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GENERALISED HESSIAN, MAX FUNCTION AND WEAK CONVEXITY

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In this paper, a second-order characterisation of η-convex $C^{1,1}$ functions is derived in a Hilbert space using a generalised second-order directional derivative. Using this result it is then shown that every $C^{1,1}$ function is locally weakly convex, that is, every $C^{1,1}$ real-valued function f can be represented as $f(x) = h(x) - \eta \|x\|^2$ on a neighbourhood of x where h is a convex function and $\eta > 0$. Moreover, a characterisation of the generalised second-order directional derivative for max functions is given.

1. INTRODUCTION

In this paper, characterisations of the generalised Hessian and the generalised second-order directional derivative introduced in [11] for certain max functions are obtained. It is shown how the twice weakly Gâteaux differentiability of max functions can be characterised. A necessary and sufficient condition for a real valued $C^{1,1}$ function to be η-convex is presented in a Hilbert space using the generalised second-order directional derivative. It is then shown that every $C^{1,1}$ function is locally weakly convex in a Hilbert space. This extends the corresponding results given in Hiriart-Urruty [3] and Vial [10].

Let X be a Banach space. The class of $C^{1,1}$ functions is defined to be the set of all real valued continuously Gâteaux differentiable functions with locally Lipschitz gradients on X, denoted by $C^{1,1}(X)$. Consider the max function of the form $f(x) = [\max\{g(x), 0\}]^2$, where $x \in X$ and $g : X \longrightarrow \mathbb{R}$. If g is twice continuously differentiable, then it is known that f is a $C^{1,1}$ function and various generalised Hessians for the function f were given, for example, in Hiriart-Urruty, Strodiot and Nguyen [4] and Yang and Jeyakumar [11]. In this paper, we study the generalised Hessian introduced in [11] for f when g is a $C^{1,1}$ function. It is worth noting that squares of max functions appear in augmented Lagrangian function methods and smoothing

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approximation methods (see [**9, 13, 14**]). Thus generalised differentiabilities of max functions may be useful in studying optimisation methods.

In Hiriart-Urruty [**3**] and Vial [**10**], it was shown that in a finite dimensional space every $C^{1,1}$ function is locally weakly convex. This result is useful in establishing relations between $C^{1,1}$ functions and so-called lower-C^2 functions. We reprove this result in a Hilbert space by first obtaining a necessary and sufficient condition for η-convex functions. This generalises a corresponding characterisation for convex $C^{1,1}$ functions in [**11**] to generalised convex functions and extends a result of finite dimensional spaces in [**10**] to infinite dimensional spaces.

2. A Generalised Second-Order Directional Derivative

Let X^* be the dual space of X and $\langle\cdot,\cdot\rangle$ be the canonical pair between X^* and X. Let $g : X \longrightarrow \mathbb{R}$ be a locally Lipschitz function and $x \in X$. The Michel-Penot generalised directional derivative of g at x in the direction $u \in X$ is defined by

$$g^\diamond(x;u) = \sup_{z\in X} \limsup_{s\downarrow 0} \frac{g(x+sz+su)-g(x+sz)}{s},$$

and g is said to be semi-regular at x if the one-sided direction derivative

$$g'(x;u) = \lim_{s\downarrow 0} \frac{g(x+su)-g(x)}{s},$$

exists and is equal to $g^\diamond(x;u)$ for every $u \in X$. (See Michel and Penot [**7**].)

It is known that the max function of semi-regular functions is semi-regular and that the semi-regularity condition can be used to establish strong calculus rules. We now give the following notion of a second-order directional derivative of a $C^{1,1}$ function f in terms of the gradient function ∇f. (See Yang and Jeyakumar [**11**] and Yang [**12**].)

Definition 1: Let $f : X \longrightarrow \mathbb{R}$ be a $C^{1,1}$ function and let $x \in X$. Then the *generalised second-order directional derivative* of f at x in the directions $(u,v) \in X\times X$, denoted by $f^{\diamond\diamond}(x;u,v)$, is defined by

$$f^{\diamond\diamond}(x;u,v) = \sup_{z\in X} \limsup_{s\downarrow 0} \frac{\langle\nabla f(x+sz+su),v\rangle - \langle\nabla f(x+sz),v\rangle}{s}. \tag{1}$$

The *generalised Hessian* of f at $x \in X$ for each $u \in X$, denoted by $\partial^{\diamond\diamond} f(x)(u)$, is defined by

$$\partial^{\diamond\diamond} f(x)(u) = \{x^* \in X^* : f^{\diamond\diamond}(x;u,v) \geqslant \langle x^*,v\rangle,\ \forall v \in X\}. \tag{2}$$

The following proposition summarises some basic properties of the generalised second-order directional derivative and the generalised Hessian which are used in the sequel (see [**11**]).

