

EXTENDED CENTROIDS OF POWER SERIES RINGS

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(Received 19 June, 1989; revised 10 November, 1989)

Prime rings came into prominence when Posner characterized prime rings satisfying a polynomial identity [9]. The scarcity of invertible central elements made it difficult to generalize results from central simple and primitive algebras to prime rings. For example, we do not automatically have tensor products at our disposal. In [5], the first author introduced the Martindale ring of quotients $Q(R)$ of a prime ring R in his theorem characterizing prime rings satisfying a generalized polynomial identity (GPI). $Q(R)$ is a prime ring containing R whose center C is a field called the extended centroid of R . The central closure of R is the subring RC of $Q(R)$ generated by R and C . RC is a closed prime ring since its extended centroid equals its center C . Hence we have a useful procedure for proving results about an arbitrary prime ring R . We first answer the question for closed prime rings and then apply to R the information obtained from RC . It should be noted that simple rings and free algebras of rank at least 2 are closed prime rings. For these reasons, closed prime rings are natural objects to study.

The extended centroid has been computed for many classes of rings, such as some group rings [2], skew polynomial rings of automorphism and derivation type [7], [10], coproducts of algebras, and free algebras [6]. The ring of quotients $Q(R)$ and its various subrings have had numerous applications. As mentioned above, they played a key role in the solution of the GPI problem. In [4], Kharchenko developed a Galois theory of semiprime rings which makes extensive use of these ideas. We refer the reader to [8] for a simplified account of this work over prime rings.

In this paper, we determine the extended centroid of the power series ring $R[[x]]$ over a closed prime ring R . One motivation for this problem comes from commutative algebra. In this case, finding the extended centroid of a prime ring is equivalent to finding its field of quotients. If F is a field, then the field of quotients of $F[[x]]$ is $F((x))$, the field of Laurent series over F . However, if D is an integral domain then the field of quotients of $D[[x]]$ is not known in general. In particular, it is rarely the case that the field of quotients of $D[[x]]$ is $F((x))$, where F is the field of quotients of D . We will provide an example where $D = \mathbb{Z}$ (the integers) and refer the reader to [1], [3] for more general examples. If R is a closed prime ring over C , we prove the extended centroid of $R[[x]]$ is $C((x))$. This theorem generalizes the corresponding result for $F[[x]]$. As a corollary, we show that if R is a closed prime ring then so is the ring of Laurent series $R((x))$.

1. Preliminaries. Let $R[[x]]$ denote the power series ring over R . If $f = \sum_{i=0}^{\infty} r_i x^i$ is a nonzero power series in $R[[x]]$ then the degree of f , denoted $\deg(f)$, is the nonnegative integer n such that $r_i = 0$ for $i < n$ and $r_n \neq 0$. Let $R((x))$ denote the ring of Laurent series over R , i.e.

$$R((x)) = \left\{ \sum_{i=n}^{\infty} r_i x^i \mid r_i \in R, n \in \mathbb{Z} \right\}.$$

If R is a prime ring then $R[[x]]$ and $R((x))$ are also prime.

Glasgow Math. J. **32** (1990) 371–375.

We now summarize the definition of the extended centroid of a prime ring R with 1. We refer the reader to [5] for a more detailed account. Let $\mu = \{U\}$ be the collection of all nonzero two-sided ideals of R and consider the totality T of all right module homomorphisms $\phi : U_R \rightarrow R_R$, where $U \in \mu$ and U and R are regarded as right R -modules. Write (ϕ, U) for an element of T and define an equivalence relation \sim on T as follows: $(\phi, U) \sim (\psi, V)$ if $\phi = \psi$ on some $W \in \mu$, where $W \subseteq U \cap V$. Let $\overline{(\phi, U)}$ denote the equivalence class of (ϕ, U) . The Martindale ring of quotients $Q(R)$ is defined to be the set of these equivalence classes. $Q(R)$ is made into a ring as follows:

$$\begin{aligned} \overline{(\phi, U)} + \overline{(\psi, V)} &= \overline{(\phi + \psi, U \cap V)}, \\ \overline{(\phi, U)}\overline{(\psi, V)} &= \overline{(\phi \circ \psi, VU)} \quad (\text{composition acting on the left}). \end{aligned}$$

