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ON SOME TWISTED CHEVALLEY GROUPS OVER LAURENT POLYNOMIAL RINGS

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0. Introduction. We let **Z** denote the ring of rational integers, **Q** the field of rational numbers, **R** the field of real numbers, and **C** the field of complex numbers.

For elements e and f of a Lie algebra, $[e, f]$ denotes the bracket of e and f.

A generalized Cartan matrix $C = (c_{ij})$ is a square matrix of integers satisfying $c_{ii} = 2$, $c_{ij} \leqq 0$ if $i \neq j$, $c_{ij} = 0$ if and only if $c_{ji} = 0$. For any generalized Cartan matrix $C = (c_{ij})$ of size $l \times l$ and for any field F of characteristic zero, $\mathfrak{L}_F(C)$ denotes the Lie algebra over F generated by $3l$ generators $e_1, \ldots, e_l, h_1, \ldots, h_l, f_1, \ldots, f_l$ with the defining relations

$$[h_i, h_j] = 0,\ [e_i, f_j] = \delta_{ij}h_i,\ [h_i, e_j] = c_{ji}e_j,\ [h_i, f_j] = -c_{ji}f_j$$

for all i, j,

$$(\text{ad } e_i)^{-c_{ji}+1}e_j = 0,\ (\text{ad } f_i)^{-c_{ji}+1}f_j = 0$$

for distinct i, j. Let A be the Cartan matrix arising from a choice of ordered simple roots of a finite dimensional complex semisimple Lie algebra $\mathfrak{g}_{\mathbf{C}}$ with respect to a Cartan subalgebra $\mathfrak{h}_{\mathbf{C}}$. Then $\mathfrak{L}_{\mathbf{C}}(A)$ is isomorphic to $\mathfrak{g}_{\mathbf{C}}$ (cf. [**3**, p. 99]). Such a matrix A is called a finite Cartan matrix.

Let $\mathfrak{G} = \mathfrak{G}_F(C)$ be the subgroup of Aut $(\mathfrak{L}_F(C))$ generated by exp (ad te_i) and exp (ad tf_i) for all $t \in F$ and $i = 1, \ldots, l$. Then $\mathfrak{G}$ has a BN-pair structure, i.e., a Tits system (cf. [**10**]).

A generalized Cartan matrix C is called a Euclidean Cartan matrix if C is singular and possesses the property that removal of any row and the corresponding column leaves a finite Cartan matrix. Euclidean Cartan matrices are classified (cf. [**8**]).

From now on we assume that C is a Euclidean Cartan matrix. The algebra $\mathfrak{L}_F(C)$ has a one dimensional center, denoted by $\mathfrak{Z}$. Let $\mathfrak{E} = \mathfrak{L}_F(C)/\mathfrak{Z}$, called a Euclidean Lie algebra. Any Euclidean Lie algebra $\mathfrak{E}$ owns the constant r associated with the structure of its root system, which is named the tier number and is dependent only on C. It is known that r equals one of 1, 2, or 3 (cf. [**8**]). We suppose that F has a primitive cubic root of unity if the tier number r of $\mathfrak{E}$ is 3. Let $F[T, T^{-1}]$ be the ring of Laurent polynomials in T and T^{-1} with coefficients in F. Then the algebra $\mathfrak{E}$ is isomorphic to the subalgebra of fixed points of $F[T, T^{-1}] \otimes_F \mathfrak{L}_F(A)$

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under $\tau \otimes \sigma$ for some finite Cartan matrix A, where τ is a Galois automorphism of $F[T, T^{-1}]$ over $F[T^r, T^{-r}]$ and σ is a diagram automorphism of $\mathfrak{L}_F(A)$, and both are of order r. The canonical Lie algebra homomorphism of $\mathfrak{L}_F(C)$ onto $\mathfrak{E}$ induces a group homomorphism ϕ of Aut $(\mathfrak{L}_F(C))$ into Aut $(\mathfrak{E})$. Then we can view $\phi(\mathfrak{G})$ as the twisted subgroup, associated with τ and σ, of the elementary subgroup of a Chevalley group of adjoint type over $F[T, T^{-1}]$. We note that $\mathfrak{G}$ and $\phi(\mathfrak{G})$ are isomorphic. In this paper, we will consider not only the group $\phi(\mathfrak{G})$ of adjoint type but non-adjoint types as follows.

Let Φ be a reduced irreducible root system (cf. [**2**]). Let G be a Chevalley group over $K[T, T^{-1}]$ of type Φ, and E the elementary subgroup of G (cf. [**11**]), where $K[T, T^{-1}]$ is the ring of Laurent polynomials in T and T^{-1} with coefficients in a field K and the characteristic of K does not need to be zero. We fix a diagram automorphism σ of Φ (cf. [**2**], [**3**]). We say a pair (Φ, σ) is of r-type if σ is of order r. We assume that K has a primitive rth root of unity when (Φ, σ) is of r-type. Let τ be a Galois automorphism (with the same order as σ) of $K[T, T^{-1}]$ over $K[T^r, T^{-r}]$. Then we can construct the twisted subgroup E' of E associated with τ and σ. Of course, if $r = 1$, i.e., σ is trivial, then $E = E'$.

Our assertion is that E' has a BN-pair structure (cf. Theorem 3.1/3.4). In [**11**], it is confirmed that E has a BN-pair structure, therefore we will assume $r = 2$ or 3, i.e., Φ is of type A_n $(n \geqq 2)$, D_n $(n \geqq 4)$ or E_6, and σ is not trivial (cf. Table 1). In Section 1 we introduce the twisted root system Φ_σ defined by (Φ, σ) and argue about the connection between twisted root systems and affine Weyl groups of type B_l, C_l, F_4 and G_2. We will construct twisted Lie algebras in Section 2 and twisted Chevalley groups in Section 3 respectively. Our assertion can be reduced to the case of rank 1, which is essential and considered in Section 4. In Section 5 we complete the proof of our assertion.

Let x and y be elements of a group, then $[x, y]$ denotes the commutator $xyx^{-1}y^{-1}$ of x and y. For two subgroups G_2 and G_3 of a group G_1, let $[G_2, G_3]$ be the subgroup of G_1 generated by $[x, y]$ for all $x \in G_2$ and $y \in G_3$. We shall write $G_1 = G_2 \cdot G_3$ when a group G_1 is a semidirect product of two groups G_2 and G_3, and G_3 normalizes G_2.

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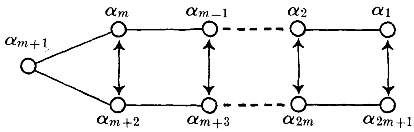
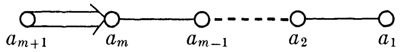
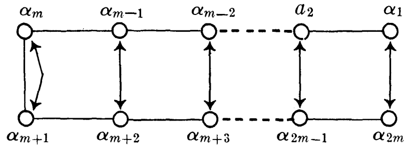
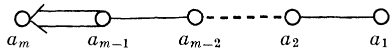
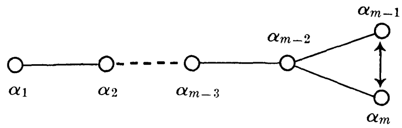
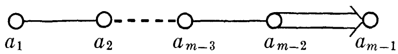
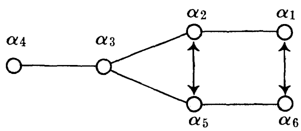
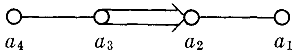
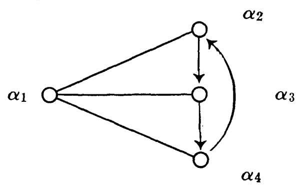
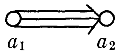
1. Twisted root systems. Let Φ be a reduced irreducible root system in a Euclidean space V (over $\mathbf{R}$) of dimension n with an inner product $(\, , \,)$, and $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ a simple system of Φ (cf. [**2**], [**3**]). For any nonzero element α in V, let w_α be the orthogonal transformation of V defined by $w_\alpha(v) = v - \langle v, \alpha \rangle \alpha$ for all $v \in V$, where $\langle v, \alpha \rangle = 2(v, \alpha)/(\alpha, \alpha)$. Let Φ be of type A_n $(n \geqq 2)$, D_n $(n \geqq 4)$ or E_6. We fix a nontrivial diagram automorphism σ of Φ (cf. Table 1). The automorphism induces

TABLE 1.