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Proposition 1. *Let $f : X \longrightarrow \mathbb{R}$ be $C^{1,1}$ and $x, u, v \in X$. Then the following properties hold*

(i) $f^{\infty}(x; u, v)$ *is finite and bi-sublinear as a function of* u *and* v*;*
(ii) $\partial^{\infty} f(x)(u)$ *is a nonempty, convex and weak*-compact subset of* X^**;*
(iii) $(-f)^{\infty}(x; u, v) = f^{\infty}(x; -u, v) = f^{\infty}(x; u, -v)$*;*
(iv) $f^{\infty}(x; u, \alpha v) = f^{\infty}(x; \alpha u, v), \quad \forall \alpha \in \mathbb{R} \setminus \{0\}$.

The function f is said to be *twice weakly Gâteaux differentiable* at x [1] if f is continuously Gâteaux differentiable near x and its gradient function ∇f is weakly Gâteaux differentiable at x, that is, there exists a linear function $D^2 f(x) : X \longrightarrow X^*$ such that for each $v \in X^{**}$, $u \in X$, the following holds:

$$\lim_{s \to 0} \frac{\langle \nabla f(x + su), v\rangle - \langle \nabla f(x), v\rangle}{s} = \langle D^2 f(x)(u), v\rangle.$$

Examples of $C^{1,1}$ functions appear, for example, in penalty function methods, augmented Lagrangian methods, proximal point methods and smooth approximation methods. We now give some examples of $C^{1,1}$ functions.

Example 1. Let $X = \mathbb{R}$ and let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a locally Lipschitz function. Then the function $f : \mathbb{R} \longrightarrow \mathbb{R}$, defined by

$$f(x) = \int_0^x g(t)\,dt, \quad x \in \mathbb{R},$$

is a $C^{1,1}$ function. If in addition g is increasing, then f is a convex $C^{1,1}$ function.

Example 2. Let X be a *Hilbert space* and let

$$h(x) = \frac{1}{2} \|x\|^2, \quad x \in X.$$

Then h is $C^{1,1}$. Furthermore, it is twice weakly Gâteaux differentiable. We have

$$h^{\infty}(x; u, v) = \langle u, v\rangle, \quad \forall u, v \in X. \tag{3}$$

Example 3. Let C be a subset of X. Define the following functions, for each $x \in X$,

$$\begin{aligned} d_C(x) &= \inf\{\|x - y\| : y \in C\}, \\ \phi(x) &= \frac{1}{2} d_C^2(x), \\ P_C(x) &= \{y \in C : \|x - y\| = \inf_{z \in C} \|x - z\|\}. \end{aligned}$$

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Two special cases:

(i) $C = \{0\}$, we have $\phi(x) = 1/2\,\|x\|^2$ which was considered in Example 2;

(ii) $C = E_i$, a closed interval in $\mathbb{R}$ (bounded or unbounded), then $d^2_{E_i}(x)$ can be used in formulating exterior point methods and augmented Lagrangian methods, see [9]. In particular, if $C = (-\infty, 0]$, then $\phi(x) = 1/2[\max\{x, 0\}]^2$.

If C is a closed convex subset of a *Hilbert space*, then $P_C(\cdot)$ is single-valued, Lipschitz with Lipschitz constant $L(P_C(\cdot)) = 1$ and

$$\nabla\phi(\cdot) = (I - P_C)(\cdot), \tag{4}$$

see Holmes [5]. Hence $\phi(x)$ is a $C^{1,1}$ function. The generalised second-order directional derivative of $\phi(x)$ was calculated in [12] under certain regularity conditions. We now obtain an estimate of the generalised second-order directional derivative for this function without regularity conditions.

