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DISTANCE TRANSITIVE GRAPHS WITH SYMMETRIC OR ALTERNATING AUTOMORPHISM GROUP

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The paper classifies all distance transitive graphs Γ such that $A_n \leq \mathrm{Aut}\,\Gamma \leq \mathrm{Aut}\,A_n$ for some alternating group A_n, and $\mathrm{Aut}\,\Gamma$ acts primitively on the vertices of Γ. This result forms part of our programme for determining all finite primitive distance transitive graphs.

1. Introduction and statement of results

In this paper we classify the finite distance transitive graphs whose automorphism group is a symmetric group S_n or an alternating group A_n for some n, acting primitively on the set of vertices. This forms a part of the programme for the classification of all finite primitive distance transitive graphs begun in [16]; for in [16] this classification is reduced to the determination of all such graphs whose automorphism group G is either almost simple (that is, $T \triangleleft G \leq \mathrm{Aut}\,T$ for some nonabelian simple group T) or affine (that is, $V \triangleleft G \leq AGL(V)$, the group of affine transformations of a finite vector space V). Thus in this paper we deal with part of the almost simple case, namely the case where $T = A_n$. When T is a linear group of dimension at least 7, a classification is obtained in [7]; discussion of the remaining almost simple cases can be

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found in [10]. The case where G is affine and $|V|$ is large is treated in [12]. Note that the primitivity of G is a natural assumption, since by [19] (see also [3]), there is a simple procedure for obtaining a primitive distance transitive graph from an imprimitive one.

It is well known that the permutation character of the automorphism group G of a distance transitive graph acting on vertices must be multiplicity-free, since all the suborbits are self-paired (see [14, Theorem 8]). Our proof is based on [17, Theorem, p340], where all such characters are determined for $G = S_n$ with $n > 18$.

Before stating our result, we describe some classes of distance transitive graphs Γ. Denote by $V\Gamma$ the set of vertices of Γ, and by Ω a set of n points, where $n \geq 5$.

(1.1) Johnson graphs $J(n,k)$. Here $V\Gamma = \Omega^{\{k\}}$, the set of k-subsets of Ω, where $k < n/2$. Two vertices A and B are joined if and only if $|A \cap B| = k - 1$. (Note that $J(n,1)$ is just the complete graph K_n.)

(1.2) Graphs $\overline{J(n,2)}$. These are the complements of the rank 3 graphs $J(n,2)$.

(1.3) Odd graphs O_k. Here $n = 2k + 1$ and $V\Gamma = \Omega^{\{k\}}$; two vertices A and B are joined if and only if $A \cap B = \emptyset$.

(1.4) Graphs $J(2k,k)'$, $k \geq 4$. These are the derived graphs of the antipodal graphs $J(2k,k)$ (see [1, p152]). They can also be described as follows: $n = 2k$, $V\Gamma$ is the set of partitions of Ω into two subsets $\{A,\bar{A}\}$ of size k, and two vertices $\{A,\bar{A}\},\{B,\bar{B}\}$ are joined if and only if either $|A \cap B| = k - 1$ or $|A \cap \bar{B}| = k - 1$.

(1.5) Graphs $\overline{J(2k,k)'}$, $k = 4,5$. These are the complements of the rank 3 graphs $J(8,4)'$, $J(10,5)'$.

(1.6) Graphs Σ_{120}, $\bar{\Sigma}_{120}$. Here Σ_{120} is the rank 3 graph of valency 56 on 120 vertices obtained from the rank 3 action of A_9 on the 120 cosets of a subgroup $P\Gamma L_2(8)$ (see [4]). (Note that there are two conjugacy classes of subgroups $P\Gamma L_2(8)$ in A_9, but the corresponding actions of A_9 of degree 120 are conjugate in S_{120}; hence Σ_{120} is unique.)

(1.7) Graph Σ_{36}. This is a rank 4 graph of valency 5 on 36 vertices; its vertices are the 36 subgroups of order 20 in S_6, and two vertices A,B are joined if and only if $|A \cap B| = 4$. Another description can be found in [1, p153].

