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ON THE EIGENVALUES OF REDHEFFER'S MATRIX, II

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Abstract

The Redheffer matrix $A_n = (a_{ij})_{n\times n}$ defined by $a_{ij} = 1$ when $i|j$ or $j = 1$ and $a_{ij} = 0$ otherwise has many interesting number theoretic properties. In this paper we give fairly precise estimates for its eigenvalues in punctured discs of small radius centred at 1.

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1. Introduction

We continue our examination of the non-trivial eigenvalues of Redheffer's matrix A_n, the $n \times n$ matrix (a_{ij}) defined by

$$a_{ij} = \begin{cases} 1 & \text{when } i \mid j \text{ or } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We adopt the notation and terminology introduced in Paragraphs 1 and 2 of part I (Vaughan [6]). Let $D_k(m)$ denote the number of choices of $m_1, \dots, m_k$ with $m_1 \dots m_k = m$ and $m_i \geq 2$ for each i, let

$$S_k(n) = \sum_{m=1}^{n} D_k(m)$$

and let

$$L = [\log_2 n], \qquad N = L + 1.$$

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Then we are concerned, for large n, with the $N-2$ roots of

$$P_n(\lambda) = (\lambda-1)^N - \sum_{k=1}^{L} (\lambda-1)^{L-k} S_k(n) \tag{1}$$

with $|\lambda| = o(\sqrt{n})$. Numerical calculations (see Barrett and Jarvis [2]) indicate that they all lie in the open disc $\mathscr{D} = \{\lambda : |\lambda| < 1\}$, have a preponderance with $\Re\lambda > 0$, and that there are roots near the point 1. In this memoir we concentrate on the neighbourhood of this point.

2. An elementary argument

Put

$$Q_n(z) = -z^N + \sum_{k=1}^{L} z^{L-k} S_k(n) \tag{2}$$

so that

$$Q_n(w^{-1}) = w^{-N}\Big(-1 + \sum_{k=1}^{L} w^{k+1} S_k(n)\Big). \tag{3}$$

Let $\lambda_1, \ldots, \lambda_N$ denote the zeros of $P_n(\lambda)$, and let $w_i = 1/(\lambda_i - 1)$. Then

$$\sum_{i=1}^{N} w_i = -X \quad \text{and} \quad \sum_{i=1}^{N} w_i^2 = -Y$$

where

$$X = S_{L-1}(n)/S_L(n) \tag{4}$$

and

$$Y = \frac{2S_{L-2}(n)}{S_L(n)} - \left(\frac{S_{L-1}(n)}{S_L(n)}\right)^2. \tag{5}$$

Therefore, by Cauchy's inequality

$$\sum_{i=1}^{N} |\lambda_i - 1|^{-2} \geq \max(X^2 N^{-1}, Y). \tag{6}$$

The value of $S_L(n)$ is easily found. When $m \leq n$ we have $m \leq 2^{L+1}$. Thus $D_L(m) = 0$ unless $m = 2^L$ or $m = 2^{L-1}3$ in which case $D_L(m) = 1$ or L respectively. Thus

$$S_L(n) = 1 \text{ or } N \text{ according as } \tfrac{1}{2}2^N \leq n < \tfrac{3}{4}2^N \text{ or } \tfrac{3}{4}2^N \leq n < 2^N. \tag{7}$$

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We need to estimate $S_{L-1}(n)$ from above and below. We have $D_{L-1}(m) = 0$ when $m < 2^{L-1}$. In the range $2^{L-1} \le m < 2^N$, the only m with at least $L-1$ prime factors are

$$2^{L-1},\ 2^{L-2}3,\ 2^{L-3}3^2,\ 2^{L-2}5,\ 2^{L-4}3^3,\ 2^{L-2}7,\ 2^{L-3}3.5,\ 2^L,\ 2^{L-1}3.$$

In order of magnitude they are

$$\tfrac{1}{4}2^N,\ \tfrac{3}{8}2^N,\ \tfrac{1}{2}2^N,\ \tfrac{9}{16}2^N,\ \tfrac{5}{8}2^N,\ \tfrac{3}{4}2^N,\ \tfrac{27}{32}2^N,\ \tfrac{7}{8}2^N,\ \tfrac{15}{16}2^N$$

and $D_{L-1}(m)$ then has the corresponding values

$$1,\ L-1,\ L-1,\ \tfrac{1}{2}(L-1)(L-2),\ L-1,\ (L-1)^2,\ \tfrac{1}{6}(L-1)(L-2)(L-3),$$
$$L-1,\ (L-1)(L-2).$$