R may be considered a subring of $Q(R)$ via the mapping $a \rightarrow \overline{(a, R)}$, where a is the left multiplication by a acting on R . $Q(R)$ is a prime ring whose center C is a field called the extended centroid of R . The pair (ϕ, U) is permissible if $\phi : {}_R U_R \rightarrow {}_R R_R$ is a bimodule homomorphism. C may also be characterized as the set

$$\{\overline{(\phi, U)} \in Q(R) \mid (\phi, U) \text{ is permissible}\}.$$

We may now form the central closure RC of R . RC is a prime ring with center C . A prime algebra over a field F is said to be closed if F is already its extended centroid. Note that RC is closed over C .

LEMMA 1.1. *Let R be a prime ring with center $Z(R)$ and extended centroid C . Let F be the field of fractions of $Z(R)$. Suppose for any $(\phi, U) \in C$, there exists nonzero $u \in U$ such that $\phi(u) = \alpha u$, where $\alpha \in F$. Then C is isomorphic to F .*

Proof. Set $\lambda = \overline{(\phi, U)}$. Thus $\phi(u) = \lambda u = \alpha u$, implying $(\lambda - \alpha)u = 0$. Viewing this equation in RC , we see that $\lambda = \alpha$.

EXAMPLE. We claim that $\mathbb{Q}((x))$ is not the extended centroid of $\mathbb{Z}[[x]]$, where \mathbb{Z} and \mathbb{Q} denote the integers and rationals respectively.

Let $f = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots$. It suffices to show that fg is not in $\mathbb{Z}[[x]]$ for any nonzero $g \in \mathbb{Z}[[x]]$. Suppose otherwise with $g = \sum_{i=n}^{\infty} a_i x^i$. Thus for any prime p , the coefficient of x^{n+p} in fg is $a_{n+p-1} + \frac{1}{2}a_{n+p-2} + \dots + \frac{1}{p-1}a_{n+1} + \frac{1}{p}a_n \in \mathbb{Z}$. Multiplying by $(p-1)!$, we can see that $p \mid a_n$, a contradiction.

2. Main theorem. In the following four results, let R be a prime ring with extended centroid C . Let D denote the extended centroid of $R[[x]]$ and assume $(\phi, U) \in D$, $(\phi, U) \neq 0$. We remark that ϕ is one-to-one on U .

LEMMA 2.1. *If $f, g \in U$, $f, g \neq 0$ then $\deg(f) + \deg(\phi(g)) = \deg(g) + \deg(\phi(f))$.*

Proof. Let $f = ax^n + \dots$ and $g = bx^k + \dots$, where $a, b \neq 0$. Set $\phi(f) = cx^m + \dots$ and $\phi(g) = dx^l + \dots$, where $c, d \neq 0$. Assume $k + m < n + l$. Since R is prime, there

exists $r \in R$ such that $crb \neq 0$. Since ϕ is bimodule,

$$\phi(frg) = (cx^m + \dots)r(bx^k + \dots) = (ax^n + \dots)r(dx^l + \dots).$$

Equating coefficients, we obtain $crb = 0$, a contradiction. Similarly $n + l \geq k + m$.

The next two corollaries are immediate consequences of Lemma 2.1.