Φ/Φ_σ	Π/Π_σ	σ
A_{2m+1} $(m \geqq 1)$		$\sigma(\alpha_i) = \alpha_{2m+2-i}$ $(1 \leqq i \leqq 2m+1)$
C_{m+1}		$a_j = \frac{1}{2}(\alpha_j + \alpha_{2m+2-j})$ $(1 \leqq j \leqq m+1)$
A_{2m} $(m \geqq 1)$		$\sigma(\alpha_i) = \alpha_{2m+1-i}$ $(1 \leqq i \leqq 2m)$
BC_m		$a_j = \frac{1}{2}(\alpha_j + \alpha_{2m+1-j})$ $(1 \leqq j \leqq m)$ $2a_m = \alpha_m + \alpha_{m+1}$
D_m $(m \geqq 4)$		$\sigma(\alpha_i) = \alpha_i$ $(1 \leqq i \leqq m-2)$ $\sigma(\alpha_{m-1}) = \alpha_m$ $\sigma(\alpha_m) = \alpha_{m-1}$
B_{m-1}		$a_j = \alpha_j$ $(1 \leqq j \leqq m-2)$ $a_{m-1} = \frac{1}{2}(\alpha_{m-1} + \alpha_m)$
E_6		$\sigma(\alpha_1) = \alpha_6$ $\sigma(\alpha_2) = \alpha_5$ $\sigma(\alpha_3) = \alpha_3$ $\sigma(\alpha_4) = \alpha_4$ $\sigma(\alpha_5) = \alpha_2$ $\sigma(\alpha_6) = \alpha_1$
F_4		$a_1 = \frac{1}{2}(\alpha_1 + \alpha_6)$ $a_2 = \frac{1}{2}(\alpha_2 + \alpha_5)$ $a_3 = \alpha_3$, $a_4 = \alpha_4$
D_4		$\sigma(\alpha_1) = \alpha_1$ $\sigma(\alpha_2) = \alpha_3$ $\sigma(\alpha_3) = \alpha_4$ $\sigma(\alpha_4) = \alpha_2$
G_2		$a_1 = \alpha_1$ $a_2 = \frac{1}{3}(\alpha_2 + \alpha_3 + \alpha_4)$

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an automorphism of V, also denoted σ. Let V_σ be the subspace of fixed points of V under σ and $l = \dim V_\sigma$, and let Π be the natural projection of V onto V_σ. We let Φ_σ (resp. Π_σ) denote the image of Φ (resp. Π) under the projection π. Then Φ_σ is an irreducible root system with a simple system Π_σ in V_σ, but it is not necessarily reduced (cf. Table 1). Let $\Phi_\sigma{}^+$ be the positive system of Φ_σ with respect to Π_σ, and $\Phi_\sigma{}^- = \Phi_\sigma - \Phi_\sigma{}^+$. We note $\Phi_\sigma{}^+ = \pi(\Phi^+)$ and $\Phi_\sigma{}^- = \pi(\Phi^-)$, where Φ^+ is the positive system of Φ with respect to Π, and $\Phi^- = \Phi - \Phi^+$.

We shall identify the set of σ-orbits in Φ with the set Φ_σ. Then we have the following four types of roots in Φ_σ. Let $c \in \Phi_\sigma$.

(R-1) $c = \{\gamma\}, \gamma = \sigma(\gamma)$

(R-2) $c = \{\gamma_1, \gamma_2\}, \gamma_1 \neq \gamma_2 = \sigma(\gamma_1), \gamma_1 + \gamma_2 \notin \Phi_\sigma$

(R-3) $c = \{\gamma_1, \gamma_2\}, \gamma_1 \neq \gamma_2 = \sigma(\gamma_1), \gamma_1 + \gamma_2 \in \Phi_\sigma$

(R-4) $c = \{\gamma_1, \gamma_2, \gamma_3\}, \gamma_1 \neq \gamma_2 \neq \gamma_3 \neq \gamma_1, \gamma_2 = \sigma(\gamma_1),$
$\gamma_3 = \sigma(\gamma_2), \gamma_1 = \sigma(\gamma_3).$

For each $c \in \Phi_\sigma{}^+$, we fix an order of elements in c according to the action of σ, so we sometimes view the set c as an ordered pair (γ_1, γ_2) (resp. an ordered triple $(\gamma_1, \gamma_2, \gamma_3)$) if c is of type (R-2) or (R-3) (resp. of type (R-4)). Then we let $-c = (-\gamma_1, -\gamma_2)$ or $(-\gamma_1, -\gamma_2, -\gamma_3)$ if $c = (\gamma_1, \gamma_2)$ or $(\gamma_1, \gamma_2, \gamma_3)$ respectively.

If Φ_σ is of type B_l $(l \geqq 3)$, C_l $(l \geqq 2)$, F_4, BC_1 or G_2, then Φ_σ has two root lengths, and we distinguish long roots from short roots. If Φ_σ is of type BC_l $(l \geqq 2)$, then Φ_σ has three root lengths, and we differentiate long roots, middle roots and short roots (cf. Table 2).

TABLE 2.

	Φ_σ	roots	lengths
(a)	B_l $(l \geqq 3)$, C_l $(l \geqq 2)$, F_4	(R − 1)	long
		(R − 2)	short
(b)	BC_1	(R − 1)	long
		(R − 3)	short
(c)	BC_l $(l \geqq 2)$	(R − 1)	long
		(R − 2)	middle
		(R − 3)	short
(d)	G_2	(R − 1)	long
		(R − 4)	short

Now we consider the subset $\Omega = \Omega_1 \cup \Omega_2$ of $\Phi_\sigma \times \mathbf{Z}$ defined as follows.

Type (a):

$$\Omega_1 = \{(c, 2n); c \text{ is long}, n \in \mathbf{Z}\}$$
$$\Omega_2 = \{(c, n); c \text{ is short}, n \in \mathbf{Z}\}$$

Type (b):

$$\Omega_1 = \{(c, 2n + 1); c \text{ is long}, n \in \mathbf{Z}\}$$
$$\Omega_2 = \{(c, n); c \text{ is short}, n \in \mathbf{Z}\}$$

Type (c):

$$\Omega_1 = \{(c, 2n + 1); c \text{ is long}, n \in \mathbf{Z}\}$$
$$\Omega_2 = \{(c, n); c \text{ is middle or short}, n \in \mathbf{Z}\}$$

Type (d):

$$\Omega_1 = \{(c, 3n); c \text{ is long}, n \in \mathbf{Z}\}$$
$$\Omega_2 = \{(c, n); c \text{ is short}, n \in \mathbf{Z}\}.$$

We see that Ω corresponds to an affine root system, denoted $S(\Phi_\sigma)^\vee$ (cf. [**11**, Proposition 2.1/Theorem 5.2]), and that an element (c, n) of Ω can be regarded as an element $c + n\xi$ of the corresponding Euclidean root system (cf. [**8**, Table 2]).

For each $(a, n) \in \Omega$, let $w_{a,n}$ be a permutation on Ω defined by

$$w_{a,n}(b, m) = (w_a b, m - \langle b, a\rangle \mathrm{n})$$

for all $(b, m) \in \Omega$. Let $W(\Omega)$ be the permutation group on Ω generated by $w_{a,n}$ for all $(a, n) \in \Omega$. We note that $W(\Omega)$ acts on $\Phi_\sigma \times \mathbf{Z}$ similarly. For each $(a, n) \in \Omega$, set

$$h_{a,n} = w_{a,n}w_{a,0}^{-1} \text{ if } \tfrac{1}{2}a \notin \Phi_\sigma,$$

and set

$$h_{a,n} = w_{a,n}w_{b,0}^{-1} \text{ if } b = \tfrac{1}{2}a \in \Phi_\sigma.$$

Let I be the subgroup of $W(\Omega)$ generated by $h_{a,n}$ for all $(a, n) \in \Omega$, and let J be the subgroup of $W(\Omega)$ generated by $w_{a,0}$ for all $a \in \mathrm{Red}\,(\Phi_\sigma)$, where

$$\mathrm{Red}\,(\Phi_\sigma) = \{b \in \Phi_\sigma; \tfrac{1}{2}b \notin \Phi_\sigma\}.$$

We see that J is isomorphic to the Weyl group W of Φ_σ.

LEMMA 1.1. (1) *Let* (a, n) *and* (b, m) *be in* Ω. *Then*

$$h_{a,n}(b, m) = (b, m + \langle b, a\rangle n).$$

(2) *Suppose that Φ_σ is of type BC_l. Let a be in Φ_σ and of type* (R-3). *Then* $h_{a,1} = (h_{2a,1})^2$.

