top \boldsymbol{x}| < 1\}$ for the process $\{\boldsymbol{Y}_t^\varepsilon\}_{t \ge n-1}$ for any nonzero vector $\boldsymbol{v} \in \mathbb{R}^n$; then, by choosing $\boldsymbol{v} = (1, 0, \dots, 0)^\top$, the result in Theorem 1 is deduced. By using a general vector $\boldsymbol{v}$, the same proof can be used for the following formulation of the theorem.

Theorem 2. *For the AR(n) process* $\{X_t^\varepsilon\}_{t \ge 0}$*, the exit time*

$$\tau^\varepsilon := \min\{t \ge n : |\boldsymbol{v}^\top \boldsymbol{Y}_t^\varepsilon| \ge 1\},$$

where $\boldsymbol{Y}_t^\varepsilon = (X_t^\varepsilon, \dots, X_{t-n+1}^\varepsilon)^\top$ *and* $\boldsymbol{v} \in \mathbb{R}^n$ *is a nonzero vector, satisfies*

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log(\mathbb{E}[\tau^\varepsilon]) = \frac{1}{2\boldsymbol{v}^\top \boldsymbol{\Sigma}_\infty \boldsymbol{v}},$$

where $\varepsilon^2 \boldsymbol{\Sigma}_\infty$ *is the covariance matrix of the invariant distribution of the process* $\{\boldsymbol{Y}_t^\varepsilon\}_{t \ge n-1}$.

3. Exit times for ARMA processes

Consider now an ARMA(n, m) process, that is, an autoregressive moving average model with n autoregressive terms and m moving average terms, defined by the recursive formula

$$X_t^\varepsilon = \varepsilon \xi_t + \sum_{i=1}^n b_i X_{t-i}^\varepsilon + \varepsilon \sum_{i=1}^m c_i \xi_{t-i}, \qquad t \ge \max(n, m). \tag{2}$$

In (2), $\{\xi_t\}_{t \ge \max(n,m)}$ is a sequence of i.i.d. standard normal random variables, and we assume that

$$X_t^\varepsilon = \xi_t = 0, \qquad 0 \le t \le \max(n, m) - 1. \tag{3}$$

We can write the ARMA process in a more compact form by using the lag operator L, where $LX_t^\varepsilon = X_{t-1}^\varepsilon$ and $L^i X_t^\varepsilon = X_{t-i}^\varepsilon$ for $i \ge 1$. Let

$$b(L) := 1 - \sum_{i=1}^n b_i L^i, \qquad c(L) := 1 + \sum_{i=1}^m c_i L^i. \tag{4}$$

Then the ARMA process defined by (2) is expressible as

$$b(L) X_t^\varepsilon = \varepsilon c(L) \xi_t.$$

Define the characteristic polynomials (in notation following [11])

$$b(x) := 1 - \sum_{i=1}^n b_i x^i, \qquad c(x) := 1 + \sum_{i=1}^m c_i x^i, \tag{5}$$

and assume that these polynomials have no common roots (hence, the process is uniquely defined and cannot be written in a simpler form by variable transformation) and that the roots of $b(x)$ lie outside of the unit disk.

Consider now the exit time from the interval $(-1, 1)$ for the ARMA(n, m) process: define τ^ε by

$$\tau^\varepsilon := \min\{t \ge \max\{n, m\} : |X_t^\varepsilon| \ge 1\}. \tag{6}$$

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We first show that the ARMA(n, m) process is expressible as a sum of elements of an AR process. It then follows that the result for the exit time of the AR(n) process in Theorem 2 can be applied to the exit time of the ARMA(n, m) process.

Start by noting that the AR(n) process defined in (1) can be written as

$$b(L)X_t^\varepsilon = \varepsilon\xi_t, \quad t \geq n, \qquad X_t^\varepsilon = 0, \quad 0 \leq t \leq n-1,$$

where $b(L)$ is as in (4).

Lemma 1. *The ARMA(n, m) process $\{X_t^\varepsilon\}_{t\geq 0}$ defined by (2) and (3) is expressible as*

$$X_t^\varepsilon = \begin{cases} c(L)U_t^\varepsilon & \text{for } t \geq \max\{n, m\}, \\ 0 & \text{for } 0 \leq t \leq \max\{n, m\} - 1, \end{cases}$$

where $\xi_t = 0$ for $0 \leq t \leq \max\{n, m\} - 1$ and $\{U_t^\varepsilon\}_{t\geq 0}$ is an AR(n) process for which

$$\begin{aligned} b(L)U_t^\varepsilon &= \varepsilon\xi_t \quad \text{for } t \geq \max\{n, m\}, \\ U_t^\varepsilon &= 0 \quad \text{for } 0 \leq t \leq \max\{n, m\} - 1. \end{aligned} \tag{7}$$