COROLLARY 2.2. *If $f \in U$ is of minimal degree then $\phi(f)$ is of minimal degree in $\phi(U)$.*

COROLLARY 2.3. *If there exists $f \in U$ such that $\deg(f) = \deg(\phi(f)) = 0$ then $\deg(g) = 0$ implies $\deg(\phi(g)) = 0$ for $g \in U$.*

COROLLARY 2.4. *Suppose $g \in U$ such that $\deg(g) = \deg(\phi(g)) = 0$. If a_0, b_0 are the degree 0 coefficients of g and $\phi(g)$ respectively then there exists $\lambda \in C$ such that $b_0 = \lambda a_0$.*

Proof. $I = \{0\} \cup \{r \in R \mid r \text{ is the degree 0 coefficient of some element in } U\}$ is a nonzero ideal of R . Choose $r \neq 0$ in I and suppose $r + r_1x + \dots$ and $r + s_1x + \dots$ are distinct elements of U . Let

$$\phi(r + r_1x + \dots) = t_0 + t_1x + \dots \quad \text{and} \quad \phi(r + s_1x + \dots) = u_0 + u_1x + \dots$$

By Corollary 2.3, $t_0, u_0 \neq 0$. Applying Lemma 2.1 to g and $(r_1 - s_1)x + \dots$, we get $t_0 = u_0$. Define $\lambda: I \rightarrow R$ by $0 \rightarrow 0$ and $r \rightarrow t$, where t is the unique degree 0 coefficient of the image under ϕ of any element in U having r as its degree 0 coefficient. λ is a bimodule homomorphism. Hence $\lambda \in C$ and $b_0 = \lambda a_0$.

The following theorem is the key result needed for determining D in the case where R is closed prime.

THEOREM 2.5. *Let R be a closed prime ring with extended centroid C . Let D denote the extended centroid of $R[[x]]$ and suppose $(\overline{\phi}, U) \in D$. Then there exists nonzero $f \in U$ such that $\phi(f) = hf$, where $h \in C((x))$.*

Proof. Choose $f \in U$ of minimal degree n and let $m = \deg(\phi(f))$. $V = x^{-n}U$ is a nonzero ideal of $R[[x]]$. By Corollary 2.2, $(\overline{\eta_0}, V) \in D$, where $\eta_0(x^{-n}k) = x^{-m}\phi(k)$ for all $k \in U$. Setting $g = x^{-n}f$, we have $\deg(g) = \deg(\eta_0(g)) = 0$. Let

$$g = \sum_{i=0}^{\infty} a_i x^i \quad \text{and} \quad \eta_0(g) = \sum_{i=0}^{\infty} b_i x^i.$$

We use induction to find $\lambda_i \in C$ such that

$$b_i = \lambda_0 a_i + \lambda_1 a_{i-1} + \dots + \lambda_i a_0 \quad \text{for all } i.$$

By Corollary 2.4, there exists $\lambda_0 \in C$ such that $b_0 = \lambda_0 a_0$. Since R is closed prime, λ_0 acts on R and hence on $R[[x]]$ as left multiplication. We claim that $\eta_1 = x^{-1}(\eta_0 - \lambda_0)$ acts on $W = R[[x]]gR[[x]]$. It suffices to apply η_1 to pgq , where $p, q \in R[[x]]$. Now

$$\begin{aligned} \eta_1(pgq) &= x^{-1}(p\eta_0(g)q - p\lambda_0 gq) \\ &= x^{-1}(p(\lambda_0 a_0 + b_1 x + \dots)q - p(\lambda_0 a_0 + \lambda_0 a_1 x + \dots)q) \in R[[x]]. \end{aligned}$$

Thus $(\overline{\eta_1}, W) \in D$. In particular, $\eta_1(g) = (b_1 - \lambda_0 a_1) + (b_2 - \lambda_0 a_2)x + \dots$. If $b_1 - \lambda_0 a_1 = 0$, choose $\lambda_1 = 0$. Otherwise, by Corollary 2.4, there exists $\lambda_1 \in C$ such that $b_1 - \lambda_0 a_1 = \lambda_1 a_0$

or $b_1 = \lambda_0 a_1 + \lambda_1 a_0$. Assume there exist $\lambda_2, \dots, \lambda_{i-1} \in C$ and corresponding $\eta_2, \dots, \eta_{i-1}$ (acting on W) such that

$$b_{i-1} = \lambda_0 a_{i-1} + \lambda_1 a_{i-2} + \dots + \lambda_{i-1} a_0$$

and

$$\eta_{i-1}(g) = (b_{i-1} - \lambda_0 a_{i-1} - \lambda_1 a_{i-2} - \dots - \lambda_{i-2} a_1) + (b_i - \lambda_0 a_i - \lambda_1 a_{i-1} - \dots - \lambda_{i-2} a_2)x + \dots$$