(3) *Let (a, n) and (b, m) be in Ω, and set $c = w_a b$. Then*

$$w_{a,n} h_{b,m} w_{a,n}^{-1} = h_{c,m}.$$

Let Ω_I be the subset of Ω defined below, where notation is as in Table 1:

$$\Omega_I = \{(a_i, 1), (a_{m+1}, 2); 1 \leqq i \leqq m\} \text{ if } \Phi_\sigma \text{ is of type } C_{m+1},$$
$$\Omega_I = \{(a_i, 1), (2a_m, 1); 1 \leqq i \leqq m-1\} \text{ if } \Phi_\sigma \text{ is of type } BC_m,$$
$$\Omega_I = \{(a_i, 2), (a_{m-1}, 1); 1 \leqq i \leqq m-2\} \text{ if } \Phi_\sigma \text{ is of type } B_{m-1},$$
$$\Omega_I = \{(a_1, 1), (a_2, 1), (a_3, 2), (a_4, 2)\} \text{ if } \Phi_\sigma \text{ is of type } F_4,$$
$$\Omega_I = \{(a_1, 3), (a_2, 1)\} \text{ if } \Phi_\sigma \text{ is of type } G_2.$$

Then I is the free abelian group generated by $h_{a,n}$ for all $(a, n) \in \Omega_I$, so $W(\Omega) = I \cdot J$.

Let $\Pi_\sigma = \{a_1, \ldots, a_l\}$ and let a_0 be as follows:

(1) a_0 is the highest short root in Φ_σ with respect to Π_σ if Φ_σ is of type B_l, C_l, F_4, or G_2,

(2) a_0 is the highest root in Φ_σ with respect to Π_σ if Φ_σ is of type BC_l. Set $a_{l+1} = -a_0$.

Let Δ be the dual root system of Red (Φ_σ) and $\Delta_0 = \{\delta_1, \ldots, \delta_l\}$ be a simple system of Δ. Let W^* be the affine Weyl group of Δ, and let δ_0 be the highest root in Δ with respect to Δ_0. Put $\delta_{l+1} = -\delta_0$. Let $\Delta_1 = \Delta \times \mathbf{Z}$, and an element of Δ_1 is denoted by $\delta^{(n)}$, where $\delta \in \Delta$ and $n \in \mathbf{Z}$.

For each $\delta^{(n)} \in \Delta_1$, let $w_{\delta^{(n)}}$ be the permutation on Δ_1 defined by

$$w_{\delta^{(n)}} \chi^{(m)} = (w_\delta \chi)^{(m - \langle \chi, \delta \rangle n)}$$

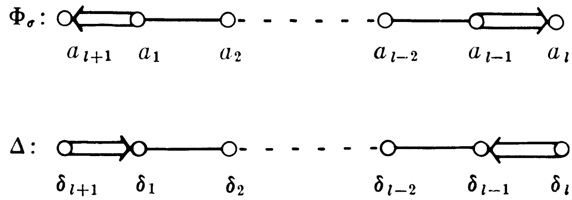
for all $\chi^{(m)} \in \Delta_1$. Let W_1 be the permutation group on Δ_1 generated by $w_{\delta^{(n)}}$ for all $\delta^{(n)} \in \Delta_1$, and W_0 the subgroup of W_1 generated by $w_{\delta^{(0)}}$ for all $\delta \in \Delta$. Set

$$h_{\delta^{(n)}} = w_{\delta^{(n)}} w_{\delta^{(0)}}^{-1}$$

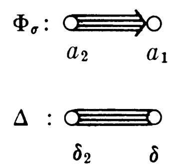
and H_1 be the subgroup of W_1 generated by $h_{\delta^{(n)}}$ for all $\delta^{(n)} \in \Delta_1$. Then W_0 is isomorphic to the Weyl group of Δ, and H_1 is the free abelian group generated by $h_{\delta_i^{(1)}}$ for all $\delta_i \in \Delta_0$, hence $W_1 = H_1 \cdot W_0$ and $W_1 \simeq W^*$ (cf. [**11**, Lemma 1.1/Proposition 1.2]). Clearly $I \simeq H_1 \simeq \mathbf{Z}^l$ and $J \simeq W_0 \simeq W$.

We fix simple roots of Φ_σ and Δ as follows, then we have a_{l+1} and δ_{l+1} as above. (We add the vertices of a_{l+1} and δ_{l+1}, and the corresponding edges.)

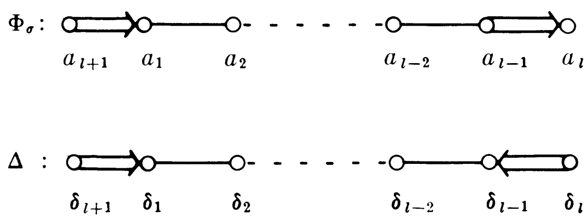
(i) The case $\Phi_\sigma = B_l$ and $\Delta = C_l$ $(l \geqq 3)$:



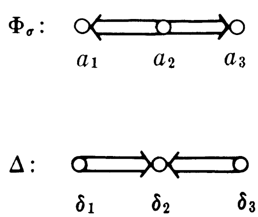
(ii) The case $\Phi_\sigma = BC_l$ and $\Delta = A_1$:



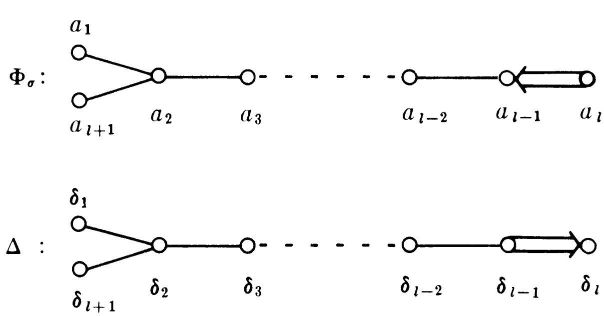
(iii) The case $\Phi_\sigma = BC_l$ and $\Delta = C_l$ $(l \geqq 2)$:



(iv) The case $\Phi_\sigma = C_2$ and $\Delta = B_2$:

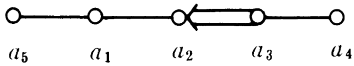


(v) The case $\Phi_\sigma = C_l$ and $\Delta = B_l$ $(l \geqq 3)$:

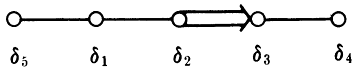


(vi) The case $\Phi_\sigma = F_4$ and $\Delta = F_4$:

Φ_σ:

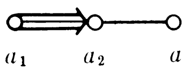


Δ :

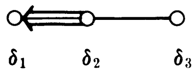


(vii) The case $\Phi_\sigma = G_2$ and $\Delta = G_2$:

Φ_σ:



Δ :



The map ψ defined by

$$\psi(w_{\delta_i}{}^{(0)}) = w_{a_i,0}$$

for $1 \leqq i \leqq l$ and

$$\psi(w_{\delta_{l+1}}{}^{(1)}) = w_{a_{l+1},1}$$

induces an isomorphism, again called ψ, of W^* onto $W(\Omega)$. This fact is easily verified by the next lemma and proposition.

LEMMA 1.2. *Let (a, m) be in Ω and w in $W(\Omega)$, and set $(b, n) = w(a, m)$. Then $w\, w_{a,m} w^{-1} = w_{b,n}$* (cf. [**11**, Lemma 1.3]).

Set

$$\Omega_0 = \{(a_0, 1), (-a_i, 0); 1 \leqq i \leqq l\} \text{ and}$$
$$Y' = \{w_{a,n}; (a, n) \in \Omega_0\}.$$

PROPOSITION 1.3. *Let $W(\Omega)$ and Y' be as above. Then $W(\Omega)$ is generated by Y'* (cf. [**11**, Proposition 1.4]).

Thus, the following result has been proved.

PROPOSITION 1.4. *The group $W(\Omega)$ is isomorphic to the affine Weyl group of type Δ as in the following table.*

TABLE 3.

Φ°	B_l	BC_l	C_l	F_4	G_2
Δ	C_l	C_l	B_l	F_4	G_2

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When $w \in W(\Omega)$ is written as $w_1 w_2 \ldots w_k$ ($w_j \in Y'$, k minimal), we write $l(w) = k$: this is the length of w. Set

$$\Omega^+ = \Omega \cap (\Phi_\sigma^+ \times \mathbf{Z}_{>0} \cup \Phi_\sigma^- \times \mathbf{Z}_{\geqq 0})$$

and

$$\Omega^- = \Omega - \Omega^+.$$

For each $w \in W(\Omega)$, set

$$\Gamma(w) = \{(a, n) \in \Omega^+; w(a, n) \in \Omega^-\}$$

and

$$N(w) = \text{Card } \Gamma(w).$$

The following two propositions hold (cf. [**4**, Lemma 2.1/2.2] and [**11**, Proposition 1.5/1.8]).