Proposition 2. *Let X be a Hilbert space. If C is a closed convex subset of X, then*

$$\phi^{\infty}(x; u, u) \leqslant 0, \quad \forall u \in X. \tag{5}$$

Proof: Since P_C is Lipschitz with Lipschitz constant $L(P_C(\cdot)) = 1$ (see Example 3), we have from (4)

$$\begin{aligned}
&\left(d^2_C\right)^{\infty}(x; u, u)\\
&\quad = \sup_{z\in X}\limsup_{s\downarrow 0} \frac{\langle 2(P_C - I)(x + su + sz), u\rangle - \langle 2(P_C - I)(x + sz), u\rangle}{s}\\
&\quad = \sup_{z\in X}\limsup_{s\downarrow 0} \frac{2\langle P_C(x + su + sz) - P_C(x + sz), -u\rangle - 2s\langle u, u\rangle}{s}\\
&\quad = 2\sup_{z\in X}\limsup_{s\downarrow 0} \frac{\langle P_C(x + su + sz) - P_C(x + sz), -u\rangle}{s} - 2\langle u, u\rangle\\
&\quad \leqslant 0, \quad \forall x, u \in X.
\end{aligned}$$

Then (5) holds. ∎

3. Max Function and Generalised Hessian

In this section, we study generalised differentiability properties of the max functions of the form

$$m_p(x) = [\max\{g(x), 0\}]^p, \quad x \in X, \tag{6}$$

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where X is a Banach space, $g : X \longrightarrow \mathbb{R}$ and $p \geqslant 2$. It is known that the max function $m_p(x)$ is (Gâteaux) differentiable if g is (Gâteaux) differentiable. Indeed, we have

$$\nabla m_p(x) = p[\max\{g(x), 0\}]^{p-1}\nabla g(x), \quad \forall x \in X. \tag{7}$$

When g has twice differentiability properties and $p = 2$, various generalised Hessians of the function m_2 have been obtained, for example, in [**2, 4, 11, 14**]. We are now able to obtain a characterisation of the generalised Hessian of m_p in terms of the generalised Hessians of g when g is $C^{1,1}$ function. Moreover, we obtain necessary and sufficient conditions for m_p to be twice weakly Gâteaux differentiable.

THEOREM 1. *Let $g : X \longrightarrow \mathbb{R}$ be $C^{1,1}$ and $p \geqslant 2$. Then $m_p(x) = [\max\{g(x), 0\}]^p$ is $C^{1,1}$ and for each $u \in X$, the generalised second-order directional derivative of m_p at x is given by*

$$m_p^{\infty}(x; u, v) = \begin{cases} pg(x)g^{\infty}(x; u, v) + p\langle\nabla g(x), u\rangle\langle\nabla g(x), v\rangle, & \text{if } g(x) > 0; \\ 0, & \text{if } g(x) < 0; \\ p\max\{\langle\nabla g(x), u\rangle\langle\nabla g(x), v\rangle, 0\}, & \text{if } g(x) = 0. \end{cases}$$

PROOF: Since g is $C^{1,1}$, it is clear from (7) that m_p is $C^{1,1}$. For simplicity, we prove the results for the case $p = 2$. We shall consider the following three cases:

CASE I. Let $g(x) > 0$.Then we have from (7) that the equality, $\nabla f(x) = 2g(x)\nabla g(x)$, holds in a neighbourhood of x. Since g is $C^{1,1}$, it is semi-regular and so, we get

$$\begin{aligned} &m_2^{\infty}(x; u, v) \\ &= \sup_{z\in X}\limsup_{s\downarrow 0}\frac{1}{s}\{2g(x+su+sz)\langle\nabla g(x+su+sz), v\rangle \\ &\qquad - 2g(x+sz)\langle\nabla g(x+sz), v\rangle\} \\ &= \sup_{z\in X}\limsup_{s\downarrow 0}\frac{1}{s}\{2g(x+sz)(\langle\nabla g(x+su+sz) - \nabla g(x+sz), v\rangle) \\ &\qquad +2(g(x+su+sz) - g(x+sz))\langle\nabla g(x+su+sz), v\rangle\} \\ &= \sup_{z\in X}\limsup_{s\downarrow 0}\frac{1}{s}\{2g(x)(\langle\nabla g(x+su+sz) - \nabla g(x+sz), v\rangle) \\ &\qquad +2(g(x+su+sz) - g(x+sz))\langle\nabla g(x), v\rangle\} \\ &= \sup_{z\in X}\limsup_{s\downarrow 0}\frac{1}{s}2g(x)(\langle\nabla g(x+su+sz) - \nabla g(x+sz), v\rangle) \\ &\qquad + \lim_{s\downarrow 0}\frac{1}{s}2(g(x+su) - g(x))\langle\nabla g(x), v\rangle \\ &= 2g(x)g^{\infty}(x; u, v) + 2\langle\nabla g(x), u\rangle\langle\nabla g(x), v\rangle, \end{aligned}$$

thus the result holds.

