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REVERSES OF THE SCHWARZ INEQUALITY GENERALISING A KLAMKIN–MCLENAGHAN RESULT

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New reverses of the Schwarz inequality in inner product spaces that incorporate the classical Klamkin–McLenaghan result for the case of positive n-tuples are given. Applications for Lebesgue integrals are also provided.

1. INTRODUCTION

In 2004, the author [1] (see also [3]) proved the following reverse of the Schwarz inequality:

THEOREM 1. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $x, a \in H$, $r > 0$ such that*

$$\|x-a\| \leqslant r < \|a\|. \tag{1.1}$$

Then

$$\|x\| \left(\|a\|^2 - r^2\right)^{1/2} \leqslant \mathrm{Re}\langle x, a\rangle \tag{1.2}$$

or, equivalently,

$$\|x\|^2 \|a\|^2 - \left[\mathrm{Re}\langle x, a\rangle\right]^2 \leqslant r^2 \|x\|^2. \tag{1.3}$$

The case of equality holds in (1.2) *or* (1.3) *if and only if*

$$\|x-a\| = r \quad \text{and} \quad \|x\|^2 + r^2 = \|a\|^2. \tag{1.4}$$

If above one chooses

$$a = \frac{\Gamma+\gamma}{2}\cdot y \quad \text{and} \quad r = \frac{1}{2}|\Gamma-\gamma|\,\|y\|$$

then the condition (1.1) is equivalent to

$$\left\|x - \frac{\Gamma+\gamma}{2}\cdot y\right\| \leqslant \frac{1}{2}|\Gamma-\gamma|\,\|y\| \quad \text{and} \quad \mathrm{Re}(\Gamma\overline{\gamma}) > 0. \tag{1.5}$$

Therefore, we can state the following particular result as well:

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Corollary 1. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be as above,* $x, y \in H$ *and* $\gamma, \Gamma \in \mathbb{K}$ *with* $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$. *If*

$$\left\| x - \frac{\Gamma+\gamma}{2} \cdot y \right\| \leqslant \frac{1}{2} |\Gamma - \gamma| \, \|y\| \tag{1.6}$$

or, equivalently,

$$\operatorname{Re}\langle \Gamma y - x, x - \gamma y \rangle \geqslant 0, \tag{1.7}$$

then

$$\begin{aligned} \|x\| \, \|y\| &\leqslant \frac{\operatorname{Re}[(\overline{\Gamma}+\overline{\gamma})\langle x, y\rangle]}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}} \\ &= \frac{\operatorname{Re}(\Gamma+\gamma)\operatorname{Re}\langle x,y\rangle + \operatorname{Im}(\Gamma+\gamma)\operatorname{Im}\langle x,y\rangle}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}} \\ &\left(\leqslant \frac{|\Gamma+\gamma|}{\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}} |\langle x, y\rangle| \right). \end{aligned} \tag{1.8}$$

The case of equality holds in (1.8) *if and only if the equality case holds in* (1.6) *(or* (1.7)*) and*

$$\|x\| = \sqrt{\operatorname{Re}(\Gamma\overline{\gamma})} \, \|y\| . \tag{1.9}$$

If the restriction $\|a\| > r$ is removed from Theorem 1, then a different reverse of the Schwarz inequality may be stated [**2**] (see also [**3**]):

Theorem 2. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}$ *and* $x, a \in H$, $r > 0$ *such that*

$$\|x - a\| \leqslant r. \tag{1.10}$$

Then

$$\|x\| \, \|a\| - \operatorname{Re}\langle x, a\rangle \leqslant \frac{1}{2} r^2. \tag{1.11}$$

The equality holds in (1.11) *if and only if the equality case is realised in* (1.10) *and* $\|x\| = \|a\|$.

