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ON COEFFICIENTS OF ARTIN L FUNCTIONS AS DIRICHLET SERIES

BY

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ABSTRACT. The paper is motivated by a result of Ankeny [1] above Dirichlet L functions in 1952. We generalize this from Dirichlet L functions to Artin L functions of relative abelian extensions, by complementing the ingenious proof of Ankeny's theorem given by Iwasaki [4]. Moreover, we characterize Dirichlet L functions in the class of all Artin L functions in terms of coefficients as Dirichlet series.

We use **Z** for the ring of all integers and **Q** for the field of all rational number. A general character f of a finite group G is defined by

$$f = \sum_{\chi} a(\chi)\chi \qquad (a(\chi) \in \mathbf{Z}),$$

where χ runs over all irreducible characters on G. We note that f is a character of a representation if and only if all $a(\chi)$ are non-negative integers. Let M/K be a Galois extension of algebraic number fields. The Artin L function for a general character f of the Galois group $Gal(M/K)$ is defined by

$$L(s,f,M/K) = \prod_{\chi} L(s,\chi,M/K)^{a(\chi)}.$$

We denote by $\zeta_K(s)$ the Dedekind zeta function of an algebraic number field K.

Motivated by Suetuna [6], Ankeny [1] asserted that: Set $Z(s) = \prod_{j=1}^{n} L(s,\chi_j)$, where $L(s,\chi_j)$ are Dirichlet L functions for primitive Dirichlet characters χ_j. If the coefficients of $Z(s)$ as Dirichlet series are non-negative and real and if at most one of $L(s,\chi_j)$ is the Riemann zeta function $\zeta_{\mathbf{Q}}(s)$ then $Z(s)$ is the Dedekind zeta function for an abelian extension field of **Q**.

Iwasaki [4] corrected and simplified Ankeny's proof. We take an interest in Theorem 1 compared with Dedekind's conjecture that for any extension M/K of algebraic number fields, the quotient $\zeta_M(s)/\zeta_K(s)$ is holomorphic. This conjecture is still open. For details, see [3, van der Waall, Holomorphy of quotients of zeta functions]. Now we prove

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THEOREM 1. *Let M/K be an abelian extension of algebraic number fields, and let f be a non-zero general character of $Gal(M/K)$ (not necessarily a character of a representation). Suppose that f is non-negative real valued and the quotient $L(s,f,M/K)/\zeta_K(s)$ is holomorphic. Then the Artin L function $L(s,f,M/K)$ is the Dedekind zeta function $\zeta_E(s)$ for an intermediate field E of M/K.*

PROOF. We can write

$$f = \sum_{\chi} a(\chi)\chi \quad . (a(\chi) \in \mathbf{Z}),$$

where χ runs over all characters of degree 1 of $G = Gal(M/K)$. Using the inner product $<,>$, we obtain

$$a(\chi) = \langle f, \chi \rangle = (1/g) \sum_{\sigma \in G} f(\sigma)\chi(\sigma^{-1}).$$

where $g = [M : K]$. In particular

$$a(1) = \langle f, 1 \rangle = (1/g) \sum f(\sigma) > 0,$$

since f is non-zero and non-negative valued. The quotient

$$L(s,f)/\zeta_K(s) = \prod_{\chi \neq 1} L(s, \chi)^{a(\chi)} \zeta_K(s)^{a(1)-1}$$

is holomorphic at $s = 1$, so that it must hold that $a(1) = 1$. Also, we have $|a(\chi)| = |\langle \chi, f \rangle| \leqq \langle 1, f \rangle = a(1) = 1$. We set $S_{\pm} = \{\chi; a(\chi) = \pm 1\}$. Following [4], we can prove that S_+ is a group. Since f is real valued, we have $\langle f, \chi \rangle = \langle f, Re(x) \rangle$, where Re means the real part. If χ belongs to S_+, then we obtain