Let $\eta_i = x^{-1}(\eta_{i-1} - \lambda_{i-1})$. Note that $(\overline{\eta_i}, \overline{W}) \in D$ and the degree 0 coefficient of $\eta_i(g)$ is $b_i - \lambda_0 a_i - \lambda_1 a_{i-1} - \dots - \lambda_{i-1} a_1$. If this term is 0, choose $\lambda_i = 0$. Otherwise, by Corollary 2.4, there exists $\lambda_i \in C$ such that $b_i = \lambda_0 a_i + \dots + \lambda_i a_0$. Thus

$$\eta_0(g) = \lambda_0 a_0 + (\lambda_0 a_1 + \lambda_1 a_0)x + (\lambda_0 a_2 + \lambda_1 a_1 + \lambda_2 a_0)x^2 + \dots = (\lambda_0 + \lambda_1 x + \lambda_2 x^2 + \dots)g.$$

Recall that $g = x^{-n}f$ and $\eta_0(g) = x^{-m}\phi(f)$. Therefore

$$x^{-m}\phi(f) = (\lambda_0 + \lambda_1 x + \dots)x^{-n}f \quad \text{or} \quad \phi(f) = (\lambda_0 + \lambda_1 x + \dots)x^{m-n}f = hf,$$

where $h \in C((x))$.

By Lemma 1.1 and Theorem 2.5, we obtain the main result of this paper.

THEOREM 2.6. *Let R be a closed prime ring with extended centroid C . Then the extended centroid of $R[[x]]$ is isomorphic to $C((x))$.*

Note that the example in Section 1 shows Theorem 2.6 fails if R is not closed. We end with two results on the extended centroid of $R((x))$.

THEOREM 2.7. *For any prime ring R , $R[[x]]$ and $R((x))$ have isomorphic extended centroids.*

Proof. Let D and E denote the extended centroids of $R[[x]]$ and $R((x))$ respectively. Choose $(\overline{\phi}, \overline{U}) \in D$ and define $U' = \{x^{-m}f \mid f \in U, m \geq 0\}$. If $x^{-m}f, x^{-n}g \in U'$ with $m > n$ then $x^{-m}f + x^{-n}g = x^{-m}(f + x^{m-n}g) \in U'$. If $h \in R((x))$ such that $h = x^{-l}k$, where $l \geq 0$ and $k \in R[[x]]$, then $x^{-m}fh = x^{-(m+l)}fk \in U'$. Hence U' is a nonzero ideal of $R((x))$. Define $\phi' : U' \rightarrow R((x))$ by $\phi'(x^{-m}f) = x^{-m}\phi(f)$. If $x^{-m}f = x^{-n}g$ with $m > n$ then $\phi(f) = x^{m-n}\phi(g)$, which shows that $x^{-m}\phi(f) = x^{-n}\phi(g)$. Using the above computations, one can verify that $(\overline{\phi'}, \overline{U'}) \in E$. The mapping $(\overline{\phi}, \overline{U}) \rightarrow (\overline{\phi'}, \overline{U'})$ gives the desired isomorphism from D to E . The fact that every nonzero ideal of $R((x))$ intersects $R[[x]]$ nontrivially implies this mapping is onto.

COROLLARY 2.8. *Let R be a closed prime ring with extended centroid C . Then $R((x))$ is closed prime with extended centroid $C((x))$.*

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