As a corollary of the above, we can state:

Corollary 2. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be as above,* $x, y \in H$ *and* $\gamma, \Gamma \in \mathbb{K}$ *with* $\Gamma \neq -\gamma$. *If either* (1.6) *or, equivalently,* (1.7) *hold true, then*

$$\|x\| \, \|y\| - \frac{\operatorname{Re}(\Gamma+\gamma)\operatorname{Re}\langle x,y\rangle + \operatorname{Im}(\Gamma+\gamma)\operatorname{Im}\langle x,y\rangle}{|\Gamma+\gamma|} \leqslant \frac{1}{4} \cdot \frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|} \|y\|^2. \tag{1.12}$$

The equality holds in (1.12) *if and only if the equality case is realised in either* (1.6) *or* (1.7) *and*

$$\|x\| = \frac{1}{2} |\Gamma + \gamma| \, \|y\| . \tag{1.13}$$

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As pointed out in [**4**], the above results are motivated by the fact that they generalise to the case of real or complex inner product spaces some classical reverses of the Cauchy–Bunyakovsky–Schwarz inequality for positive n-tuples due to Polya and Szegö [**8**], Cassels [**10**], Shisha and Mond [**9**] and Greub and Rheinboldt [**6**].

The main aim of this paper is to establish a new reverse of Schwarz's inequality similar to the ones in Theorems 1 and 2 which will reduce, for the particular case of positive n-tuples, to the Klamkin and McLenaghan result from [**7**].

2. The Results

The following result may be stated.

Theorem 3. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $x,a\in H$, $r>0$ with $\langle x,a\rangle\neq 0$ and*

$$\|x-a\|\leqslant r<\|a\|. \tag{2.1}$$

Then

$$\frac{\|x\|^2}{|\langle x,a\rangle|}-\frac{|\langle x,a\rangle|}{\|a\|^2}\leqslant\frac{2r^2}{\|a\|(\|a\|+\sqrt{\|a\|^2-r^2})}, \tag{2.2}$$

with equality if and only if the equality case holds in (2.1) *and*

$$\operatorname{Re}\langle x,a\rangle=\left|\langle x,a\rangle\right|=\|a\|\left(\|a\|^2-r^2\right)^{1/2}. \tag{2.3}$$

The constant 2 *is best possible in* (2.2) *in the sense that it cannot be replaced by a smaller quantity.*

Proof: The first condition in (2.1) is obviously equivalent with

$$\frac{\|x\|^2}{|\langle x,a\rangle|}\leqslant\frac{2\operatorname{Re}\langle x,a\rangle}{|\langle x,a\rangle|}-\frac{\|a\|^2-r^2}{|\langle x,a\rangle|} \tag{2.4}$$

with equality if and only if $\|x-a\|=r$.

Subtracting from both sides of (2.4) the same quantity $|\langle x,a\rangle|/\|a\|^2$ and performing some elementary calculations, we get the equivalent inequality:

$$\frac{\|x\|^2}{|\langle x,a\rangle|}-\frac{|\langle x,a\rangle|}{\|a\|^2}\leqslant 2\cdot\frac{\operatorname{Re}\langle x,a\rangle}{|\langle x,a\rangle|}-\left(\frac{|\langle x,a\rangle|^{1/2}}{\|a\|}-\frac{(\|a\|^2-r^2)^{1/2}}{|\langle x,a\rangle|^{1/2}}\right)^2-\frac{2\sqrt{\|a\|^2-r^2}}{\|a\|}. \tag{2.5}$$

Since, obviously

$$\operatorname{Re}\langle x,a\rangle\leqslant\left|\langle x,a\rangle\right|\quad\text{and}\quad\left(\frac{|\langle x,a\rangle|^{1/2}}{\|a\|}-\frac{(\|a\|^2-r^2)^{1/2}}{|\langle x,a\rangle|^{1/2}}\right)^2\geqslant 0,$$

hence, by (2.5) we get

$$\frac{\|x\|^2}{|\langle x,a\rangle|} - \frac{|\langle x,a\rangle|}{\|a\|^2} \leqslant 2\left(1 - \frac{\sqrt{\|a\|^2 - r^2}}{\|a\|}\right) \tag{2.6}$$

with equality if and only if

$$\|x-a\| = r, \quad \operatorname{Re}\langle x,a\rangle = \left|\langle x,a\rangle\right| \quad \text{and} \quad \left|\langle x,a\rangle\right| = \|a\|\left(\|a\|^2 - r^2\right)^{1/2}. \tag{2.7}$$

Observe that (2.6) is equivalent with (2.2) and the first part of the theorem is proved.