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CASE II. Let $g(x) < 0$. Then we obtain $m_2(x) = 0$ in a neighbourhood of x. Hence the result is true.

CASE III. Let $g(x) = 0$. In fact, when $p = 2$, (7) becomes

$$\nabla m_2(x) = 2\max\{g(x), 0\}\nabla g(x), \quad \forall x \in X.$$

For each $z \in X$, we get

$$\lim_{s\downarrow 0} \frac{\max\{g(x+sz), 0\}(\langle \nabla g(x+su+sz), v\rangle - \langle \nabla g(x+sz), v\rangle)}{s} = 0.$$

Thus we have

$$\begin{aligned}
&m_2^{\infty}(x; u, v)\\
&= \sup_{z\in X}\limsup_{s\downarrow 0} \frac{1}{s}\{2\max\{g(x+su+sz), 0\}\langle \nabla g(x+su+sz), v\rangle\\
&\qquad -2\max\{g(x+sz), 0\}\langle \nabla g(x+sz), v\rangle\}\\
&= \sup_{z\in X}\limsup_{s\downarrow 0} \frac{1}{s}[2\max\{g(x+sz), 0\}(\langle \nabla g(x+su+sz), v\rangle - \langle \nabla g(x+sz), v\rangle)\\
&\qquad +2(\max\{g(x+su+sz), 0\} - \max\{g(x+sz), 0\})\langle \nabla g(x+su+sz), v\rangle]\\
&= \sup_{z\in X}\limsup_{s\downarrow 0} \frac{2(\max\{g(x+su+sz), 0\} - \max\{g(x+sz), 0\})\langle \nabla g(x), v\rangle}{s}.
\end{aligned}$$

Since g is $C^{1,1}$, $\max\{g, 0\}$ is semi-regular, thus we obtain

$$\begin{aligned}
&m_2^{\infty}(x; u, v)\\
&\quad = 2\lim_{s\downarrow 0} \frac{\max\{g(x+su)\langle \nabla g(x), v\rangle, 0\} - \max\{g(x)\langle \nabla g(x), v\rangle, 0\}}{s}\\
&\quad = 2\max\{\langle \nabla g(x), u\rangle\langle \nabla g(x), v\rangle, 0\}.
\end{aligned}$$

Then the proof is complete. ∎

REMARK 1. From the Hahn-Banach Theorem [5], we get the following inclusions of the generalised Hessian,

$$\partial^{\infty} m_p(x)(u) \tag{8}$$
$$= \begin{cases} \{pg(x)^{p-1}x + p(p-1)g(x)^{p-2}\langle \nabla g(x), u\rangle \nabla g(x) : \\ \qquad\qquad x^* \in \partial^{\infty} g(x)(u)\}, & \text{if } g(x) > 0;\\ \{0\}, & \text{if } g(x) < 0;\\ \{\beta p(p-1)g(x)^{p-2}\langle \nabla g(x), u\rangle \nabla g(x) : \beta \in [0,1]\}, & \text{if } g(x) = 0. \end{cases}$$

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REMARK 2. It follows from a second-order chain rule (see [**12**, Theorem 2]) that

$$(9)\qquad \partial^{\infty} m_p(x)(u) \subseteq \begin{cases} \{pg(x)^{p-1}x + p(p-1)g(x)^{p-2}\langle \nabla g(x), u\rangle \nabla g(x) : \\ \qquad\qquad x^* \in \partial^{\infty} g(x)(u)\}, & \text{if } g(x) > 0; \\ \{0\}, & \text{if } g(x) < 0; \\ \{\beta p(p-1)g(x)^{p-2}\langle \nabla g(x), u\rangle \nabla g(x) : \beta \in [0,1]\}, & \text{if } g(x) = 0. \end{cases}$$

and that (9) holds with equality if $\nabla g(x)$ is onto. Comparing (8) with (9), we see that the onto condition used in [**11**] is only sufficient.