PROPOSITION 1.5. *Let (a, n) be in Ω_0 and w in $W(\Omega)$. Then:*
(1) $\Gamma(w_{a,n}) = \{(a, n)\}$,
(2) $w_{a,n}(\Gamma(w) - \{(a, n)\}) = \Gamma(w\, w_{a,n}) - \{(a, n)\}$,
(3) *(a, n) is in precisely one of $\Gamma(w)$ or $\Gamma(w, w_{a,n})$,*
(4) $N(w\, w_{a,n}) = N(w) - 1$ *if* $(a, n) \in \Gamma(w)$, $N(w\, w_{a,n}) = N(w) + 1$ *if* $(a, n) \notin \Gamma(w)$.

PROPOSITION 1.6. *Let w be in $W(\Omega)$. Then $N(w) = l(w)$.*

2. Twisted Lie algebras. Let Φ be a reduced irreducible root system with a simple system $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ and $\mathfrak{g}_\mathbf{C}$ a finite dimensional complex simple Lie algebra of type Φ. Then there is a Chevalley basis $\{h_i, e_\alpha; 1 \leqq i \leqq n, \alpha \in \Phi\}$ of $\mathfrak{g}_\mathbf{C}$ satisfying

$$(1) \qquad [h_i, e_\alpha] = \langle \alpha, \alpha_i \rangle e_\alpha,$$

$$(2) \qquad [e_\alpha, e_\beta] = \begin{cases} N_{\alpha,\beta} e_{\alpha+\beta} \text{ if } \alpha + \beta \in \Phi, \\ h_\alpha \text{ if } \alpha + \beta = 0, \\ 0 \text{ otherwise,} \end{cases}$$

(3) $N_{\alpha,\beta} = \pm(p + 1)$ if $\beta - p\alpha, \ldots, \beta, \ldots, \beta + q\alpha$ is the α-string through β, $N_{\alpha,\beta} = -N_{\beta,\alpha} = -N_{-\alpha,-\beta}$,
(4) h_α is a $\mathbf{Z}$-linear combinations of h_i's, $h_{\alpha_i} = h_i$, for any $\alpha, \beta \in \Phi$ and $1 \leqq i \leqq n$. We set

$$\mathfrak{h}_\mathbf{Z} = \sum_{i=1}^n \mathbf{Z} h_i \quad \text{and} \quad \mathfrak{g}_\mathbf{Z} = \mathfrak{h}_\mathbf{Z} + \sum_{\alpha \in \Phi} \mathbf{Z} e_\alpha.$$

Let $K[T, T^{-1}]$ be the ring of Laurent polynomials in T and T^{-1} with coefficients in a field K, i.e.,

$$K[T, T^{-1}] = \left\{ \sum_{m \in \mathbf{Z}} t_m T^m \text{ (finite sum)}; t_m \in K \right\},$$

and set

$$L = K[T, T^{-1}] \otimes_{\mathbf{Z}} \mathfrak{g}_{\mathbf{Z}} \quad \text{and} \quad \mathfrak{h} = K[T, T^{-1}] \otimes_{\mathbf{Z}} \mathfrak{h}_{\mathbf{Z}}.$$

From now on we will assume that Φ is of type A_n $(n \geqq 2)$, D_n $(n \geqq 4)$ or E_6. We fix a nontrivial diagram automorphism σ of Φ (cf. Table 1). Associated to σ, we can find an automorphism of $\mathfrak{g}_{\mathbf{Z}}$, again denoted σ, such that

$$\sigma(h_{\alpha_i}) = h_{\beta_i},\ \sigma(e_{\pm\alpha_i}) = e_{\pm\beta_i}$$

for all $\alpha_i \in \Pi$, where $\beta_i = \sigma(\alpha_i)$. We write

$$\sigma(e_\alpha) = k_\alpha e_{\sigma(\alpha)}$$

for each $\alpha \in \Phi$, where $k_\alpha \in \mathbf{Z}$. Then we have $k_\alpha = \pm 1$ for all $\alpha \in \Phi$.

PROPOSITION 2.1. *Let (Φ, σ) be of 2-type. Then we can choose a Chevalley basis which satisfies the following condition:*

(1) $k_\alpha = -1$ *if Φ is of type A_{2n} $(n \geqq 1)$ and $\sigma(\alpha) = \alpha$;*

(2) $k_\alpha = 1$ *otherwise* (*cf.* [**1**, *Proposition* 3.1]).

PROPOSITION 2.2. *Let (Φ, σ) be of 3-type. Then we can choose a Chevalley basis such that $k_\alpha = 1$ for all $\alpha \in \Phi$.*

Proof. We have $k_\alpha = k_{-\alpha}$ as $\sigma(h_\alpha) = h_{\sigma(\alpha)}$, so we may assume α is positive. Suppose $\sigma(\alpha) = \alpha$. Then $(k_\alpha)^3 = 1$ and $k_\alpha = 1$. Next suppose $\sigma(\alpha) \neq \alpha$, and set $\beta = \sigma(\alpha)$ and $\gamma = \sigma^2(\alpha)$. Then $k_\alpha k_\beta k_\gamma = 1$, and $(k_\alpha, k_\beta, k_\gamma) = (1, 1, 1), (1, -1, -1), (-1, 1, -1)$, or $(-1, -1, 1)$. To establish this proposition, we may assume $(k_\alpha, k_\beta, k_\gamma) = (1, -1, -1)$. Replacing e_γ by $-e_\gamma$, we have $\sigma(e_\alpha) = e_\beta$, $\sigma(e_\beta) = e_\gamma$ and $\sigma(e_\gamma) = e_\alpha$. Arrange the bases for negative roots similarly, and $k_\alpha = 1$ for all $\alpha \in \Phi$.

We shall fix a Chevalley basis of $\mathfrak{g}_{\mathbf{C}}$ with the properties of Proposition 2.1 or 2.2. We assume that K has a primitive rth root of unity when (Φ, σ) is of r-type. Therefore, in particular, we have char $K \neq r$. If $r = 3$, we let ω denote a primitive cubic root of unity in K. Let τ be the Galois automorphism of $K[T, T^{-1}]$ over $K[T^r, T^{-r}]$ defined by

(1) $\tau(T^{\pm 1}) = -T^{\pm 1}$ if $r = 2$,

(2) $\tau(T^{\pm 1}) = (\omega T)^{\pm 1}$ if $r = 3$.

Let L' (resp. $\mathfrak{h}'$) be the subalgebra of fixed points of L (resp. $\mathfrak{h}$) under $\tau \otimes \sigma$. (For more general cases, see [**5**], [**6**]).

For each $(c, m) \in \Omega$, we define an element $e_{c,m}$ of L' as follows.

Type (a):

$$e_{c,m} = T^m e_\gamma \text{ if } c = (\gamma) \text{ is of type (R-1) and } m \equiv 0\ (2)$$
$$e_{c,m} = T^m e_{\gamma_1} + T^m e_{\gamma_2} \text{ if } c = (\gamma_1, \gamma_2) \text{ is of type (R-2) and } m \equiv 0\ (2)$$
$$e_{c,m} = T^m e_{\gamma_1} - T^m e_{\gamma_2} \text{ if } c = (\gamma_1, \gamma_2) \text{ is of type (R-2) and } m \equiv 1\ (2).$$

Type (b):

$$e_{c,m} = T^m e_\gamma \text{ if } c = (\gamma) \text{ is of type (R-1) and } m \equiv 1\ (2)$$
$$e_{c,m} = T^m e_{\gamma_1} + T^m e_{\gamma_2} \text{ if } c = (\gamma_1, \gamma_2) \text{ is of type (R-3) and } m \equiv 0\ (2)$$
$$e_{c,m} = T^m e_{\gamma_1} - T^m e_{\gamma_2} \text{ if } c = (\gamma_1, \gamma_2) \text{ is of type (R-3) and } m \equiv 1\ (2).$$

Type (c):

$$e_{c,m} = T^m e_\gamma \text{ if } c = (\gamma) \text{ is of type (R-1) and } m \equiv 1\ (2)$$
$$e_{c,m} = T^m e_{\gamma_1} + T^m e_{\gamma_2} \text{ if } c = (\gamma_1, \gamma_2) \text{ is of type (R-2) or (R-3), and } m \equiv 0\ (2)$$
$$e_{c,m} = T^m e_{\gamma_1} - T^m e_{\gamma_2} \text{ if } c = (\gamma_1, \gamma_2) \text{ is of type (R-2) or (R-3), and } m \equiv 1\ (2).$$