To prove the sharpness of the constant, let us assume that there is a $C > 0$ such that

$$\frac{\|x\|^2}{|\langle x,a\rangle|} - \frac{|\langle x,a\rangle|}{\|a\|^2} \leqslant \frac{Cr^2}{\|a\|(\|a\| + \sqrt{\|a\|^2 - r^2})}, \tag{2.8}$$

provided $\|x-a\| \leqslant r < \|a\|$.

Now, consider $\varepsilon \in (0,1)$ and let $r = \sqrt{\varepsilon}$, $a, e \in H$, $\|a\| = \|e\| = 1$ and $a \perp e$. Define $x := a + \sqrt{\varepsilon}e$. We observe that $\|x-a\| = \sqrt{\varepsilon} = r < 1 = \|a\|$, which shows that the condition (2.1) of the theorem is satisfied. We also observe that

$$\|x\|^2 = \|a\|^2 + \varepsilon\|e\|^2 = 1 + \varepsilon, \quad \langle x,a\rangle = \|a\|^2 = 1$$

and utilising (2.8) we get

$$1 + \varepsilon - 1 \leqslant \frac{C\varepsilon}{(1 + \sqrt{1-\varepsilon})},$$

giving $1 + \sqrt{1-\varepsilon} \leqslant C$ for any $\varepsilon \in (0,1)$. Letting $\varepsilon \to 0+$, we get $C \geqslant 2$, which shows that the constant 2 in (2.2) is best possible. ∎

Remark 1. In a similar manner, one can prove that if $\operatorname{Re}\langle x,a\rangle \neq 0$ and if (2.1) holds true, then:

$$\frac{\|x\|^2}{|\operatorname{Re}\langle x,a\rangle|} - \frac{|\operatorname{Re}\langle x,a\rangle|}{\|a\|^2} \leqslant \frac{2r^2}{\|a\|(\|a\| + \sqrt{\|a\|^2 - r^2})} \tag{2.9}$$

with equality if and only if $\|x-a\| = r$ and

$$\operatorname{Re}\langle x,a\rangle = \|a\|\left(\|a\|^2 - r^2\right)^{1/2}. \tag{2.10}$$

The constant 2 is best possible in (2.9).

Remark 2. Since (2.2) is equivalent with

$$\|x\|^2\|a\|^2 - \left|\langle x,a\rangle\right|^2 \leqslant \frac{2r^2\|a\|^2}{\|a\|(\|a\| + \sqrt{\|a\|^2 - r^2})}\left|\langle x,a\rangle\right| \tag{2.11}$$

and (2.9) is equivalent to

$$\|x\|^2\|a\|^2 - \left[\operatorname{Re}\langle x,a\rangle\right]^2 \leqslant \frac{2r^2\|a\|^2}{\|a\|(\|a\| + \sqrt{\|a\|^2 - r^2})}\left|\operatorname{Re}\langle x,a\rangle\right| \tag{2.12}$$

hence (2.12) is a tighter inequality than (2.11), because in complex spaces, in general $\left|\langle x,a\rangle\right| > \left|\operatorname{Re}\langle x,a\rangle\right|$.

The following corollary is of interest.