Using Theorem 1, we obtain characterisations of twice weakly Gâteaux differentiability of the max function m_p when the function g is $C^{1,1}$.

PROPOSITION 3. *Let X be a reflexive Banach space and let g be $C^{1,1}$ and $x \in X$ be a point satisfying $g(x) = 0$. Then the function m_p is twice weakly Gâteaux differentiable at x if and only if $\nabla g(x) = 0$ and g is twice weakly Gâteaux differentiable at x.*

PROOF: From Theorem 1, $\partial^{\infty} m_p(x)(u)$ is single-valued for all $u \in X$ if and only if $\nabla g(x) = 0$ and $\partial^{\infty} g(x)(u)$ is single-valued for all $u \in X$. Then the conclusion holds. ∎

We finish this section with a couple of numerical examples to show the structure of the generalised Hessian of max functions.

EXAMPLE 4. Let $m_p(x) = [\max\{x,0\}]^p$, $x \in \mathbb{R}$ and $p \geqslant 2$. Then we have

$$\partial^{\infty} m_p(x)(u) = \begin{cases} \{p(p-1)x^{p-2}u\}, & \text{if } x > 0; \\ \{0\}, & \text{if } x < 0; \\ \{\beta p(p-1)x^{p-2}u : \beta \in [0,1]\}, & \text{if } x = 0. \end{cases}$$

EXAMPLE 5. Let $m_2(x) = \left[\max\left\{\int_0^x t^2 \sin(1/t)\, dt + 1, 0\right\}\right]^2$, $x \in \mathbb{R}$. Then our generalised Hessian $\partial^{\infty} m_2(x)(u)$ at $x = 0$ is

$$\partial^{\infty} m_2(0)(u) = \{0\}.$$

4. Weak Convexity and Generalised Second-Order Derivative

In this section, we obtain a characterisation of η-convexity and show that every $C^{1,1}$ function is locally weakly convex in a Hilbert space using the generalised second-order directional derivative $f^{\infty}(x; u, v)$.

We first recall the definition of η-convexity.

Definition 2. *Let C be a convex subset of X and let $f : C \longrightarrow \mathbb{R}$. The function f is said to be η-convex on C if there exist a real number η and a convex function $h : C \longrightarrow \mathbb{R}$ such that $f(x) = h(x) + \eta \|x\|^2$, $\forall x \in C$.*

Note that if $\eta > 0$, then f is said to be strongly convex on C; if $\eta = 0$, then f is convex on C; if $\eta < 0$, then f is said to be *weakly convex* on C, see Vial [**10**] and Jeyakumar [**6**].

Definition 3. *(i) $f : X \longrightarrow \mathbb{R}$ is said to be locally weakly convex on X if for each $x \in X$, there exists $r > 0$ such that f is weakly convex on an open ball centred at x with radius r, denoted by $U^\circ(x, r)$;*

(ii) f is said to be globally weakly convex if f is weakly convex on X.

The following characterisation for a $C^{1,1}$ function to be convex is given in [**11**].

Lemma 1. *Let X be a Banach space and let $f : X \longrightarrow \mathbb{R}$. Then f is convex on X if and only if*

$$f^{\infty}(x; u, -u) \geqslant 0, \quad \forall x, u \in X.$$

We first obtain a characterisation of η-convexity in terms of the generalised second-order directional derivative. It is worth noting that this result paves the way to establishing and generalising connections between a $C^{1,1}$ function and weak convexity in a Hilbert space.

Theorem 2. *Let X be a Hilbert space and let $f : X \longrightarrow \mathbb{R}$ be $C^{1,1}$. Then f is η-convex on X if and only if*

$$f^{\infty}(x; u, -u) \geqslant -2\eta \|u\|^2, \quad \forall x, u \in X. \tag{10}$$