Thus

$$\text{(8)} \qquad S_{L-1}(n) \ge \tfrac{1}{2}N(N-1) \qquad \text{when } n \ge \tfrac{9}{16}2^N,$$
$$\text{(9)} \qquad S_{L-1}(n) \ge \tfrac{1}{6}(N^3+5N-12) \qquad \text{when } n \ge \tfrac{27}{32}2^N,$$
$$\text{(10)} \qquad S_{L-1}(n) \le 2N-3 \qquad \text{when } n < \tfrac{9}{16}2^N,$$
$$\text{(11)} \qquad S_{L-1}(n) \le \tfrac{1}{2}(3N^2-7N+4) \qquad \text{when } n < \tfrac{27}{32}2^N.$$

Finally we require a lower bound for $S_{L-2}(n)$ when $n < (9/16)2^N$ and when $(3/4)2^N \le n < (27/32)2^N$. We have $(9/32)2^N = 2^{N-5}3^2$ and $(27/64)2^N = 2^{N-6}3^3$ so that $D_{L-2}((9/32)2^N) = \binom{L-2}{2}$ and $D_{L-2}((27/64)2^N) = \binom{L-2}{3}$. Thus

$$\text{(12)} \qquad S_{L-2}(n) \ge \binom{N-2}{3} \quad \text{when } \tfrac{1}{2}2^N \le n < \tfrac{9}{16}2^N.$$

We also have $(81/128)2^N = 2^{N-7}3^4$. Thus

$$\text{(13)} \qquad S_{L-2}(n) \ge \binom{N-3}{4} \qquad \text{when } \tfrac{3}{4}2^N \le n < \tfrac{27}{32}2^N.$$

Suppose that $(9/16)2^N \le n < (3/4)2^N$. Then, by (4), (7) and (8) we have

$$X \ge \frac{N(N-1)}{2} > \frac{N^2+4}{6}.$$

If instead $(27/32)2^N \le n < 2^N$, then by (4), (7) and (9) we have

$$X \ge \frac{N^3+5N-12}{6N} > \frac{N^2+4}{6}.$$

Similarly when $(1/2)2^N \le n < (9/16)2^N$ we have, by (7), (10) and (12)

$$Y \ge 2\binom{N-2}{3} - (2N-3)^2 > \frac{N^3+8N}{36}$$

and when $(3/4)2^N \le n < (27/32)2^N$ we have, by (7), (11) and (13)

$$Y \ge \frac{2}{N}\binom{N-3}{4} - \left(\frac{3N^2-7N+4}{2N}\right)^2 > \frac{N^3+8N}{36}.$$

Hence, by (6), it follows that $\sum_{i=1}^{N} |\lambda_i - 1|^2 > (N^3+8N)/36$. Moreover, by Theorem 2 of I, the dominant eigenvalues $\lambda_{\pm}$ contribute $O(n^{-1})$ to the sum above. Without loss of generality we may suppose that λ_N and λ_{N-1} are the dominant eigenvalues. Hence we have established the following theorem.

THEOREM 1. *The non-trivial eigenvalues* $\lambda_1, \dots, \lambda_{L-1}$ *of* A_n *satisfy, for* n *sufficiently large,* $\sum_{i-1}^{L-1} |\lambda_i - 1|^{-2} > N^3/36$.

COROLLARY 1. *The matrix* A_n *has eigenvalues* λ *with* $0 < |\lambda - 1| < 6\log 2/\log n$.