Corollary 3. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be a real or complex inner product space and* $x, y \in H$ *with* $\langle x,y\rangle \neq 0$, $\gamma,\Gamma \in \mathbb{K}$ *with* $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$. *If either* (1.6) *or, equivalently* (1.7) *holds true, then*

$$\frac{\|x\|^2}{|\langle x,y\rangle|} - \frac{|\langle x,y\rangle|}{\|y\|^2} \leqslant |\Gamma+\gamma| - 2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}. \tag{2.13}$$

The equality holds in (2.13) *if and only if the equality case holds in* (1.6) *(or in* (1.7)*) and*

$$\operatorname{Re}\big[(\Gamma+\gamma)\langle x,y\rangle\big] = |\Gamma+\gamma|\,\big|\langle x,y\rangle\big| = |\Gamma+\gamma|\,\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}\|y\|^2. \tag{2.14}$$

Proof: We use the inequality (2.2) in its equivalent form

$$\frac{\|x\|^2}{|\langle x,a\rangle|} - \frac{|\langle x,a\rangle|}{\|a\|^2} \leqslant \frac{2(\|a\| - \sqrt{\|a\|^2 - r^2})}{\|a\|}.$$

Choosing $a = (\Gamma + \gamma/2)\cdot y$ and $r = |\Gamma-\gamma|/2\|y\|$, we have

$$\frac{\|x\|^2}{|(\Gamma+\gamma)/2||\langle x,y\rangle|} - \frac{|(\Gamma+\gamma)/2||\langle x,y\rangle|}{|(\Gamma+\gamma)/2|^2\|y\|^2} \leqslant \frac{2(|\frac{\Gamma+\gamma}{2}|\|y\| - \sqrt{|(\Gamma+\gamma)/2|^2\|y\|^2 - (1/4)|\Gamma-\gamma|^2\|y\|^2})}{|(\Gamma-\gamma)/2|\|y\|}$$

which is equivalent to (2.13). ∎

Remark 3. The inequality (2.13) has been obtained in a different way in [**5**, Theorem 2]. However, in [**5**] the authors did not consider the equality case which may be of interest for applications.

Remark 4. If we assume that $\Gamma = M \geqslant m = \gamma > 0$, which is very convenient in applications, then

$$\frac{\|x\|^2}{|\langle x,y\rangle|} - \frac{|\langle x,y\rangle|}{\|y\|^2} \leqslant (\sqrt{M} - \sqrt{m})^2, \tag{2.15}$$

provided that either

$$\operatorname{Re}\langle My - x, x - my\rangle \geqslant 0 \tag{2.16}$$

or, equivalently,

$$\left\|x - \frac{m+M}{2}y\right\| \leqslant \frac{1}{2}(M-m)\,\|y\| \tag{2.17}$$

holds true.

The equality holds in (2.15) if and only if the equality case holds in (2.16) (or in (2.17)) and

$$\operatorname{Re}\langle x, y\rangle = \left|\langle x, y\rangle\right| = \sqrt{Mm}\,\|y\|^2 . \tag{2.18}$$

The multiplicative constant $C = 1$ in front of $(\sqrt{M} - \sqrt{m})^2$ cannot be replaced in general with a smaller positive quantity.

Now for a non-zero complex number z, we define $\operatorname{sgn}(z) := z/|z|$.

The following result may be stated:

Proposition 1. *Let $(H; \langle\cdot,\cdot\rangle)$ be a real or complex inner product space and $x, y \in H$ with $\operatorname{Re}\langle x, y\rangle \neq 0$ and $\gamma, \Gamma \in \mathbb{K}$ with $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$. If either (2.6) or, equivalently, (2.7) hold true, then*

$$\begin{aligned}(0 \leqslant \|x\|^2\|y\|^2 - \left|\langle x, y\rangle\right|^2 \leqslant)\ &\|x\|^2\|y\|^2 - \left[\operatorname{Re}\left(\operatorname{sgn}\left(\frac{\Gamma+\gamma}{2}\right)\cdot\langle x, y\rangle\right)\right]^2 \\ &\leqslant \left(|\Gamma+\gamma| - 2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}\right)\left|\operatorname{Re}\left(\operatorname{sgn}\left(\frac{\Gamma+\gamma}{2}\right)\cdot\langle x, y\rangle\right)\right|\|y\|^2 \\ &\left(\leqslant \left(|\Gamma+\gamma| - 2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}\right)\left|\langle x, y\rangle\right|\|y\|^2\right).\end{aligned} \tag{2.19}$$