$$\sum f(\sigma)(1 - Re(\chi(\sigma^{-1})) = ga(1) - ga(\chi) = 0.$$

We see from $Re(\chi(\sigma^{-1})) \leqq 1$ and $f(\sigma) \geqq 0$ that if $f(\sigma) \neq 0$ then $Re(\chi(\sigma^{-1}) = 1$, that is, $\chi(\sigma) = 1$. Therefore $S_+ = \{\chi; \chi(\sigma) = 1$ for every $\sigma \in G$ with $f(\sigma) \neq 0\}$. This implies that S_+ is a group. Similarly, $S_- = \{\chi; \chi(\sigma) = -1$ for every $\sigma \in G$ with $f(\sigma) \neq 0\}$. If S_- is empty then

$$f = \sum_{\chi \in S_+} \chi,$$

which coincides with the permutation character 1_H^G for the subgroup $H = \{\sigma \in G; \chi(\sigma) = 1$ for every $\chi \in S_+\}$ of G. Therefore we get $L(s,f) = \zeta_E(s)$ for the intermediate field E corresponding to H. Suppose now that S_- is not empty. Then by $\chi_- S_- = S_+$ with $\chi_- \in S_-$, the cardinal number of S_+ is equal to that of S_-. Denote $m = [K : \mathbf{Q}]$. From $L(1, \chi) \neq 0, \infty$ and the functional equation of $L(s, \chi)$, we see that

$L(s,\chi)$ has a zero of order $q(\chi)+r$ at $s=-1$ and a zero of order $m-r-q(\chi)$ at $s=-2$, where r is dependent only upon K. For example, see [3, Martinet, Character theory and Artin L functions]. Since

$$L(s,f)/\zeta_K(s)=\prod_{\substack{\chi_+\in S_+\\ \chi_+\neq 1}} L(s,\chi_+)\Big/\prod_{\chi_-\in S_-} L(s,\chi_-)$$

is holomorphic at $s=-1$ and also at $s=-2$, the following inequalities hold:

$$\sum_{\substack{\chi_+\in S_+\\ \chi_+\neq 1}} (q(\chi_+)+r) \geqq \sum_{\chi_-\in S_-} (q(\chi_-)+r)$$

$$\sum_{\substack{\chi_+\in S_+\\ \chi_+\neq 1}} (m-r-q(\chi_+)) \geqq \sum_{\chi_-\in S_-} (m-r-q(\chi_-)).$$

These yield a contradiction $m(\#S_+-1)\geqq m\#S_-=m\#S_+$, where # means the cardinal number. The proof is complete. □

REMARK 1. The result of Theorem 1 is not true if M/K is not abelian. Namely we have an example: We set $E=\mathbf{Q}(\sqrt{-3})$ and $F=\mathbf{Q}(\sqrt[3]{n})$, where n is a square free integer greater than 1. Let M be the composite field of E and F. Then $M/\mathbf{Q}$ is a Galois extension whose Galois group is isomorphic to the symmetric group of degree 3. This group has two non-principal irreducible characters χ and ξ of respective degrees 1 and 2. We know that $L(s,1,M/\mathbf{Q})=\zeta_{\mathbf{Q}}(s)$ and $L(s,1+\chi+2\xi,M/\mathbf{Q})=\zeta_M(s)$. Both $L(s,\chi,M/\mathbf{Q})$ and $L(s,\xi,M/\mathbf{Q})$ are holomorphic since χ is of degree 1 and ξ is monomial. Let f be a character of degree 4 defined by $1+\chi+\xi$. Then f is rational and non-negative valued. We have $L(s,f,M/\mathbf{Q})=\zeta_{\mathbf{Q}}(s)\zeta_M(s)/\zeta_F(s)$. It is easily seen that all of the quotients $L(s,f,M/\mathbf{Q})/\zeta_{\mathbf{Q}}(s)$, $\zeta_M(s)/L(s,f,M/\mathbf{Q})$, and $L(s,f,M/\mathbf{Q})/\xi_F(s)$ are holomorphic. But $L(s,f)$ coincides with no Dedekind zeta function because f is not the permutation character 1_H^G for any subgroup H of $G=Gal(M/\mathbf{Q})$. Now $Gal(M/E)$ is isomorphic to the cyclic group of order 3. Let η be any non-principal irreducible character of this group. Then we have $L(s,f,M/\mathbf{Q})=L(s,1+\eta,M/E)$, which implies that $L(s,f,M/\mathbf{Q})/\zeta_E(s)$ is holomorphic.