3. A combinatorial lemma

In order to understand better the behaviour of the eigenvalues in the neighbourhood of 1 we first require a precise estimate for $S_k(n)$ when k is near L.

LEMMA 1. *Suppose that* $1 \le k \le L$. *Then*

$$T_k(n) \le S_k(n) \le (2k+1)S_{k+1}(n) + T_k(n)$$

where

$$T_k(n) = \sum_{\substack{a=0 \\ \frac{1}{2}n < 2^{k-a}3^a \le n}}^{k} \binom{k}{a}.$$

PROOF. Since $D_k(2^{k-a}3^a) = \binom{k}{a}$, the left hand inequality is trivial. Thus we may concentrate on the one on the right.

First we consider any $m \le n$ for which $D_{k+1}(m) > 0$. Here we adopt a procedure suggested by Carl Pomerance. In this case the total number of prime factors of m is at least $k+1$. Given any sequence of $k+1$ integers $a_i \ge 2$ with $a_1 \dots a_{k+1} = m$ we form k sequences of k numbers b_{ij} by taking $b_{ij} = a_i$ when $1 \le i < j \le k$, $b_{ii} = a_i a_{i+1}$ when $1 \le i < k$ and $b_{ij} = a_{i+1}$ when $1 \le j < i \le k$. Thus every

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$k+1$-tuple $a_1, \dots, a_{k+1}$ gives rise to at most k different k-tuples $b_{1j}, \dots, b_{kj}$ in this way. On the other hand, whenever we have a k-tuple $b_1, \dots, b_k$ with $b_r \geq 2$ and $b_1 \dots b_k = m$, then at least one of the b_r, say b_i, will be composite, so that $b_i = b_{i1}b_{i2}$ with $b_{ij} \geq 2$. Thus $b_1, \dots, b_k$ will certainly arise by the construction described above from the $k+1$-tuple $b_1, \dots, b_{i-1}, b_{i1}, b_{i2}, b_{i+1}, \dots, b_k$. Hence

$$D_k(m) \leq kD_{k+1}(m) \quad \text{when } D_{k+1}(m) > 0,$$

and so

$$\sum_{\substack{m\leq n \\ D_{k+1}(m)>0}} D_k(m) \leq kS_{k+1}(n). \tag{14}$$

It remains to consider the m for which $D_{k+1}(m) = 0 < D_k(m)$. Then the total number of prime factors of m is k, that is $m = p_1 \dots p_k$ with p_i prime. Hence

$$\sum_{\substack{m\leq n \\ D_{k+1}(m)=0}} D_k(m) = \operatorname{card}\{(p_1, \dots, p_k) : p_1 \dots p_k \leq n\}.$$

Let $\mathscr{C}$ denote the set of composite numbers and let $\mathscr{C}^* = \{2, 3\} \cup \mathscr{C}$. Further, let ϕ denote the bijection which takes the jth member of the set of primes in order of magnitude to the jth member of $\mathscr{C}^*$ in order of magnitude. Then $\phi(a) \leq a$ and $\phi(p_1) \dots \phi(p_k) \leq p_1 \dots p_k$. Therefore

$$\sum_{\substack{m\leq n \\ D_{k+1}(m)=0}} D_k(m) \leq \operatorname{card}\{(c_1, \dots, c_k) : c_1 \dots c_k \leq n;\ c_i \in \mathscr{C}^*\}.$$

For each k-tuple $c_1, \dots, c_k$ counted on the right, either $D_{k+1}(c_1 \dots c_k) > 0$ or for each i we have $c_i \in \{2, 3\}$. Thus

$$\sum_{\substack{m\leq n \\ D_{k+1}(m)=0}} D_k(m) \leq \sum_{\substack{m\leq n \\ D_{k+1}(m)>0}} D_k(m) + \sum_{\substack{a=0 \\ 2^{k-a}3^a\leq n}}^{k} \binom{k}{a}.$$

Hence, by (14),

$$S_k(n) \leq 2kS_{k+1}(n) + \sum_{\substack{a=0 \\ 2^{k-a}3^a\leq n}}^{k} \binom{k}{a}.$$

When $2^{k-a}3^a \leq (1/2)n$ we have $2^{k+1-a}3^a \leq n$ and $\binom{k}{a} = \binom{k+1}{a}(k+1-a)/(k+1) \leq \binom{k+1}{a}$. Therefore

$$\sum_{\substack{a=0 \\ 2^{k-a}3^a\leq n/2}}^{k} \binom{k}{a} \leq \sum_{\substack{a=0 \\ 2^{k+1-a}3^a\leq n}}^{k} \binom{k+1}{a} \leq S_{k+1}(n),$$

which completes the proof of the lemma.