The equality holds in (2.19) if and only if the equality case holds in (2.6) (or in (2.7)) and

$$\operatorname{Re}\left[\operatorname{sgn}\left(\frac{\Gamma+\gamma}{2}\right)\cdot\langle x, y\rangle\right] = \sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}\,\|y\|^2 .$$

Proof: The inequality (2.9) is equivalent with:

$$\|x\|^2\|a\|^2 - \left[\operatorname{Re}\langle x, a\rangle\right]^2 \leqslant 2\left(\|a\| - \sqrt{\|a\|^2 - r^2}\right)\cdot\left|\operatorname{Re}\langle x, a\rangle\right|\|a\| .$$

If in this inequality we choose $a = (\Gamma+\gamma)/2 \cdot y$ and $r = |\Gamma - \gamma|/2 \mid y$, we have

$$\begin{aligned}\|x\|^2\left|\frac{\Gamma+\gamma}{2}\right|^2\|y\|^2 &- \left(\operatorname{Re}\left[\left(\frac{\Gamma+\gamma}{2}\right)\cdot\langle x, y\rangle\right]\right)^2 \\ &\leqslant 2\left(\left|\frac{\Gamma+\gamma}{2}\right|\|y\| - \sqrt{\left|\frac{\Gamma+\gamma}{2}\right|^2\|y\|^2 - \frac{1}{4}|\Gamma-\gamma|^2\|y\|^2}\right) \\ &\qquad\times\left|\operatorname{Re}\left[\left(\frac{\Gamma+\gamma}{2}\right)\cdot\langle x, y\rangle\right]\right|\left|\frac{\Gamma+\gamma}{2}\right|\|y\| ,\end{aligned}$$

which, on dividing by $\left|(\Gamma+\gamma)/2\right|^2 \neq 0$ (since $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$), is clearly equivalent to (2.19). ∎

Remark 5. If we assume that x, y, m, M satisfy either (2.16) or, equivalently (2.17), then

$$\frac{\|x\|^2}{|\operatorname{Re}\langle x, y\rangle|} - \frac{|\operatorname{Re}\langle x, y\rangle|}{\|y\|^2} \leqslant \left(\sqrt{M} - \sqrt{m}\right)^2 \tag{2.20}$$

or, equivalently

$$\|x\|^2\|y\|^2 - [\operatorname{Re}\langle x,y\rangle]^2 \leqslant (\sqrt{M}-\sqrt{m})^2|\operatorname{Re}\langle x,y\rangle|\,\|y\|^2. \tag{2.21}$$

The equality holds in (2.20) (or (2.21)) if and only if the case of equality is valid in (2.16) (or (2.17)) and

$$\operatorname{Re}\langle x,y\rangle = \sqrt{Mm}\,\|y\|^2. \tag{2.22}$$

3. Applications for Integrals

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, a σ-algebra of parts Σ and a countably additive and positive measure μ on Σ with values in $\mathbb{R} \cup \{\infty\}$.

Denote by $L^2_\rho(\Omega, \mathbb{K})$ the Hilbert space of all $\mathbb{K}$-valued functions f defined on Ω that are $2-\rho$-integrable on Ω, that is, $\int_\Omega \rho(t)|f(s)|^2 d\mu(s) < \infty$, where $\rho : \Omega \to [0,\infty)$ is a measurable function on Ω.

The following proposition contains a reverse of the Cauchy–Bunyakovsky–Schwarz integral inequality:

Proposition 2. *Let $f, g \in L^2_\rho(\Omega, \mathbb{K})$, $r > 0$ be such that*

$$\int_\Omega \rho(t)|f(t)-g(t)|^2 d\mu(t) \leqslant r^2 < \int_\Omega \rho(t)|g(t)|^2 d\mu(t). \tag{3.1}$$