Proof. Define a process $\{V_t^\varepsilon\}_{t\geq 0}$ by

$$V_t^\varepsilon = c(L)U_t^\varepsilon$$

for an AR(n) process $\{U_t^\varepsilon\}_{t\geq 0}$ defined as in (7), and let

$$V_t^\varepsilon = \xi_t = 0, \qquad 0 \leq t \leq \max\{n, m\} - 1.$$

Then

$$b(L)V_t^\varepsilon = b(L)c(L)U_t^\varepsilon = c(L)b(L)U_t^\varepsilon = \varepsilon c(L)\xi_t \quad \text{for } t \geq \max\{n, m\},$$

and the process $\{V_t^\varepsilon\}_{t\geq 0}$ has the same initial values as the process $\{X_t^\varepsilon\}_{t\geq 0}$ (given in (3)). Therefore, $\{V_t^\varepsilon\}_{t\geq 0} = \{X_t^\varepsilon\}_{t\geq 0}$, the ARMA$(n, m)$ process defined by (2) and (3). □

The following theorem gives the asymptotic result for the ARMA(n, m) process.

Theorem 3. *Let the ARMA(n, m) process $\{X_t^\varepsilon\}_{t\geq 0}$ be defined as in (2) and (3), and be such that the characteristic polynomials $b(x)$ and $c(x)$ as in (5) have no common roots, with all roots of $b(x)$ lying outside the unit disk. Then the exit time τ^ε as at (6) of this process satisfies*

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log(\mathbb{E}[\tau^\varepsilon]) = \frac{1}{2\sigma^2},$$

where $\varepsilon^2\sigma^2 = \lim_{t\to\infty} \operatorname{var} X_t^\varepsilon$.

Proof. If $m+1 = n$, Lemma 1 implies that we can write $X_t^\varepsilon = \boldsymbol{c}^\top \boldsymbol{Y}_t^\varepsilon$ for $\boldsymbol{c} = (1, c_1, \ldots, c_m)^\top$ and $\boldsymbol{Y}_t^\varepsilon = (U_t^\varepsilon, \ldots, U_{t-n+1}^\varepsilon)^\top$, where the dimensions of the vectors match because $m + 1 = n$. Now

$$\tau^\varepsilon = \min\{t \geq \max\{n, m\} : |X_t^\varepsilon| \geq 1\} = \min\{t \geq \max\{m, n\} : |\boldsymbol{c}^\top \boldsymbol{Y}_t^\varepsilon| \geq 1\}.$$

By Theorem 2,

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log(\mathbb{E}[\tau^\varepsilon]) = \frac{1}{2\boldsymbol{c}^\top \boldsymbol{\Sigma}_\infty \boldsymbol{c}},$$

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where $\varepsilon^2\mathbf{\Sigma}_\infty$ is the covariance matrix of the invariant distribution of $\{\boldsymbol{Y}_t^\varepsilon\}_{t\geq 0}$. Since $X_t^\varepsilon = \boldsymbol{c}^\top \boldsymbol{Y}_t^\varepsilon$, we have the recursive formula

$$\operatorname{var}(X_t^\varepsilon) = \mathbb{E}[(X_t^\varepsilon)^2)] = \mathbb{E}[\boldsymbol{c}^\top \boldsymbol{Y}_t^\varepsilon \boldsymbol{c}^\top \boldsymbol{Y}_t^\varepsilon] = \boldsymbol{c}^\top \mathbb{E}[\boldsymbol{Y}_t^\varepsilon (\boldsymbol{Y}_t^\varepsilon)^\top]\boldsymbol{c},$$

which implies that $\boldsymbol{c}^\top \mathbf{\Sigma}_\infty \boldsymbol{c} = \sigma^2$. The case that $m+1 < n$ can be dealt with by adding zeroes at the end of the vector $\boldsymbol{c}$ to make the dimensions match. If $m+1 > n$, add terms with coefficient zero in

$$U_t^\varepsilon = b_1 U_{t-1}^\varepsilon + \cdots + b_n U_{t-n}^\varepsilon + 0 \cdot U_{t-n-1}^\varepsilon + \cdots + 0 \cdot U_{t-m-1}^\varepsilon + \varepsilon\xi_t,$$

and let $\boldsymbol{Y}_t^\varepsilon = (U_t^\varepsilon, \ldots, U_{t-m}^\varepsilon)$. □

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