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We now apply the above lemma to $Q_n(w)$. As in the lemma, let

(15)
$$T_k(n) = \sum_{\substack{a=0 \\ n/2<2^{k-a}3^a\leq n}}^{k} \binom{k}{a},$$

and define

(16)
$$F(w) = -z^N + \sum_{k=1}^{L} z^{L-k} T_k(n).$$

Our object is to show that when z is small compared with $(\log n)^{-1}$, the polynomial $F(z)$ is the dominant part of $Q_n(z)$.

By the lemma,

$$S_k(n) = T_k(n) + \theta(2k+1)S_{k+1}(n)$$

where $0 \leq \theta \leq 1$. Thus, by (2),

$$\begin{aligned}
0 \leq Q_n(|z|) - F(|z|) &= \sum_{k=1}^{L} |z|^{L-k}(S_k(n) - T_k(n)) \leq \sum_{k=1}^{L} |z|^{L-k}(2k+1)S_{k+1}(n) \\
&\leq 3L|z| \sum_{k=2}^{L} |z|^{L-k} S_k(n) = 3L|z| \left(|z|^N - |z|^{L-1}S_1(n) + Q_n(|z|)\right) \\
&< 3L|z| Q_n(|z|),
\end{aligned}$$

provided that $|z|^2 < n-1$. The following lemma is now immediate.

LEMMA 2. *Suppose that* $3L|z| < 1$ *and* $|z| < (n-1)^{\frac{1}{2}}$. *Then*

$$|Q_n(z) - F(z)| \leq Q_n(|z|) - F(|z|) < \frac{3L|z|}{1-3L|z|} F(|z|).$$

4. The dominant terms in F

For convenience we define

(17)
$$\alpha = \log 2/\log(3/2) = 1.709511\ldots$$

and suppose that β is a real number with

(18)
$$0 \leq \beta < \alpha.$$

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Let $\mathscr{H}$ be the set of j such that

$$[\alpha j + \beta] - [\beta] > \alpha j, \tag{19}$$

that is, such that

$$\{\beta\} \geq 1 - \{\alpha j\}. \tag{20}$$

Suppose $\beta \notin \mathbb{Z}$. Since $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ the set $\mathscr{H}$ is not empty. Let $j_0 \in \mathscr{H}$. Then the number γ_0 defined by $\gamma_0 = ([\alpha j_0 + \beta] - [\beta])/j_0$ satisfies $\gamma_0 > \alpha$. Moreover if $([\alpha j + \beta] - [\beta])/j \geq \gamma_0$, then $\{\beta\} - \{\alpha j + \beta\} \geq j(\gamma_0 - \alpha)$. Thus there are only a finite number of j such that $([\alpha j + \beta] - [\beta])/j \geq \gamma_0$. Hence we can define $\gamma(\beta)$ by $\gamma(\beta) = \max_j([\alpha j + \beta] - [\beta])/j$. By (19), we have, for $j \in \mathscr{H}$, $[\alpha j + \beta] - [\beta] = 1 + [\alpha j]$. Thus

$$\gamma(\beta) = \max\left\{\frac{1 + [\alpha j]}{j} : j \in \mathbb{N}, \{\beta\} \geq 1 - \{\alpha j\}\right\} \quad (\beta \neq 0, 1). \tag{21}$$