Then

$$\begin{aligned}
&\int_\Omega \rho(t)|f(t)|^2 d\mu(t) \int_\Omega \rho(t)|g(t)|^2 d\mu(t) - \left|\int_\Omega \rho(t)f(t)\overline{g(t)}d\mu(t)\right|^2 \\
&\qquad \leqslant 2\left(\int_\Omega \rho(t)|g(t)|^2 d\mu(t)\right)^{1/2}\left|\int_\Omega \rho(t)f(t)\overline{g(t)}d\mu(t)\right| \\
&\qquad\quad \times \left[\left(\int_\Omega \rho(t)|g(t)|^2 d\mu(t)\right)^{1/2} - \left(\int_\Omega \rho(t)|g(t)|^2 d\mu(t) - r^2\right)^{1/2}\right].
\end{aligned} \tag{3.2}$$

The constant 2 is sharp in (3.2).

The proof follows from Theorem 3 applied for the Hilbert space $(L^2_\rho(\Omega,\mathbb{K}), \langle\cdot,\cdot\rangle_\rho)$ where

$$\langle f, g\rangle_\rho := \int_\Omega \rho(t)f(t)\overline{g(t)}d\mu(t).$$

Remark 6. We observe that if $\int_\Omega \rho(t)d\mu(t) = 1$, then a simple sufficient condition for (3.1) to hold is

$$|f(t)-g(t)| \leqslant r < |g(t)| \quad \text{for} \quad \mu-\text{almost every} \quad t \in \Omega. \tag{3.3}$$

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The second general integral inequality is incorporated in:

Proposition 3. *Let $f, g \in L^2_\rho(\Omega, \mathbb{K})$ and $\Gamma, \gamma \in \mathbb{K}$ with $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$. If either*

$$\int_\Omega \operatorname{Re}\left[(\Gamma g(t) - f(t))\left(\overline{f(t)} - \overline{\gamma}\overline{g(t)}\right)\right]\rho(t)d\mu(t) \geqslant 0 \tag{3.4}$$

or, equivalently,

$$\left(\int_\Omega \rho(t)\left|f(t) - \frac{\Gamma+\gamma}{2}g(t)\right|^2 d\mu(t)\right)^{1/2} \leqslant \frac{1}{2}|\Gamma - \gamma|\left(\int_\Omega \rho(t)|g(t)|^2 d\mu(t)\right)^{1/2} \tag{3.5}$$

holds, then

$$\begin{aligned}\int_\Omega \rho(t)|f(t)|^2 d\mu(t)\int_\Omega \rho(t)|g(t)|^2 d\mu(t) - \left|\int_\Omega \rho(t)f(t)\overline{g(t)}d\mu(t)\right|^2 \\ \leqslant \left[|\Gamma+\gamma| - 2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}\right]\left|\int_\Omega \rho(t)f(t)\overline{g(t)}d\mu(t)\right|\int_\Omega \rho(t)|g(t)|^2 d\mu(t).\end{aligned} \tag{3.6}$$

The proof is obvious by Corollary 3.

Remark 7. A simple sufficient condition for the inequality (3.4) to hold is:

$$\operatorname{Re}\left[(\Gamma g(t) - f(t))\left(\overline{f(t)} - \overline{\gamma}\overline{g(t)}\right)\right] \geqslant 0, \tag{3.7}$$

for μ-almost every $t \in \Omega$.

A more convenient result that may be useful in applications is:

Corollary 4. *If $f, g \in L^2_\rho(\Omega, \mathbb{K})$ and $M \geqslant m > 0$ such that either*

$$\int_\Omega \operatorname{Re}\left[(Mg(t) - f(t))\left(\overline{f(t)} - m\overline{g(t)}\right)\right]f(t)d\mu(t) \geqslant 0 \tag{3.8}$$

or, equivalently,

$$\left(\int_\Omega \rho(t)\left|f(t) - \frac{M+m}{2}g(t)\right|^2 d\mu(t)\right)^{1/2} \leqslant \frac{1}{2}(M-m)\left(\int_\Omega \rho(t)|g(t)|^2 d\mu(t)\right)^{1/2}, \tag{3.9}$$

holds, then

$$\begin{aligned}\int_\Omega \rho(t)|f(t)|^2 d\mu(t)\int_\Omega \rho(t)|g(t)|^2 d\mu(t) - \left|\int_\Omega \rho(t)f(t)\overline{g(t)}d\mu(t)\right|^2 \\ \leqslant \left(\sqrt{M} - \sqrt{m}\right)^2\left|\int_\Omega \rho(t)f(t)\overline{g(t)}d\mu(t)\right|\int_\Omega \rho(t)|g(t)|^2 d\mu(t).\end{aligned} \tag{3.10}$$