Type (d):

$$e_{c,m} = T^m e \text{ if } c = (\gamma) \text{ is of type (R-1) and } m \equiv 0\ (3)$$
$$e_{c,m} = T^m e_{\gamma_1} + T^m e_{\gamma_2} + T^m e_{\gamma_3} \text{ if } c = (\gamma_1, \gamma_2, \gamma_3) \text{ is of type (R-4) and } m \equiv 0\ (3)$$
$$e_{c,m} = T^m e_{\gamma_1} + \omega T^m e_{\gamma_2} + \omega^2 T^m e_{\gamma_3} \text{ if } c = (\gamma_1, \gamma_2, \gamma_3) \text{ is of type (R-4) and } m \equiv 1\ (3)$$
$$e_{c,m} = T^m e_{\gamma_1} + \omega^2 T^m e_{\gamma_2} + \omega T^m e_{\gamma_3} \text{ if } c = (\gamma_1, \gamma_2, \gamma_3) \text{ is of type (R-4) and } m \equiv 2\ (3).$$

Then $L' = \mathfrak{h}' \oplus \sum_{(c,m)\in\Omega} Ke_{c,m}$. For each $c \in \Phi_\sigma$, set $h_c = h_\gamma$ if $c = (\gamma)$ is of type (R-1), $h_c = h_{\gamma_1} + h_{\gamma_2}$ if $c = (\gamma_1, \gamma_2)$ is of type (R-2) or (R-3), and $h_c = h_{\gamma_1} + h_{\gamma_2} + h_{\gamma_3}$ if $c = (\gamma_1, \gamma_2, \gamma_3)$ is of type (R-4). Let

$$\mathfrak{h}'' = \sum_{c\in\Phi_\sigma} Kh_c.$$

For each $(c, m) \in \Omega$, we have $[h, e_{c,m}] = c(h)e_{c,m}$ for all $h \in \mathfrak{h}''$, where c is regarded as an element of $(\mathfrak{h}'')^*$, the dual of $\mathfrak{h}''$.

PROPOSITION 2.3. *Let (c, m) be in Ω. Then:*
(1) $[h_c, e_{c,m}] = 2e_{c,m}$ *if c is of type* (R-1), (R-2) *or* (R-4),
(2) $[h_c, e_{c,m}] = e_{c,m}$ *if c is of type* (R-3),
(3) $[e_{c,m}, e_{-c,-m}] = h_c$.

Proof. The case when c is of type (R-1), (R-2), or (R-4) is easy. Assume $c = (\gamma_1, \gamma_2)$ is of type (R-3). Then $h_c = h_{\gamma_1} + h_{\gamma_2}$, and $e_{c,m} = T^m e_{\gamma_1} + T^m e_{\gamma_2}$ (resp. $T^m e_{\gamma_1} - T^m e_{\gamma_2}$) if $m \equiv 0\ (2)$ (resp. $m \equiv 1\ (2)$). Hence,

$$[h_{\gamma_1} + h_{\gamma_2}, T^m e_{\gamma_1} \pm T^m e_{\gamma_2}] = 2T^m e_{\gamma_1} - T^m e_{\gamma_1} \mp T^m e_{\gamma_2} \pm 2T^m e_{\gamma_2} = T^m e_{\gamma_1} \pm T^m e_{\gamma_2}$$

and

$$[T^m e_{\gamma_1} \pm T^m e_{\gamma_2}, T^{-m} e_{-\gamma_1} \pm T^{-m} e_{-\gamma_2}] = h_{\gamma_1} + h_{\gamma_2}.$$

3. Twisted Chevalley groups. Let ρ be a finite dimensional complex faithful representation of $\mathfrak{g}_{\mathbf{C}}$. We let G be a Chevalley group over $K[T, T^{-1}]$ associated with $\mathfrak{g}_{\mathbf{C}}$ and ρ. Set $\Phi_1 = \Phi \times \mathbf{Z}$. For each $(\alpha, n) \in \Phi_1$, there exists a group isomorphism

$$t \mapsto x_\alpha^{(n)}(t)$$

of the additive group K^+ of K onto a subgroup $X_\alpha^{(n)}$ of G (for the definition, see [**11**]). The elementary subgroup E of G is generated by $X_\alpha^{(n)}$ for all $(\alpha, n) \in \Phi_1$. Let K^* be the multiplicative group of K. For each $(\alpha, n) \in \Phi_1$ and $t \in K^*$, we define

$$w_\alpha^{(n)}(t) = x_\alpha^{(n)}(t)x_{-\alpha}^{(-n)}(-t^{-1})x_\alpha^{(n)}(t),$$
$$h_\alpha^{(n)}(t) = w_\alpha^{(n)}(t)w_\alpha^{(0)}(1)^{-1}.$$

Let N be the subgroup of E generated by $w_\alpha^{(n)}(t)$ for all $(\alpha, n) \in \Phi_1$ and $t \in K^*$, and let H_0 be the subgroup of E generated by $h_\alpha^{(0)}(t)$ for all $\alpha \in \Phi$ and $t \in K^*$. Let U be the subgroup of E generated by $x_\alpha^{(n)}(t)$ for all $(\alpha, n) \in \Phi_1^+$ and $t \in K$, where

$$\Phi_1^+ = (\Phi^+ \times \mathbf{Z}_{>0}) \cup (\Phi^- \times \mathbf{Z}_{\geqq 0}).$$

Let B be the subgroup of E generated by U and H_0.

THEOREM 3.1. *Notation is as above. Then:*

(1) (E, B, N) *is a Tits system,*

(2) $N/(B \cap N)$ *is isomorphic to the affine Weyl group of* Φ (*cf.* [**11**, *Theorem* 2.1]).

For any $(c, m) \in \Omega$ and $t \in K$, we define $x_{c,m}(t)$ as follows.

Type (a):

$$x_{c,m}(t) = x_\gamma^{(m)}(t) \text{ if } c = (\gamma) \text{ is of type (R-1) and } m \equiv 0\ (2)$$
$$x_{c,m}(t) = x_{\gamma_1}^{(m)}(t)x_{\gamma_2}^{(m)}(t) \text{ if } c = (\gamma_1, \gamma_2) \text{ is of type (R-2) and } m \equiv 0\ (2)$$
$$x_{c,m}(t) = x_{\gamma_1}^{(m)}(t)x_{\gamma_2}^{(m)}(-t) \text{ if } c = (\gamma_1, \gamma_2) \text{ is of type (R-2) and } m \equiv 1\ (2).$$

Type (b):

$$x_{c,m}(t) = x_\gamma^{(m)}(t) \text{ if } c = (\gamma) \text{ is of type (R-1) and } m \equiv 1\ (2)$$
$$x_{c,m}(t) = x_{\gamma_1}^{(m)}(t)x_{\gamma_2}^{(m)}(t)x_{\gamma_1+\gamma_2}^{(2m)}(\tfrac{1}{2}N_{\gamma_2,\gamma_1}t^2)$$
$$\text{if } c = (\gamma_1, \gamma_2) \text{ is of type (R-3) and } m \equiv 0\ (2)$$
$$x_{c,m}(t) = x_{\gamma_1}^{(m)}(t)x_{\gamma_2}^{(m)}(-t)x_{\gamma_1+\gamma_2}^{(2m)}(-\tfrac{1}{2}N_{\gamma_2,\gamma_1}t^2)$$
$$\text{if } c = (\gamma_1, \gamma_2) \text{ is of type (R-3) and } m \equiv 1\ (2).$$

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Type (c):

$$x_{c,m}(t) = x_{\gamma}{}^{(m)}(t) \text{ if } c = (\gamma) \text{ is of type (R-1) and } m \equiv 1\ (2)$$
$$x_{c,m}(t) = x_{\gamma_1}{}^{(m)}(t)x_{\gamma_2}{}^{(m)}(t) \text{ if } c = (\gamma_1, \gamma_2) \text{ is of type (R-2) and } m \equiv 0\ (2)$$
$$x_{c,m}(t) = x_{\gamma_1}{}^{(m)}(t)x_{\gamma_2}{}^{(m)}(-t) \text{ if } c = (\gamma_1, \gamma_2) \text{ is of type (R-2) and } m \equiv 1\ (2)$$
$$x_{c,m}(t) = x_{\gamma_1}^{(m)}(t)x_{\gamma_2}^{(m)}(t)x_{\gamma_1+\gamma_2}^{(2m)}(\tfrac{1}{2}N_{\gamma_2,\gamma_1}t^2) \quad \text{if } c = (\gamma_1, \gamma_2) \text{ is of type (R-3) and } m \equiv 0\ (2)$$
$$x_{c,m}(t) = x_{\gamma_1}^{(m)}(t)x_{\gamma_2}^{(m)}(-t)x_{\gamma_1+\gamma_2}^{(2m)}(-\tfrac{1}{2}N_{\gamma_2,\gamma_1}t^2) \quad \text{if } c = (\gamma_1, \gamma_2) \text{ is of type (R-3) and } m \equiv 1\ (2).$$