When $\beta \in \mathbb{Z}$ we define $\gamma(\beta)$ by

$$\gamma(\beta) = \alpha \quad (\beta = 0, 1). \tag{22}$$

In either case we have

$$[\alpha j + \beta] - [\beta] \leq j\gamma(\beta). \tag{23}$$

By (21) and (22) $\gamma(\beta)$ is a periodic function of β with period 1 which satisfies, in particular,

$$\gamma(\beta) = \begin{cases} 2 & \text{when } 2 - \alpha < \{\beta\}, \\ \frac{7}{4} = 1.75 & \text{when } 7 - 4\alpha < \{\beta\} \leq 2 - \alpha, \\ \frac{12}{7} = 1.71428\ldots & \text{when } 12 - 7\alpha < \{\beta\} \leq 7 - 4\alpha. \\ \frac{53}{31} = 1.70967\ldots & \text{when } 53 - 31\alpha < \{\beta\} \leq 12 - 7\alpha. \\ \frac{359}{210} = 1.70952\ldots & \text{when } 359 - 210\alpha < \{\beta\} \leq 53 - 31\alpha. \\ \frac{665}{389} = 1.70951\ldots & \text{when } 665 - 389\alpha < \{\beta\} \leq 359 - 210\alpha. \end{cases}$$

By (15),

$$T_k(n) = \binom{k}{a_k} + \theta\binom{k}{a_k - 1} \tag{24}$$

where

$$a_k = \left[\log\left(n2^{-k}\right) / \log(3/2)\right]$$

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and $\theta = 0$ or 1 according as $\{\log(n2^{-k})/\log(3/2)\} \geq 1 - \alpha$ or otherwise. Let

$$\beta_n = \log(n2^{-L})/\log(3/2) \tag{25}$$

and define

$$\gamma_n = \gamma(\beta_n). \tag{26}$$

Then

$$a_k = [\alpha(L - k) + \beta_n]. \tag{27}$$

Now define

$$G(z) = \sum_{k=1}^{L} z^{L-k} \binom{k}{a_k}. \tag{28}$$

Then

$$\begin{aligned} |F(z) - G(z)| &\leq |z|^N + \sum_{k=1}^{L} |z|^{L-k} \binom{k}{a_k - 1} \\ &\leq |z|^N + \sum_{\substack{j=0 \\ [\alpha j + \beta_n] \geq 1}}^{L-1} |z|^j \frac{L^{[\alpha j + \beta_n] - 1}}{([\alpha j + \beta_n] - 1)!} \\ &\leq |z|^N + L^{[\beta_n]-1} \sum_{\substack{j=0 \\ [\alpha j + \beta_n] \geq 1}}^{L-1} |z|^j \frac{L^{\gamma_n j}}{([\alpha j + \beta_n] - 1)!}. \end{aligned}$$

We observe that if the term $j = 0$ occurs in the above sum, then $\beta_n \geq 1$ and so $[\alpha j + \beta_n] - 1 \geq [\alpha j] \geq j$ for each $j \geq 0$. Otherwise the first term is the term $j = 1$, and we have $[\alpha + \beta_n] \geq 1$, so that $([\alpha + \beta_n] - 1)! \geq 1$. Moreover, then for $j \geq 2$ we have $[\alpha j + \beta_n] - 1 \geq [\alpha(j - 1)] \geq j$. Thus

$$|F(z) - G(z)| \leq |z|^N + L^{[\beta_n]-1} \exp(|z| L^{\gamma_n}).$$

The next lemma is now an easy consequnece.

LEMMA 3. *Suppose that* $L|z| < 1$. *Then*

$$|F(z) - G(z)| < 2L^{[\beta_n]-1} \exp(|z| L^{\gamma_n}).$$

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5. The zero-free annulus

By (27) and (28) we have

$$G(z) = \sum_{j=0}^{L-1} z^j \binom{L-j}{[\alpha j + \beta_n]}. \tag{29}$$

We also have $[\beta_n]! = 1$. Hence

$$\begin{aligned}\left|G(z) - L^{[\beta_n]}\right| &\le \sum_{j=1}^{\infty} |z|^j \frac{L^{[\alpha j + \beta_n]}}{[\alpha j + \beta_n]!} \le L^{[\beta_n]} \sum_{j=1}^{\infty} \frac{(|z|L^{\gamma_n})^j}{j!} \\ &\le L^{[\beta_n]}\big(\exp(|z|L^{\gamma_n}) - 1\big).\end{aligned}$$