(1.8) Graph Σ_{45}. This is a rank 5 graph of valency 4 on 45 vertices; its vertices are the 45 Sylow 2-subgroups of $\mathrm{Aut}\, A_6$, two vertices A,B being joined if and only if $|A \cap B| = 8$. The graph Σ_{45} can also be described as the line graph of the trivalent Tutte 8-cage (see [20, Chapter 8]).

THEOREM. *Let G be the group A_n or S_n $(n \geq 5)$ and suppose that G acts distance transitively on a graph Γ and is primitive on $V\Gamma$. Then Γ is one of the graphs in* (1.1)-(1.6) *above. Further, if Γ is in* (1.1)-(1.5) *and Γ is not a complete graph, then* $\mathrm{Aut}\, \Gamma \cong S_n$ *; and in* (1.6), $G = A_9 < \mathrm{Aut}\, \Gamma \cong O_8^+(2)$.

The statement that G acts distance transitively on Γ means that whenever $\alpha,\beta,\gamma,\delta \in V\Gamma$ with the distance (= length of shortest path) between α and β being the same as the distance between γ and δ, there is an automorphism $g \in G$ such that $\alpha^g = \gamma$, $\beta^g = \delta$. Note that we exclude complete graphs Γ in the last sentence of the theorem in view of the 2-transitive actions of A_5, A_6, A_7 and A_8 of degrees 6,10,15 and 15 respectively.

Our classification result follows immediately from the theorem, together with the observations on $\mathrm{Aut}\, A_6$ in Section 5:

COROLLARY. *Suppose that Γ is a distance transitive graph with full automorphism group G, where $A_n \triangleleft G \leq \mathrm{Aut}\, A_n (n \geq 5)$. Then either*

(i) $G = S_n$ *and Γ is as in* (1.1)-(1.5) *above, or*

(ii) $n = 6$, $G = \mathrm{Aut}\, A_6$ *and Γ is as in* (1.7) *or* (1.8).

Proof of the theorem. Let G and Γ be as in the statement of the theorem. For $\alpha,\beta \in V\Gamma$ let $d(\alpha,\beta)$ be the distance between α and β, and put $d = \max\{d(\alpha,\beta) \mid \alpha,\beta \in V\Gamma\}$, the diameter of Γ. Choose $\alpha \in V\Gamma$ and for $1 \leq i \leq d$ let $\Gamma_i(\alpha) = \{\beta \mid d(\alpha,\beta) = i\}$, so that $\Gamma_i(\alpha)$ are the orbits

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of G_α on $V\Gamma\backslash\{\alpha\}$. Define $k_i = |\Gamma_i(\alpha)|$. Some well-known properties of the integers k_i are given in [16, 1.1]; in particular,

(1.9) $k_1 < k_i$ *for all* i *such that* $2 \leq i \leq d-1$

so that $\Gamma_1(\alpha)$ is one of the shortest two orbits $\Gamma_i(\alpha)$.

Let $\Omega = \{1,\ldots,n\}$ be a set of n points permuted naturally by G , and write $H = G_\alpha$, a maximal subgroup of G . The proof is carried out in three sections, according as H^Ω is intransitive (Section 2), transitive and imprimitive (Section 3), or primitive (Section 4). As remarked above, the permutation character $\pi = 1_H^G$ is multiplicity-free, and hence

(1.10) $$|G:H| \leq \sum \chi(1) ,$$

where the summation is over all irreducible characters χ of G .