REMARK 2. Let $Z(s)$ be a product of integral powers of the Dirichlet L functions for primitive Dirichlet characters. Assume now that the coefficients of $Z(s)$ as Dirichlet series are rational. Since $Z(s)$ is the Artin L function $L(s,f,K/\mathbf{Q})$ for an abelian extension $K/\mathbf{Q}$, our assumption implies that the general character f is rational valued. Since $G=Gal(K/\mathbf{Q})$ is abelian, the rational character f can be represented as linear combination with integral (not rational) coefficients in the permutation characters 1_H^G for some subgroups H of G. (If G is not abelian then this is not always true. See Serre [5, Section 13.1].) Namely $Z(s)=L(s,\,f,\,K/\mathbf{Q})$ can be represented as product of integral powers of Dedekind zeta functions for some abelian extension fields of $\mathbf{Q}$. This is a generalization of Suetuna [5]. For, a character of degree 3 of an abelian group is non-negative real valued if and only if the character is rational valued. This is also an answer to Ankeny's remark [1, p. 390] and Iwasaki [4, Remark].

The following theorem gives a characterization of Dirichlet L functions in the class of all Artin L functions.

THEOREM 2. *If the coefficients of an Artin L function as Dirichlet series are periodic, then the function is the Dirichlet L function for a primitive Dirichlet character.*

PROOF. It is enough to verify the theorem for an Artin L function of a Galois extension over $\mathbf{Q}$. We put

$$L(s, f, K/\mathbf{Q}) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Let N be the primitive period of the coefficients a_n. By the Euler-Fermat theorem, for every prime p not dividing N, we find a positive integer m such that $p^m \equiv 1 \; mod \; N$. By the Dirichlet theorem on primes in arithmetic progressions, we can take distinct primes $p_1, \; p_2, \; \ldots, \; p_m$ with $p_j \equiv p \; mod \; N$. So we have

$$a_p^m = a_{p_1} a_{p_2} \ldots a_{p_m} = a_{p_1 p_2 \ldots p_m} = a_{p^m} = a_1 = 1.$$

Therefore we obtain $|a_p| = 1$ for almost all primes p. Namely we have $|f(\sigma_{\mathfrak{p}})| = 1$ for almost all prime ideals $\mathfrak{p}$, where $\sigma_{\mathfrak{p}}$ means the Frobenius automorphism for $\mathfrak{p}$ in K. By the Chebotarev density theorem, we get $|\, f(\sigma)| = 1$ for all $\sigma \in \; Gal(K)/\mathbf{Q})$. Now we write $f = \sum_\chi a(\chi)\chi$ ($a(\chi) \in \mathbf{Z}$), where χ runs over all irreducible characters χ of $G = Gal(K/\mathbf{Q})$. Since we have

$$\sum_\chi a(\chi)^2 = \langle f, f \rangle = (1/g) \sum_{\sigma \in G} f(\sigma) f(\sigma^{-1}) = (1/g) \sum_{\sigma \in G} 1 = 1,$$

where $g = [K : \mathbf{Q}]$, we obtain $f = \pm\chi$ for some χ. It follows from $\chi(1) = |f(1)| = 1$ that χ is of degree 1. Thus $L(s, \chi, K/\mathbf{Q})$ coincides with the Dirichlet L function for a primitive Dirichlet character ψ. Since the coefficients of $L(s, -\chi, K/\mathbf{Q}) = L(s, \chi, K/\mathbf{Q})^{-1}$ as Dirichlet series are given by $\mu(m)\psi(m)$, where μ is the Möbius function, the coefficients are not periodic. Hence $f = \chi$, which completes the proof.

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