Hence, by Lemmas 2 and 3, when $3L|z| < 1$ and $|z| < (n-1)^{\frac{1}{2}}$ we have

$$\left|Q_n(z) - L^{[\beta_n]}\right| < \left(\frac{3L|z|}{1 - 3L|z|} 3L^{[\beta_n]} + 2L^{[\beta_n]-1}\right) \exp\big(|z|L^{\gamma_n}\big) + L^{[\beta_n]}\Big(\exp\big(|z|L^{\gamma_n}\big) - 1\Big).$$

The next theorem is an immediate consequence.

THEOREM 2. *There is a positive number c such that for each natural number n each non-trivial eigenvalue λ of A_n satisfies*

$$|\lambda - 1| > c(\log n)^{-\gamma_n},$$

where γ_n is defined by (21), (22), (25) *and* (26).

Thus we have the peculiar phenomenon that when, for example,

$$\left(\frac{3}{2}\right)^{2-\alpha} < n2^{-[\log_2 n]} < \frac{3}{2}$$

our bound for the eigenvalues is appreciably smaller than when

$$1 < n2^{-[\log_2 n]} \le \left(\frac{3}{2}\right)^{7-4\alpha}.$$

We shall see below that this bound is usually close to best possible.

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6. Non-trivial eigenvalues close to 1

There is apparently a connection between the non-trivial eigenvalues of A_n and the irrationality measure for α. That α is irrational is completely trivial, of course, and its transcendence follows from the Gelfond-Schneider theorem since otherwise $(3/2)^\alpha$ would be transcendental. Before proceeding we state the following irrationality measure for α.

LEMMA 4. *There are positive numbers A and B such that for each integer a and natural number q we have $|\alpha - a/q| > Bq^{-A}$.*

This is immediate from Feldman's theorem, Feldman [3, 4]. See Baker [1], Theorem 3.1. Any number greater than 2 might well suffice for A in the lemma at least for all sufficiently large q and certainly A cannot be any smaller than 2, but currently available methods will only give something appreciably larger. Since α is equivalent to $(\log 3)/(\log 2)$ there would be some interest in having relatively small values for A. Methods of Galochkin [5], based on the use of G-functions, would seem to be the most appropriate, but the author has been unable to find any explicit values for A in the literature.

Our next theorem shows that Theorem 2 is essentially best possible for the large majority of matrices A_n.

THEOREM 3. *Let β_n be given by* (25) *and γ_n by* (21), (22) *and* (26), *and A by Lemma* 4. *Then there are positive numbers c_1 and c_2 such that for each sufficiently large n with*

$$\{\beta_n\} \geq c_1 (\log\log\log n/\log\log n)^{1/(2A)} \tag{30}$$

the matrix A_n has eigenvalues λ for which

$$0 < |\lambda - 1| < c_2\{\beta_n\}^{-A\gamma_n}(\log n)^{-\gamma_n}.$$

We remark that, by (25), the condition (30) is satisfied by almost all n, and the number $E(X)$ of n not exceeding X for which (30) is false satisfies

$$E(X) \ll X\,(\log\log X/\log\log\log X)^{-1/2A}.$$

We need to modify the argument of the previous section so as to show that on a disk somewhat larger than the annulus there our polynomial is dominated by a non-constant term. To this end we first need to examine γ_n. Let

$$\delta = c\{\beta_n\}^A, \tag{31}$$

where $c = \frac{1}{2}\left(2+3B^{-1/(A-1)}\right)^{-A}$ and A and B are as in Lemma 4, and let

$$(32) \qquad Q = \delta^{-1+1/A}.$$

Then $\{\beta_n\} > 2\delta Q + 3/(BQ)^{1/(A-1)}$. Choose natural numbers a and q so that $(a, q) = 1$, $|\alpha - a/q| \le q^{-1}Q^{-1}$ and $q \le Q$. Then, by Lemma 4 we have $q > (BQ)^{1/(A-1)}$ and so $\{\beta_n\} > 2\delta Q + 3/q$. Now choose the integer b so that