2. The intransitive case

In this section we deal with the case where H^Ω is intransitive, so that by the maximality of H in G , we have $H = (S_k \times S_{n-k}) \cap G$ for some k with $1 \leq k < n/2$. We can therefore identify $V\Gamma$ with $\Omega^{\{k\}}$, the set of k-subsets of Ω . If $k = 1$ then Γ is the complete graph $K_n = J(n,1)$; and if $k = 2$ then G has rank 3 on $\Omega^{\{k\}}$, so Γ is $J(n,2)$ or $\overline{J(n,2)}$, as in (1.1) and (1.2). Thus we assume that $k \geq 3$. If $\alpha = A \in \Omega^{\{k\}}$ then the H-orbits on $\Omega^{\{k\}}$ are

$$\Delta_i(A) = \{B\in\Omega^{\{k\}} : |A \cap B| = k-i\}$$

for $0 \leq i \leq k$. Let $\Gamma_1(\alpha) = \Delta_i(A)$ for some $i \geq 1$. If $i = 1$ then Γ is $J(n,k)$, so assume that $i \geq 2$. Let $A = \{1,\ldots,k\}$ and $B = B_0 \cup \{k+1,\ldots,k+i\}$, where B_0 is $\emptyset$ if $i = k$ and $\{1,\ldots,k-i\}$ if $i < k$. Then $B \in \Gamma_1(A)$. Also

$$C = (A\backslash\{k\}) \cup \{k+i+1\} \in \Gamma_1(B) \cap \Delta_1(A) ,$$

and, provided that $n \geq k+i+2$,

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$$D = (A\backslash\{k-1,k\}) \cup \{k+i+1,\ k+i+2\} \in \Gamma_1(B) \cap \Delta_2(A) ,$$

and, provided that $i < k$,

$$E = (A\backslash\{1,k-1,k\}) \cup \{k+1,k+i+1,k+i+2\} \in \Gamma_1(B) \cap \Delta_3(A) .$$

This shows that Γ is not distance transitive unless $i = k$ and $n = 2k + 1$; in this latter case Γ is the odd graph O_k as in (1.3).

To complete the proof in the intransitive case, we show that $\text{Aut}\,\Gamma \cong S_n$ for the graphs Γ in (1.1), (1.2) and (1.3). For suppose that this is false, so that $\text{Aut}\,\Gamma > G \cong S_n$ for some such graph Γ . Then $\text{Aut}\,\Gamma$ is given by [6]; in each case we see that $\text{Aut}\,\Gamma$ has smaller rank on $V\Gamma$ than G , which is impossible since G acts distance transitively on Γ .

3. The imprimitive case

We next deal with the case where H^{Ω} is transitive and imprimitive, so that by the maximality of H in G , we have $H = (S_k\ wr\ S_\ell) \cap G$ with $k\ell = n$, $k > 1$ and $\ell > 1$. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ of n we denote by Ω^λ the set of cosets of the subgroup $S_{\lambda_1} \times S_{\lambda_2} \times \ldots$ in S_n , and by π^λ the permutation character of S_n on Ω^λ , as in [17]; we also denote by χ^λ the irreducible character of S_n corresponding to λ , as in [17].

LEMMA 3.1. *One of the following holds:*

(i) $\ell = 2$;

(ii) $k = 2$;

(iii) (k,ℓ) *is one of* (3,3), (3,4), (4,3) *and* (5,3).

Proof. It is well-known (see [17], 2.1) that for $1 \le r \le n/2$ the permutation character $\pi^{(n-r,r)}$ of G on $\Omega^{\{r\}}$ is given by

$$\pi^{(n-r,r)} = 1 + \chi^{(n-1,1)} + \ldots + \chi^{(n-r,r)} = \pi^{(n-r+1,r-1)} + \chi^{(n-r,r)}$$

Consequently since $\pi = 1_H^G$ is multiplicity-free,

$$(\pi,\pi^{(n-r,r)})_G \le 1 + (\pi,\pi^{(n-r+1,r-1)})_G \le \ldots \le r-1 + (\pi,\pi^{(n-1,1)})_G = r \ , \quad (*)$$

the last equality holding since H is transitive on Ω. In particular H has at most r orbits on $\Omega^{\{r\}}$.