Type (d):

$$x_{c,m}(t) = x_{\gamma}{}^{(m)}(t) \text{ if } c = (\gamma) \text{ is of type (R-1) and } m \equiv 0\ (3)$$
$$x_{c,m}(t) = x_{\gamma_1}{}^{(m)}(t)x_{\gamma_2}{}^{(m)}(t)x_{\gamma_3}{}^{(m)}(t) \quad \text{if } c = (\gamma_1, \gamma_2, \gamma_3) \text{ is of type (R-4) and } m \equiv 0\ (3)$$
$$x_{c,m}(t) = x_{\gamma_1}{}^{(m)}(t)x_{\gamma_2}{}^{(m)}(\omega t)x_{\gamma_3}{}^{(m)}(\omega^2 t) \quad \text{if } c = (\gamma_1, \gamma_2, \gamma_3) \text{ is of type (R-4) and } m \equiv 1\ (3)$$
$$x_{c,m}(t) = x_{\gamma_1}{}^{(m)}(t)x_{\gamma_2}{}^{(m)}(\omega^2 t)x_{\gamma_3}{}^{(m)}(\omega t) \quad \text{if } c = (\gamma_1, \gamma_2, \gamma_3) \text{ is of type (R-4) and } m \equiv 2\ (3).$$

For each $(c, m) \in \Omega$, let $X_{c,m}$ be the subgroup of E generated by $x_{c,m}(t)$ for all $t \in K$. Then $X_{c,m}$ is isomorphic to the additive group K^+ of K. Let E' be the subgroup of E generated by $X_{c,m}$ for all $(c, m) \in \Omega$. For each $(c, m) \in \Omega$ and $t \in K^*$, we define

$$w_{c,m}(t) = x_{c,m}(t)x_{-c,-m}(-t^{-1})x_{c,m}(t)$$

if c is of type (R-1), (R-2) or (R-4),

$$w_{c,m}(t) = x_{c,m}(t)x_{-c,-m}(-2t^{-1})x_{c,m}(t)$$

if c is of type (R-3) and $m \equiv 0\ (2)$,

$$w_{c,m}(t) = x_{c,m}(t)x_{-c,-m}(2t^{-1})x_{c,m}(t)$$

if c is of type (R-3) and $m \equiv 1\ (2)$.

Let N' be the subgroup of E' generated by $w_{c,m}(t)$ for all $(c, m) \in \Omega$ and $t \in K^*$.

LEMMA 3.2. *Let (c, m) be in Ω and t in K^*. Then:*

(1) $w_{c,m}(t) = w_{\gamma}{}^{(m)}(t)$ *if $c = (\gamma)$ is of type* (R-1),

(2) $w_{c,m}(t) = w_{\gamma_1}{}^{(m)}(t)w_{\gamma_2}{}^{(m)}(t)$ *if $c = (\gamma_1, \gamma_2)$ is of type* (R-2) *and $m \equiv 0\ (2)$*,

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(3) $w_{c,m}(t) = w_{\gamma_1}{}^{(m)}(t)w_{\gamma_2}{}^{(m)}(-t)$ *if* $c = (\gamma_1, \gamma_2)$ *is of type* (R-2) *and* $m \equiv 1\ (2)$,

(4) $w_{c,m}(t) = h_{\gamma_1}^{(0)}(-1)w_{\gamma_1+\gamma_2}^{(2m)}(\tfrac{1}{2}N_{\gamma_2,\gamma_1}t^2)$ *if* $c = (\gamma_1, \gamma_2)$ *is of type* (R-3) *and* $m \equiv 0\ (2)$,

(5) $w_{c,m}(t) = h_{\gamma_1}^{(0)}(-1)w_{\gamma_1+\gamma_2}^{(2m)}(-\tfrac{1}{2}N_{\gamma_2,\gamma_1}t^2)$ *if* $c = (\gamma_1, \gamma_2)$ *is of type* (R-3) *and* $m \equiv 1\ (2)$,

(6) $w_{c,m}(t) = w_{\gamma_1}{}^{(m)}(t)w_{\gamma_2}{}^{(m)}(t)w_{\gamma_3}{}^{(m)}(t)$ *if* $c = (\gamma_1, \gamma_2, \gamma_3)$ *is of type* (R-4) *and* $m \equiv 0\ (3)$,

(7) $w_{c,m}(t) = w_{\gamma_1}{}^{(m)}(t)w_{\gamma_2}{}^{(m)}(\omega t)w_{\gamma_3}{}^{(m)}(\omega^2 t)$ *if* $c = (\gamma_1, \gamma_2, \gamma_3)$ *is of type* (R-4) *and* $m \equiv 1\ (3)$,

(8) $w_{c,m}(t) = w_{\gamma_1}{}^{(m)}(t)w_{\gamma_2}{}^{(m)}(\omega^2 t)w_{\gamma_3}{}^{(m)}(\omega t)$ *if* $c = (\gamma_1, \gamma_2, \gamma_3)$ *is of type* (R-4) *and* $m \equiv 2\ (3)$.

Proof. (1), (2), (3), (6), (7), and (8) are easy. Here we shall establish (4). By the Jacobi identity, we have

$$N_{\gamma_1+\gamma_2,-\gamma_1}N_{\gamma_2,\gamma_1} = N_{-\gamma_1-\gamma_2,\gamma_1}N_{-\gamma_2,-\gamma_1} = 1 \text{ and}$$
$$N_{\gamma_1+\gamma_2,-\gamma_2}N_{\gamma_2,\gamma_1} = N_{-\gamma_1-\gamma_2,\gamma_2}N_{-\gamma_2,-\gamma_1} = -1.$$

Thus,

$$\begin{aligned} w_{c,m}(t) &= x_{c,m}(t)x_{-c,-m}(-2t^{-1})x_{c,m}(t) \\ &= x_{\gamma_1}^{(m)}(t)x_{\gamma_2}^{(m)}(t)x_{\gamma_1+\gamma_2}^{(2m)}(\tfrac{1}{2}N_{\gamma_2,\gamma_1}t^2)x_{-\gamma_1}^{(-m)}(-2^{-1})x_{-\gamma_2}^{(-m)}(-2t^{-1}) \\ &\times x_{-\gamma_1-\gamma_2}^{(-2m)}(2N_{-\gamma_2,-\gamma_1}t^{-2})x_{\gamma_1}^{(m)}(t)x_{\gamma_2}^{(m)}(t)x_{\gamma_1+\gamma_2}^{(2m)}(\tfrac{1}{2}N_{\gamma_2,\gamma_1}t^2) \\ &= x_{\gamma_1}^{(m)}(t)x_{-\gamma_1}^{(-m)}(-2t^{-1})x_{\gamma_1}^{(m)}(t)x_{-\gamma_1}^{(-m)}(-2t^{-1})x_{\gamma_1+\gamma_2}^{(2m)}(\tfrac{1}{2}N_{\gamma_2,\gamma_1}t^2) \\ &\times x_{-\gamma_1-\gamma_2}^{(-2m)}(2N_{-\gamma_2,-\gamma_1}t^2)x_{\gamma_1+\gamma_2}^{(2m)}(\tfrac{1}{2}N_{\gamma_2,\gamma_1}t^2) \\ &= w_{\gamma_1}^{(m)}(t)x_{\gamma_1}^{(m)}(-t)w_{-\gamma_1}^{(-m)}(-t^{-1})x_{-\gamma_1}^{(-m)}(-t^{-1})w_{\gamma_1+\gamma_2}^{(2m)}(\tfrac{1}{2}N_{\gamma_2,\gamma_1}t^2) \\ &= w_{\gamma_1}^{(m)}(t)x_{\gamma_1}^{(m)}(-t)w_{\gamma_1}^{(m)}(t)x_{-\gamma_1}^{(-m)}(-t^{-1})w_{\gamma_1+\gamma_2}^{(2m)}(\tfrac{1}{2}N_{\gamma_2,\gamma_1}t^2) \\ &= w_{\gamma_1}^{(m)}(t)^2 w_{\gamma_1+\gamma_2}^{(2m)}(\tfrac{1}{2}N_{\gamma_2,\gamma_1}t^2) \\ &= h_{\gamma_1}^{(m)}(t)h_{\gamma_1}^{(-m)}(-t^{-1})w_{\gamma_1+\gamma_2}^{(2m)}(\tfrac{1}{2}N_{\gamma_2,\gamma_1}t^2) \\ &= h_{\gamma_1}^{(0)}(-1)w_{\gamma_1+\gamma_2}^{(2m)}(\tfrac{1}{2}N_{\gamma_2,\gamma_1}t^2). \end{aligned}$$

(5) is similarly shown.

By Lemma 3.2 and [**11**, Lemma 2.3 (2)], the next lemma can be established.