Proof: Let f be a $C^{1,1}$ function. If f is η-convex on X, then there exist a real number η and a convex function $h : X \longrightarrow \mathbb{R}$ such that $f(x) = h(x) + \eta \|x\|^2$, $\forall x \in X$. Since f and $\eta \|\cdot\|^2$ are $C^{1,1}$, the function h is also $C^{1,1}$. Note from (3) that $\left(\|\cdot\|^2\right)^{\infty}(x; u, -u) = -2\|u\|$, $\forall u \in X$. Hence from the triangle inequality, we obtain

$$\begin{aligned} f^{\infty}(x; u, -u) &\leqslant h^{\infty}(x; u, -u) + \left(\eta \|\cdot\|^2\right)^{\infty}(x; u, -u) \\ &\leqslant h^{\infty}(x; u, -u) - 2\eta \|u\|^2, \quad \forall x, u \in X. \end{aligned}$$

From Lemma 1, $h^{\infty}(x; u, -u) \leqslant 0$, $\forall x, u \in X$, so we have

$$f^{\infty}(x; u, -u) \leqslant -2\eta \|u\|^2, \quad \forall x, u \in X.$$

Conversely, if (10) holds, then

$$f(x) = \left(f(x) - \eta \|x\|^2\right) + \eta \|x\|^2, \quad \forall x, u \in X,$$

and the function $f(x) - \eta \|x\|^2$ is convex on X since

$$\left(f - \eta \|\cdot\|^2\right)^{\infty}(x; u, -u) \leqslant f^{\infty}(x; u, -u) + 2\eta \|u\|^2 \leqslant 0, \quad \forall x, u \in X.$$

Thus f is η-convex on X. □

Clearly Theorem 2 is an extension of Lemma 1. Moreover, when $\eta = 0$, Theorem 2 reduces to Lemma 1. As an immediate application of Theorem 2, let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a locally Lipschitz function. Then the function f defined in Example 1 is η-convex if and only if $g^{\circ}(x; -1) \leqslant 2\eta$, $\forall x \in \mathbb{R}$. The following corollary shows that Theorem 2 generalises a result in [**10**, Proposition 4.11] where twice differentiability is required.

COROLLARY 1. *Let X be a Hilbert space and let $f : X \longrightarrow \mathbb{R}$ be twice weakly Gâteaux differentiable. Then f is η-convex on X if and only if*

$$\langle D^2 f(x)(u), u\rangle \geqslant 2\eta \|u\|^2, \quad \forall x, u \in X.$$

PROOF: This follows from the fact that f is twice weakly Gâteaux differentiable, thus

$$f^{\infty}(x; u, -u) = -\langle D^2 f(x)(u), u\rangle, \ \forall x, u \in X.$$ □

Now we establish that in a Hilbert space every $C^{1,1}$ function is locally weakly convex using our generalised second-order directional derivative.

THEOREM 3. *Let X be a Hilbert space. If $f : X \longrightarrow \mathbb{R}$ is a $C^{1,1}$ function, then f is locally weakly convex on X.*

PROOF: Let $f : X \longrightarrow \mathbb{R}$ be a $C^{1,1}$ function. Then for any fixed $\bar{x} \in X$, it follows from the locally Lipschitz condition of ∇f that there exist $L(\nabla f, \bar{x}) > 0$ and $r > 0$ such that

$$\|\nabla f(y) - \nabla f(x)\| \leqslant L(\nabla f, \bar{x}) \|y - x\|, \quad \forall y, x \in U^{\circ}(\bar{x}, r).$$

Let $\eta \geqslant (L(\nabla f, \bar{x}))/2$. Then for any $u \in X$, $x \in U^{\circ}(\bar{x}, r)$, we have

$$\begin{aligned} f^{\infty}(x; u, -u) &= \sup_{z \in X} \limsup_{s \downarrow 0} \frac{\langle \nabla f(x + su + sz), -u\rangle - \langle \nabla f(x + sz), -u\rangle}{s} \\ &\leqslant L(\nabla f, \bar{x}) \|u\|^2 \leqslant 2\eta \|u\|^2. \end{aligned}$$

So,

$$\begin{aligned} \left(f + \eta \|\cdot\|^2\right)^{\infty}(x; u, -u) &\leqslant f^{\infty}(x; u, -u) + \left(\eta \|\cdot\|^2\right)^{\infty}(x; u, -u) \\ &= f^{\infty}(x; u, -u) - 2\eta \|u\|^2 \\ &\leqslant 0, \quad \forall x \in U^{\circ}(\bar{x}, r), \ u \in X. \end{aligned}$$

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From Lemma 1, $f+\eta\|\cdot\|^2$ is convex on $U^\circ(\overline{x},r)$. Then $f(x)=\left(f(x)+\eta\|x\|^2\right)-\eta\|x\|^2$, in which $f+\eta\|\cdot\|^2$ is convex on $U^\circ(\overline{x},r)$. Hence f is locally weakly convex on X. ∎

It is well known that the function $-d_C^2(x)$ is globally weakly convex, where C is a closed convex subset of a Hilbert space. We present a proof of this result using our generalised second-order directional derivative. Recall that $-d_C^2(x)$ is a $C^{1,1}$ function, see Example 3.