If $k \ge 4, \ell \ge 4$ then it is easy to see that H has at least five orbits on $\Omega^{\{4\}}$, which is not so. If $\ell = 3, k \ge 6$ or $k = 3, \ell \ge 6$ then H has seven orbits on $\Omega^{\{6\}}$, which is again false. Finally, let $k = 3, \ell = 5$. We claim that if $G = S_{15}$ then $\chi^{(9,4,2)}$ appears in 1_H^G with multiplicity 2, which is a contradiction. For by the determinantal rule [9, 2.3.15], we have

$$\pi^{(9,4,2)} = \chi^{(9,4,2)} + \pi^{(9,5,1)} + \pi^{(10,3,2)} + \pi^{(11,4)} - \pi^{(10,5)} - \pi^{(11,3,1)}$$

and so, if n_λ denotes the number of orbits of H on Ω^λ, the multiplicity of $\chi^{(9,4,2)}$ in 1_H^G is

$$n_{(9,4,2)} - n_{(9,5,1)} - n_{(10,3,2)} - n_{(11,4)} + n_{(10,5)} + n_{(11,3,1)} .$$

A straightforward calculation shows that this number is 2, as claimed.

REMARK. In fact S_n is multiplicity-free on $(S_n : S_k \; wr \; S_\ell)$ with k, ℓ as in (i), (ii) or (iii) of Lemma 3.1; this is [17, 2.2 and 2.3] in cases (i) and (ii), and can be verified by calculation in case (iii).

We deal separately with cases (i), (ii) and (iii) of Lemma 3.1.

(3.2) Case $\ell = 2$. Here $n = 2k$ and $H = (S_k \; wr \; S_2) \cap G$. We identify $V\Gamma$ with the set of partitions of Ω into two subsets $\{A, \bar{A}\}$ of size k. If $\alpha = \{A, \bar{A}\}$ then the H-orbits on $V\Gamma$ are

$$\Sigma_i(\alpha) = \{\{B,\bar{B}\} \in V\Gamma : |B \cap A| = i \text{ or } |\bar{B} \cap A| = i\}$$

for $0 \le i \le [k/2]$. If $k \le 5$ then G has rank 2 or 3 on $V\Gamma$ and Γ is as in (1.1), (1.4) or (1.5). Thus we assume that $k \ge 6$. Now $|\Sigma_i(\alpha)| = \binom{k}{i}^2$, so the shortest two H-orbits on $V\Gamma \backslash \{\alpha\}$ are $\Sigma_1(\alpha)$ and $\Sigma_2(\alpha)$. Hence by (1.9), $\Gamma_1(\alpha)$ is one of these. If $\Gamma_1(\alpha) = \Sigma_1(\alpha)$ then Γ

is $J(2k,k)'$ as in (1.4), so assume that $\Gamma_1(\alpha) = \Sigma_2(\alpha)$. Write $A = \{1,\ldots,k\}$, $B = \{1,\ldots,k-2,k+1,k+2\}$, $C = \{1,\ldots,k-1,k+3\}$, $D = \{1,\ldots,k-3,k+1,k+3,k+4\}$. Then

$$\beta = \{B,\bar{B}\} \in \Gamma_1(\alpha) ,$$

$$\{C,\bar{C}\} \in \Gamma_1(\beta) \cap \Sigma_1(\alpha) ,$$

$$\{D,\bar{D}\} \in \Gamma_1(\beta) \cap \Sigma_3(\alpha) .$$

Hence $\Gamma_2(\alpha)$ contains $\Sigma_1(\alpha) \cup \Sigma_3(\alpha)$ and so Γ is not distance transitive, a contradiction.

(3.3) Case $k = 2$. Here $n = 2\ell$ and $H = (S_2 \text{ wr } S_\ell) \cap G$. First let $\ell = 3$. Here G has rank 3 on the 15 points $(G:H)$. Now G has just one other primitive action of degree 15 , namely that on $\Omega^{\{2\}}$. In each of these actions, $N_{S_{15}}(G) = N_{A_{15}}(G) \cong S_6$, and it follows that the two actions of G are conjugate in S_{15} . Hence the graphs Γ on $(G:H)$ here are $J(6,2)$ and its complement.