LEMMA 3.3. *Let* (a, n) *and* (b, m) *be in* Ω, *and* t *in* K^*, *and set* $(b', m') = w_{a,n}(b, m)$. *Then*

$$w_{a,n}(t)X_{b,m}w_{a,n}(t)^{-1} = X_{b',m'}.$$

By Lemma 3.3, we see that there is a group homomorphism ν of N' onto $W(\Omega)$ defined by $\nu(w_{a,n}(t)) = w_{a,n}$ for all $(a, n) \in \Omega$ and $t \in K^*$. Let H_0' be the kernel of ν. We sometimes identify an element of $W(\Omega)$ with a representative in N' of N'/H_0'. Let U' be the subgroup of E' generated by $X_{c,m}$ for all $(c, m) \in \Omega^+$, and let B' be the subgroup of E' generated by U' and H_0'.

THEOREM 3.4. *Let Y' be as in Section* 1. *Then* (E', B', N', Y') *is a Tits system.*

This theorem will be established in Section 5. For that purpose it is necessary to prove the next proposition. Let s be in Y'. For some $(c, n) \in \Omega_0$, we have $s = w_{c,n}$. Set

$$\Omega^+(s) = \{(a, m) \in \Omega^+; a \in \mathbf{Q}c\}.$$

Let P_s be the subgroup of U' generated by $X_{a,m}$ for all $(a, m) \in \Omega^+(s)$.

PROPOSITION 3.5. *Let s be in Y'. Then*

$$sP_s s^{-1} \subseteq B' \cup B'sB'.$$

We shall show this proposition in Section 4.

4. Proof of proposition 3.5. Let s be in Y', and write $s = w_{c,n}$ for some $(c, n) \in \Omega_0$. Let

$$\Omega(s) = \{(a, m) \in \Omega; a \in \mathbf{Q}c\}$$

and $E'(s)$ be the subgroup of E' generated by $x_{a,m}(t)$ for all $(a, m) \in \Omega(s)$ and $t \in K$. If $\Phi_\sigma \cap \mathbf{Q}c = \{\pm c\}$, then we can view $E'(s)$ as the elementary subgroup of a Chevalley group of type A_1 over $K[T, T^{-1}]$, $K[T^2, T^{-2}]$ or $K[T^3, T^{-3}]$, therefore Proposition 3.5 can be shown using the result in [**11**, Section 3]. Thus, to establish Proposition 3.5, we may assume that Φ is of type A_2 and Φ_σ is of type BC_1. In this section, from now on we assume G is a Chevalley group of type A_2 over $K[T, T^{-1}]$, so $\Phi_\sigma = \{\pm a, \pm 2a\}$,

$$\Omega^+ = \{(+a, n), (-a, m), (\pm 2a, k) \in \Omega;$$
$$n > 0, m \geqq 0, k > 0, k \equiv 1 \ (2)\}.$$

We simply write

$$w_0 = w_{-a,0} = w_{-a}(1)w_{2a,1}(2)w_{2a,1}(-1) \quad \text{and}$$
$$w_1 = w_{2a,1} = w_{2a,1}(1).$$

Let $S_\lambda = B' \cup B'w_\lambda B'$, where $\lambda = 0, 1$.

LEMMA 4.1. *The following statements hold.*

(1) $w_0 X_{\pm a,n} w_0^{-1} = X_{\mp a,n} \subseteq B'$ *if* $n \geqq 1$.

(2) $w_0 X_{\pm 2a,n} w_0^{-1} = X_{\mp 2a,n} \subseteq B'$ *if* $n \geqq 1, n \equiv 1 \ (2)$.

(3) $w_0 X_{-a,0} w_0^{-1} = X_{a,0} \subseteq S_0$.
(4) $w_1 X_{a,n} w_1^{-1} = X_{-a,n-1} \subseteq B'$ *if* $n \geqq 1$.
(5) $w_1 X_{-a,n} w_1^{-1} = X_{a,n+1} \subseteq B'$ *if* $n \geqq 0$.
(6) $w_1 X_{2a,n} w_1^{-1} = X_{-2a,n-2} \subseteq B'$ *if* $n \geqq 3, n \equiv 1\ (2)$.
(7) $w_1 X_{-2a,n} w_1^{-1} = X_{2a,n+2} \subseteq B'$ *if* $n \geqq 1, n \equiv 1\ (2)$.
(8) $w_1 X_{2a,1} w_1^{-1} = X_{-2a,-1} \subseteq S_1$.

Definition. Let x be in E'.
(1) x is called a $(QS, 0)$-*element* if x can be written as

$$x_{-a,0}(t)x_{a,0}(u)x_{b_1,m_1}(t_1)\ldots x_{b_k,m_k}(t_k)x_{-a,0}(v),$$

where $(b_j, m_j) \in \Omega^+ - \{(-a, 0)\}$, $k \geqq 0$, $t, u, t_1, \ldots, t_k \in K$, and $v \in K^*$.
(2) x is called a $(QS, 1)$-*element* if x can be written as

$$x_{2a,1}(t)x_{-2a,-1}(u)x_{b_1,m_1}(t_1)\ldots x_{b_k,m_k}(t_k)x_{2a,1}(v),$$

where $(b_j, m_j) \in \Omega^+ - \{(2a, 1)\}$, $k \geqq 0$, $t, u, t_1, \ldots, t_k \in K$, and $v \in K^*$.
(3) x is called an $(S, 0)$-*element* (resp. $(S, 1)$-*element*) if x is a $(QS, 0)$-element (resp. $(QS, 1)$-element) with $u = 0$.

LEMMA 4.2. *Let x be in E' and $\lambda = 0, 1$. If x is an (S, λ)-element, then $w_\lambda x w_\lambda \in S$.*

Proof. Set $\lambda = 0$. We proceed by induction on k. If $t = 0$, clearly $w_0 x w_0^{-1} \in S_0$ by Lemma 4.1. Assume $t \neq 0$.
Case 1: $(b_1, m_1) = (-a, m)$, $m > 0$, $m \equiv 1\ (2)$.

$$\begin{aligned} w_0 x w_0^{-1} &= w_0 x_{-a,0}(t)x_{-a,m}(t_1)x_{b_2,m_2}(t_2)\ldots \\ &\quad \ldots x_{b_k,m_k}(t_k)x_{-a,0}(v)w_0^{-1} = w_0 x_{-2a,m}(\pm 2tt_1)x_{-a,m}(t_1)x_{-a,0}(t) \\ &\quad \times x_{b_2,m_2}(t_2)\ldots x_{b_k,m_k}(t_k)x_{-a,0}(v)w_0^{-1} \in X_{2a,m}X_{a,m}w_0 x_{-a,0}(t) \\ &\quad \times x_{b_2,m_2}(t_2)\ldots x_{b_k,m_k}(t_k)x_{-a,0}(v)w_0^{-1} \subseteq B'S_0 \subseteq S_0. \end{aligned}$$

Case 2: $(b_1, m_1) = (-a, m)$, $m > 0$, $m \equiv 0\ (2)$.

$$\begin{aligned} w_0 x w_0^{-1} &= w_0 x_{-a,0}(t)x_{-a,m}(t_1)x_{b_2,m_2}(t_2)\ldots x_{b_k,m_k}(t_k)x_{-a,0}(v)w_0^{-1} \\ &= w_0 x_{-a,m}(t_1)x_{-a,0}(t)x_{b_2,m_2}(t_2)\ldots x_{b_k,m_k}(t_k)x_{-a,0}(v)w_0^{-1} \\ &\in X_{a,m}w_0 x_{-a,0}(t)x_{b_2,m_k}(t_2)\ldots x_{b_k,m_k}(t_k)x_{-a,0}(v)w_0^{-1} \\ &\subseteq B'S_0 = S_0. \end{aligned}$$

Case 3: $(b_1, m_1) = (-2a, m)$, $m > 0$, $m \equiv 1\ (2)$.

$$\begin{aligned} w_0 x w_0^{-1} &= w_0 x_{-a,0}(t)x_{-2a,m}(t_1)x_{b_2,m_2}(t_2)\ldots x_{b_k,m_k}(t_k)x_{-a,0}(v)w_0^{-1} \\ &= w_0 x_{-2a,m}(t_1)x_{-a,0}(t)x_{b_2,m_2}(t_2)\ldots x_{b_k,m_k}(t_k)x_{-a,0}(v)w_0^{-1} \\ &\in X_{2a,m}w_0 x_{-a,0}(t)x_{b_2,m_2}(t_2)\ldots x_{b_k,m_k}(t_k)x_{-a,0}(v)w_0^{-1} \\ &\subseteq B'S_0 = S_0. \end{aligned}$$