Remark 8. Since, obviously,

$$\begin{aligned}\operatorname{Re}\left[(Mg(t) - f(t))\left(\overline{f(t)} - m\overline{g(t)}\right)\right] = \left(M\operatorname{Re} g(t) - \operatorname{Re} f(t)\right)\left(\operatorname{Re} f(t) - m\operatorname{Re} g(t)\right) \\ + \left(M\operatorname{Im} g(t) - \operatorname{Im} f(t)\right)\left(\operatorname{Im} f(t) - m\operatorname{Im} g(t)\right)\end{aligned}$$

for any $t \in \Omega$, hence a very simple sufficient condition that can be useful in practical applications for (3.8) to hold is:

$$M \operatorname{Re} g(t) \geqslant \operatorname{Re} f(t) \geqslant m \operatorname{Re} g(t)$$

and

$$M \operatorname{Im} g(t) \geqslant \operatorname{Im} f(t) \geqslant m \operatorname{Im} g(t)$$

for μ-almost every $t \in \Omega$.

If the functions are in $L_\rho^2(\Omega, \mathbb{R})$ (here $\mathbb{K} = \mathbb{R}$), and $f, g \geqslant 0$, $g(t) \neq 0$ for μ-almost every $t \in \Omega$, then one can state the result:

$$\begin{aligned}(3.11)\quad \int_\Omega \rho(t) f^2(t) d\mu(t) \int_\Omega \rho(t) g^2(t) d\mu(t) - \left(\int_\Omega \rho(t) f(t) g(t) d\mu(t)\right)^2 \\ \leqslant \left(\sqrt{M} - \sqrt{m}\right)^2 \int_\Omega \rho(t) f(t) g(t) d\mu(t) \int_\Omega \rho(t) g^2(t) d\mu(t),\end{aligned}$$

provided

$$(3.12)\qquad 0 \leqslant m \leqslant \frac{f(t)}{g(t)} \leqslant M < \infty \quad \text{for} \quad \mu - \text{almost every} \quad t \in \Omega.$$

REMARK 9. We notice that (3.11) is a generalisation for the abstract Lebesgue integral of the Klamkin–McLenaghan inequality [7]

$$(3.13)\qquad \frac{\sum_{k=1}^n w_k x_k^2}{\sum_{k=1}^n w_k x_k y_k} - \frac{\sum_{k=1}^n w_k x_k y_k}{\sum_{k=1}^n w_k y_k^2} \leqslant \left(\sqrt{M} - \sqrt{m}\right)^2,$$

provided the nonnegative real numbers x_k, y_k $\left(k \in \{1, \dots, n\}\right)$ satisfy the assumption

$$(3.14)\qquad 0 \leqslant m \leqslant \frac{x_k}{y_k} \leqslant M < \infty \quad \text{for each} \quad k \in \{1, \dots, n\}$$

and $w_k \geqslant 0$, $k \in \{1, \dots, n\}$.

We also remark that Klamkin–McLenaghan inequality (3.13) is a generalisation in its turn of the Shisha–Mond inequality obtained earlier in [9]:

$$\frac{\sum_{k=1}^n a_k^2}{\sum_{k=1}^n a_k b_k} - \frac{\sum_{k=1}^n a_k b_k}{\sum_{k=1}^n b_k^2} \leqslant \left(\sqrt{\frac{A}{b}} - \sqrt{\frac{a}{B}}\right)^2$$

provided

$$0 < a \leqslant a_k \leqslant A, \quad 0 < b \leqslant b_k \leqslant B$$

for each $k \in \{1, \dots, n\}$.

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