Thus we assume now that $\ell \geq 4$. We identify $V\Gamma$ with the set of partitions of Ω into ℓ blocks of size 2 . Let $\alpha = \{A_1,\ldots,A_\ell\} \in V\Gamma$ (with $|A_i| = 2$ for all i), and for each $\beta \in V\Gamma$ define the graph $\Delta(\beta)$ to have as vertices $A_1,\ldots,A_\ell$, with A_i and A_j $(i \neq j)$ adjacent whenever some block of β consists of a point of A_i and a point of A_j . For $1 \leq i \leq \ell$ let a_i be the number of connected components of size i of $\Delta(\beta)$. Note that each such component is just a cycle of length i . Thus β corresponds to the partition $\rho_\beta = (1^{a_1}, 2^{a_2}, \ldots, \ell^{a_\ell})$ of $\ell = \sum i a_i$. It is easy to check that if $G = S_n$ then the orbits of $H = G_\alpha$ on $V\Gamma$ are the sets

$$\Sigma(\rho,\alpha) = \{\beta \in V\Gamma \mid \rho_\beta = \rho\}$$

where ρ is a partition of ℓ . If $G = A_n$, the sets $\Sigma(\rho,\alpha)$ may split into two H-orbits of equal size - however, no such splitting occurs if, for example, $a_1 \geq 1$ or $a_3 \geq 1$ (since in these cases, for $\beta \in \Sigma(\rho,\alpha)$, there

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is an odd permutation of S_n fixing both α and β). Note that $\Sigma((1^\ell),\alpha) = \{\alpha\}$. Write

$$|\Sigma(\rho,\alpha)| = \sigma_\ell(\rho) .$$

LEMMA 3.3.1. *For* $\rho = (1^{a_1},\ldots,\ell^{a_\ell})$ *a partition of* ℓ, *we have*

$$\sigma_\ell(\rho) = \ell!2^{\ell-\sum a_i} / \prod_{i=1}^{\ell} (a_i! i^{a_i}) .$$

Proof. We count the number of β in $\Sigma(\rho,\alpha)$:

(a) Choose a_1 common blocks for α and β, in $\binom{\ell}{a_1}$ ways.

(b) For $i \geq 2$, at the i^{th} step choose ia_i blocks from the remaining $\ell - \sum_1^{i-1} ja_j$ blocks of α, and distribute the $2ia_i$ points of Ω which they contain into ia_i blocks of β in such a way that a_i components of size i in $\Delta(\beta)$ are obtained. The number of ways in which this can be done is

$$\binom{\ell - \sum^{i-1} ja_j}{ia_i} \cdot \left(\prod_{j=1}^{a_i} (ji-1)\ldots(ji-i+1)\right) . 2^{(i-1)a_i}$$

$$= (\ell - \sum^{i-1} ja_j)!2^{(i-1)a_i}/(\ell - \sum^{i} ja_j)!a_i!i^{a_i}$$

Note that this formula is valid even if $a_i = 0$. The result follows.

LEMMA 3.3.2. *For* $\ell \geq 3$, *the smallest value of* $\sigma_\ell(\rho)$ *with* $\rho \neq (1^\ell)$ *is* $\ell(\ell-1)$. *Further,* $\sigma_\ell(\rho) = \ell(\ell-1)$ *if and only if* $\rho = (1^{\ell-2},2^1)$, *unless* $\ell = 4$, *when* $\sigma_4(1^2,2^1) = \sigma_4(2^2) = 12$.

Proof. The result is true for $\ell \leq 4$ by 3.3.1. Now assume that $\ell \geq 5$. We proceed by induction on ℓ. If $\rho = (\ell^1)$ then $\sigma_\ell(\rho) = (\ell-1)!2^{\ell-1} > \ell(\ell-1)$. So suppose that $\rho = (1^{a_1},\ldots,\ell^{a_\ell}) \neq (\ell^1)$.