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Case 4: $(b_1, m_1) = (a, m), m > 0,$

$$\begin{aligned} w_0xw_0^{-1} &= w_0x_{-a,0}(t)x_{a,m}(t_1)x_{b_2,m_2}(t_2) \ldots x_{b_k,m_k}(t_k)x_{-a,0}(v)w_0^{-1} \\ &= x_{a,0}(-t)x_{-a,m}(-t_1)x_2 \ldots x_kx_{a,0}(-v) \\ &= x_{-a,0}(-2t^{-1})w_{-a,0}(2t^{-1})x_{-a,0}(-2t^{-1})x_{-a,m}(-t_1) \\ &\times x_2 \ldots x_kx_{-a,0}(-2v^{-1})w_{-a,0}(2v^{-1})x_{-a,0}(-2v^{-1}) \\ &\in B'w_0x_{-a,0}(-2t^{-1})x_{-a,m}(-t_1)x_2 \ldots x_kx_{-a,0}(-2v^{-1})w_0^{-1}B' \\ &\subseteq B'S_0B' = S_0 \end{aligned}$$

$(x_j = w_0x_{b_j,m_j}(t_j)w_0^{-1}, 2 \leqq j \leqq k).$

Case 5: $(b_1, m_1) = (2a, m), m > 0, m \equiv 1\ (2).$

$$\begin{aligned} w_0xw_0^{-1} &= w_0x_{-a,0}(t)x_{2a,m}(t_1)x_{b_2,m_2}(t_2) \ldots x_{b_k,m_k}(t_k)x_{-a,0}(v)w_0^{-1} \\ &= x_{a,0}(-t)x_{-2a,m}(t_1)x_2 \ldots x_kx_{a,0}(-v) \\ &= x_{-a,0}(-2t^{-1})w_{-a,0}(2t^{-1})x_{-a,0}(-2t^{-1})x_{-2a,m}(t_1) \\ &\times x_2 \ldots x_kx_{-a,0}(-2v^{-1})w_{-a,0}(2v^{-1})x_{-a,0}(-2v^{-1}) \\ &\in B'w_0x_{-a,0}(-2t^{-1})x_{-2a,m}(t_1)x_2 \ldots x_kx_{-a,0}(-2v^{-1})w_0^{-1}B' \\ &\subseteq B'S_0B' = S_0 \end{aligned}$$

$(x_j = w_0x_{b_j,m_j}(t_j)w_0^{-1}, 2 \leqq j \leqq k).$

The case when $\lambda = 1$ is similarly shown.

LEMMA 4.3. *Let x be in E'.*

(1) *If x is an $(S, 0)$-element, then*

$$w_0xw_0^{-1} \in B'w_0X_{-a,0}X_{a,0}w_0^{-1}.$$

(2) *If x is an $(S, 1)$-element, then*

$$w_1xw_1^{-1} \in B'w_1X_{2a,1}X_{-2a,-1}w_1^{-1}.$$

Proof. Proceed by induction on k as in Lemma 4.2. Then we have (1) and (2).

LEMMA 4.4. *Let x be in E' and $\lambda = 0, 1$. If x is a (QS, λ)-element, then $w_\lambda xw_\lambda^{-1} \in S$.*

Proof. Lemma 4.2 implies this lemma as in [**11**, Lemma 3.6].

LEMMA 4.5. *Let x be in E'.*

(1) *If x is a $(QS, 0)$-element, then*

$$w_0xw_0^{-1} \in B'w_0X_{-a,0}X_{a,0}w_0^{-1}.$$

(2) *If x is a $(QS, 1)$-element, then*

$$w_1xw_1^{-1} \in B'w_1X_{2a,1}X_{-2a,-1}w_1^{-1}.$$

Proof. Lemma 4.3 implies this lemma.

These five lemmas lead to Proposition 3.5 as in [**11**, Section 3].

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5. Proof of theorem 3.4. Notation is as in Section 3. By using the commutator relations in [**11**, Lemma 2.2], we can establish the following proposition.

PROPOSITION 5.1. *Let (a, m) and (b, n) be in Ω such that $a + b \neq 0$. Then*

$$[X_{a,m}, X_{b,n}] \subseteq \langle X_{c,k};\ (c, k) \in \Omega,\ c = ia + jb,\ k = im + jn,\ i, j > 0 \rangle.$$

Let s be in Y', and let $\Omega^+(s)' = \Omega^+ - \Omega^+(s)$. Let Q_s be the subgroup of U' generated by $X_{a,m}$ for all $(a, m) \in \Omega^+(s)'$. Then, by Proposition 5.1, we have

(5.2) P_s normalizes Q_s,

(5.3) $U' = P_s Q_s$.

By the definition of H_0',

(5.4) H_0' normalizes $X_{c,m}$ for all $(c, m) \in \Omega$,

(5.5) $B' = U' \cdot H_0'$.

Clearly, $B' \cap N' \supseteq H_0'$. Conversely let x be in $B' \cap N'$. Then $\bar{x} \in W(\Omega)$, where $\bar{x}$ is the image of x under the canonical group homomorphism $^-$ of N' onto N'/H_0'. Since x is in B', we have $\bar{x}\Omega^+ \subseteq \Omega^+$, hence $N(\bar{x}) = 0$ and $x \in H_0'$. Thus,

(5.6) $B' \cap N' = H_0'$.

By Proposition 3.5, (5.3) and (5.5),

$$sB's^{-1} = s(P_s Q_s H_0')s^{-1} = (sP_s s^{-1})(sQ_s s^{-1})(sH_0' s^{-1}) \subseteq (B' \cup B'sB')B'H_0'.$$

Hence,

(5.7) $B' \cup B'sB'$ is a subgroup of E'.

We see that E acts on L via the adjoint representation (cf. [**11**, Section 4]). Then L' is stable under the action of E'. Let g be in U' and $(a, n) \in \Omega_0$, and set

$$Z_{a,n} = \textstyle\sum_{(b,m) \in \Omega^+ - \{(a,n)\}} K e_{b,m}.$$

If a is of type (R-1), (R-2), or (R-4) (resp. of type (R-3)), then we can write

$$g e_{-a,-n} = e_{-a,-n} + \zeta h_a - \zeta^2 e_{a,n} + z$$

(resp. $g e_{-a,-n} = e_{-a,-n} + \zeta h_a - \frac{1}{2}\zeta^2 e_{a,n} + z$) for some $\zeta \in K$ and $z \in Z_{a,n}$ (cf. Proposition 2.3). Let $\theta_{a,n}$ be a map of U' onto K defined by $\theta_{a,n}(g) = \zeta$. As

$$g h_a = h_a - 2\zeta e_{a,n} + z'$$

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(resp. $gh_a = h_a - \zeta e_{a,n} + z'$) and $gZ_{a,n} \subseteq Z_{a,n}$, the map $\theta_{a,n}$ is a group homomorphism of U' onto the additive group K^+ of K, where $z' \in Z_{a,n}$. Let $D_{a,n}$ be the kernel of the homomorphism $\theta_{a,n}$. By (5.7),

$$w_{a,n}D_{a,n}w_{a,n}^{-1} \subseteq B' \cup B'w_{a,n}B'.$$

For any $x \in D_{a,n}$, we have

$$(w_{a,n}xw_{a,n}^{-1})e_{a,n} = e_{a,n} + z'',$$

where $z'' \in Z_{a,n}$, so $w_{a,n}xw_{a,n}^{-1}$ can not be in $B'w_{a,n}B'$. Thus,

(5.8) $\quad w_{a,n}D_{a,n}w_{a,n}^{-1} \subseteq B'.$

If g is in U', $(a, n) \in \Omega_0$ and $\theta_{a,n}(g) = \zeta$, then

$$gx_{a,n}(-\zeta) \in D_{a,n}.$$

Hence,

(5.9) $\quad U' = D_{a,n} \cdot X_{a,n}.$

Therefore, as in [**11**, Section 4], we have

(5.10) $\quad (B'wB')(B'sB') \subseteq (B'wsB')(B'wB')$

for any $w \in W(\Omega)$ and $s \in Y'$. These facts imply Theorem 3.4.

Remark. If (Φ, σ) is of r-type, then L' has the structure of an r-tiered Euclidean Lie algebra (cf. [**5**], [**6**], [**8**], [**9**], [**13**], Table 4 below). We follow the classification in [**8**], so here we use the notation D_3 instead of A_3.

TABLE 4.

	2-type					3-type
(Φ, σ)	A_{2n+1} $(n \geqq 2)$	A_{2n} $(n \geqq 2)$	D_n $(n \geqq 3)$	E_6	A_2	D_4
L'	$C_{n+1,2}$	$BC_{n,2}$	$B_{n-1,2}$	$F_{4,2}$	$A_{1,2}$	$G_{2,3}$

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