$$1 - \{\beta_n\} + \frac{1}{q} \le \frac{b}{q} < 1 - 2\delta Q - \frac{1}{q}.$$

Then $0 < b < q$. Finally choose j so that $1 \le j \le q$ and $aj \equiv b \pmod{q}$. Then $j < q$, $1/q \le \{aj/q\} < 1 - 1/q$ and $|\{\alpha j\} - \{aj/q\}| < 1/q$. Thus $1 - \{\beta_n\} < \{\alpha j\} < 1 - 2\delta Q$, and so $[\alpha j + \beta_n] - [\beta_n] = 1 + [\alpha j]$. Hence $\left(1 + [\alpha j]\right)/j = \alpha + \left(1 - \{\alpha j\}\right)/j > \alpha + 2\delta Q/j > \alpha + 2\delta$. This shows that

$$(33) \qquad \gamma_n > \alpha + 2\delta.$$

We now advert to our polynomial Q_n. We suppose henceforward that

$$(34) \qquad |z| \le 1/(3L).$$

Then, by Lemmas 2 and 3 we have

$$(35) \qquad |Q_n(z) - G(z)| < \frac{2L^{[\beta_n]-1}}{1 - 3L|z|}\exp\left(|z|L^{\gamma_n}\right) + \frac{3L|z|}{1 - 3L|z|}G(|z|).$$

Let

$$(36) \qquad J = \delta^{-1}.$$

Then by (31) and (32) we have

$$(37) \qquad Q \le J.$$

Let

$$G^*(z) = \sum_j z^j \binom{L - j}{[\alpha j + \beta_n]}$$

where the sum is over those j in the range $0 \le j \le L - 1$ for which $[\alpha j + \beta_n] - [\beta_n] \ge j(\alpha + \delta)$. We note that, in particular, the term $j = 0$ is included in the sum. When $j \ge J$ we have

$$\frac{[\alpha j + \beta_n] - [\beta_n]}{j} \le \frac{1 + [\alpha j]}{j} = \alpha + \frac{1 - \{\alpha j\}}{j} < \alpha + \delta,$$

so that $j < J$ for each included term. Thus, by (29), we have

$$|G(z) - G^*(z)| \le L^{[\beta_n]}|z|L^{\alpha+\delta}\exp\big(|z|L^{\alpha+\delta}\big). \tag{38}$$

By (36), (31) and the hypothesis of the theorem we have $J \ll \log L/\log\log L$. Hence for $j < J$ we have

$$\binom{L-j}{[\alpha j+\beta_n]} = \frac{L^{[\alpha j+\beta_n]}}{[\alpha j+\beta_n]!}\big(1+O(L^{-1/2})\big).$$

Let

$$G^{**}(z) = \sum_j z^j \frac{L^{[\alpha j+\beta_n]}}{[\alpha j+\beta_n]!}, \tag{40}$$

where the summation conditions are as for G^*. Then

$$|G^*(z) - G^{**}(z)| \ll L^{[\beta_n]-\frac{1}{2}}\exp\big(|z|L^{\gamma_n}\big). \tag{41}$$

Now we isolate the terms in G^{**} for which $([\alpha j+\beta_n]-[\beta_n])/j$ is maximal, that is, takes on the value γ_n. We define, for $j > 0$, $\Gamma_j = ([\alpha j+\beta_n]-[\beta_n])/j$. Suppose j and k are both positive, satisfy the summation conditions and $\Gamma_j \ne \Gamma_k$. Then since Γ_j and Γ_k are rational numbers with denominators j and k respectively we have $|\Gamma_j - \Gamma_k| \ge 1/jk > \delta^2$. Let

$$H(z) = L^{[\beta_n]}\sum_j \frac{z^j L^{j\gamma_n}}{[\alpha j+\beta_n]!}$$

where now the sum is over those j for which $[\alpha j+\beta_n]-[\beta_n] = j\gamma_n$. Again we observe that the term $j = 0$ is included in the sum. Thus, by (40), we have

$$|G^{**}(z) - H(z)| \le L^{[\beta_n]}|z|L^{\gamma_n-\delta^2}\Big(\exp\big(|z|L^{\gamma_n-\delta^2}\big)\Big). \tag{42}$$