Proposition 4. *Let X be a Hilbert space. If C is a closed convex subset of X, then $-d_C^2(x)$ is globally weakly convex.*

Proof: Observe that

$$-d_C^2(x)=\left(2\|x\|^2-d_C^2(x)\right)-2\|x\|^2.$$

Thus we need to prove that $x\longrightarrow 2\|x\|^2-d_C^2(x)$ is convex on X. From Proposition 2, we have

$$\left(-d_C^2\right)^{\infty}(x;u,-u)=\left(d_C^2\right)^{\infty}(x;u,u)\leqslant 0,\quad \forall x,u\in X.$$

Then from (3)

$$\begin{aligned}\left(2\|\cdot\|^2-d_C^2\right)^{\infty}(x;u,-u)&\leqslant\left(2\|\cdot\|^2\right)^{\infty}(x;u,-u)+\left(-d_C^2\right)^{\infty}(x;u,-u)\\&\leqslant -4\langle u,u\rangle\\&\leqslant 0,\quad \forall x,u\in X.\end{aligned}$$

From Lemma 1, the function $x\longrightarrow 2\|x\|^2-d_C^2(x)$ is convex on X. Therefore $-d_C^2(x)$ is globally weakly convex. ∎

Corollary 2. *Let X be a Hilbert space and let $g:X\longrightarrow\mathbb{R}$ be a convex function. Then $m_2(x)=-[\max\{g(x),0\}]^2$ is globally weakly convex.*

Proof: Let $C=\{x\in X: g(x)\leqslant 0\}$. Thus C is a closed convex subset and $d_C^2(x)=[\max\{g(x),0\}]^2$. The conclusion follows from Proposition 4. ∎

5. Discussion

Let X be a Hilbert space and let $f:X\longrightarrow\mathbb{R}$. Then the following classes of functions are introduced and studied in [**3**, **6**, **8**, **10**]:

(i) the function f is said to be *locally difference convex* on X if for every $\overline{x}\in X$, there exist a convex neighbourhood $N(\overline{x})$ of $\overline{x}$, and convex functions $p_N,q_N:X\longrightarrow\mathbb{R}$ such that $f(x)=p_N(x)-q_N(x)$, $\forall x\in N(\overline{x})$. This class of functions is denoted by LDC(X). The function f is said to be *difference convex* on X if there exist two convex functions $p,q:X\longrightarrow\mathbb{R}$ such that $f(x)=p(x)-q(x)$, $\forall x\in X$;

(ii) the function f is said to be lower-C^2 on X if for every $\overline{x} \in X$, there exist a convex neighbourhood $N(\overline{x})$ of $\overline{x}$, a convex function p_N and a quadratic convex function q_N such that $f(x) = p_N(x) - q_N(x)$, $\forall x \in N(\overline{x})$. This class of functions is denoted by $LC^2(X)$.

It follows from the previous definitions that every locally weakly convex function is locally difference convex. In general, a quadratic convex function in a Hilbert space has the form

$$\langle A(u), u\rangle + \langle b, u\rangle + c,$$

where $b \in X$, $c \in \mathbb{R}$ and $A : X \longrightarrow X$ satisfies $\langle A(x), x\rangle \geqslant 0$, $\langle A(x), y\rangle = \langle A(y), x\rangle$. In particular $\|x\|^2 = \langle x, x\rangle$ is a quadratic convex function. Hence it follows from Theorem 3 that every $C^{1,1}$ function is lower-C^2. It is clear that every lower-C^2 function is locally difference convex. Therefore we have established that

$$C^{1,1}(X) \subset LC^2(X) \subset LDC(X),$$

where X is a Hilbert space. This result was initially given in Hiriart-Urruty [**3**] and Vial [**10**] in a finite dimensional space.

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