Let j_0 denote the largest j with $\Gamma_j = \gamma_n$, so that, in particular,

$$j_0 < 1/\delta, \tag{43}$$

and put

$$I(z) = z^{j_0}\frac{L^{[\alpha j_0+\beta_n]}}{[\alpha j_0+\beta_n]!}. \tag{44}$$

Now suppose that

$$|z| = 16\big((\alpha j_0+\alpha)L^{-1}\big)^{\gamma_n}. \tag{45}$$

We observe, by (31) and our hypothesis, that (45) certainly implies (34). Moreover the ratio $\rho = |H(z) - I(z)|/|I(z)|$ satisfies

$$\rho \le \sum_{\substack{j<j_0 \\ \Gamma_j = \gamma_n}} |z|^{j-j_0} L^{\gamma_n(j-j_0)} \frac{[\alpha j_0 + \beta_n]!}{[\alpha j + \beta_n]!}.$$

For each term in the sum we have $\big([\alpha j + \beta_n] - [\beta_n]\big)/j = \big([\alpha j_0 + \beta_n] - [\beta_n]\big)/j_0 = \gamma_n$, so that $[\alpha j_0 + \beta_n] - [\alpha j + \beta_n] = (j_0 - j)\gamma_n$. Thus the sum is bounded by

$$\sum_{j<j_0} |z|^{j-j_0} L^{\gamma_n(j-j_0)} (\alpha j_0 + \beta_n)^{\gamma_n(j_0-j)} \le \sum_{j<j_0} 16^{j-j_0} \le \tfrac{1}{15}.$$

Hence

$$|H(z) - I(z)| < \tfrac{1}{2}|I(z)|. \tag{46}$$

We now show that I dominates the polynomial Q_n on the circle (45). We have $2^{j_0} \ge 1 + j_0$ and $4 > \alpha^\alpha$. Thus $16^{j_0} > (\alpha j_0 + \alpha)^\alpha$ and so, by (44) and (45),

$$|I(z)| > L^{[\beta_n]}. \tag{47}$$

By (30), (31) and (45) we have $|z|L^{\gamma_n - \delta^2} \ll (\log\log n)^{2-c^2c_1^{2A}}$. Thus we need only impose the condition

$$c_1 = 2c^{-1/A} \tag{48}$$

in order to ensure from (42) and (47) that

$$|G^{**}(z) - H(z)| < \tfrac{1}{4}|I(z)| \tag{49}$$

and from (33), (38) and (47) that

$$|G(z) - G^*(z)| < \tfrac{1}{8}|I(z)|. \tag{50}$$

By (45) we have $|z|L^{\gamma_n} \ll \delta^{-2}$. Thus, by (30) and (31),

$$|z|L^{\gamma_n} \ll \log\log n/\log\log\log n. \tag{51}$$

Hence, by (41) and (47),

$$|G^*(z) - G^{**}(z)| < \tfrac{1}{16}|I(z)|. \tag{52}$$

By (46), (49), (50) and (52) we have $|G(z)| < 2|I(z)|$, and hence, by (30), (31), (35), (45), (47) and (51) we have

$$|Q_n(z) - G(z)| < \tfrac{1}{32}|I(z)|.$$

Hence by (46), (49), (50) and (52) we have

$$|Q_n(z) - I(z)| < |I(z)|$$

on the circle (45). Since Q_n and I are analytic in and on the corresponding disc we may appeal to Rouché's theorem, and conclude that I and $Q_n = Q_n - I + I$ have the same number of zeros in the interior of the disc. Since I has $j_0 > 0$ such zeros Theorem 3 now follows easily from (31), (43), (45) and (48).

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