Let $i < \ell$ be the smallest integer for which $a_i \neq 0$. Then $\ell - i \geq 3$.
Define $\rho^* = (1^{a_1^*}, \ldots, (\ell-i)^{a^*_{\ell-i}})$ to be the partition of $\ell - i$ with

$$a_j^* = \begin{cases} a_j & , \text{ if } i \neq j \\ a_i - 1 , & \text{if } i = j . \end{cases}$$

Then, using (3.3.1) and induction, we have

$$(1) \quad \sigma_\ell(\rho) = \sigma_{\ell-i}(\rho^*).2^{i-1}.\ell!/ia_i.(\ell-i)! \geq \sigma_{\ell-i}(1^{\ell-i-2},2^1).2^{i-1}.\ell!/ia_i.(\ell-i)!$$

Since $\sigma_{\ell-i}(1^{\ell-i-2}, 2^1) = (\ell-i)(\ell-i-1)$, we have

$$(2) \quad \sigma_\ell(\rho) \geq 2^{i-1}\ell(\ell-1) \ldots (\ell-i-1)/ia_i .$$

If $i \geq 2$ then the right hand side is greater than $\ell(\ell-1)$, since $ia_i \leq \ell$. And if $i = 1$, it is $\ell(\ell-1)(\ell-2)/a_1 \geq \ell(\ell-1)$, with equality if and only if $\rho = (1^{\ell-2},2^1)$.

COROLLARY 3.3.3. (a) *If* $\ell \geq 5$ *then the unique shortest orbit of* H *on* $V\Gamma\backslash\{\alpha\}$ *is* $\Sigma(\rho,\alpha)$ *with* $\rho = (1^{\ell-2},2^1)$; *it has size* $\ell(\ell-1)$.

(b) *If* $\ell = 4$ *and* $G = S_8$ *then the shortest two orbits of* H *on* $V\Gamma\backslash\{\alpha\}$ *are* $\Sigma(\rho,\alpha)$ *with* $\rho = (1^2,2^1)$ *or* (2^2) , *each of size* 12 .

Proof. This follows from 3.3.2 if $G = S_{2\ell}$, so assume that $G = A_{2\ell}$. For $\ell = 4$ it is easy to check that (b) holds, so we take $\ell \geq 5$. Suppose that (a) is false, so that there is a partition $\rho = (1^{a_1}, \ldots, \ell^{a_\ell})$ of ℓ different from (1^ℓ) and $(1^{\ell-2},2^1)$, and an H-orbit $\Delta \subseteq \Sigma(\rho,\alpha)$ such that $|\Delta| \leq \ell(\ell-1)$. If $a_1 \neq 0$ then $\Delta = \Sigma(\rho,\alpha)$, so $|\Delta| > \ell(\ell-1)$ by 3.3.2. Hence $a_1 = 0$ and so $i \geq 2$ in the notation of the proof of 3.3.2. But now (2) of that proof shows that $|\Delta| \geq \sigma_\ell(\rho)/2 > \ell(\ell-1)$, a contradiction.

LEMMA 3.3.4. (a) *If* $\ell \geq 7$ *then* $\Sigma((1^{\ell-3},3^1),\alpha)$ *is the unique second shortest orbit of* H *on* $V\Gamma\backslash\{\alpha\}$, *and has size* $4\ell(\ell-1)(\ell-2)/3$.

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(b) *If* $\ell = 6$ *and* $G = S_{12}$ *then* $\Sigma((2^3),\alpha)$ *is the unique second shortest orbit of* H *on* $V\Gamma\backslash\{\alpha\}$ *, of size* 120 *; if* $\ell = 6$ *and* $G = A_{12}$ *the shortest three orbits of* H *have sizes* 30,60,60 *and their union is* $\Sigma((1^4,2^1),\alpha) \cup \Sigma((2^3),\alpha)$.

(c) *If* $\ell = 5$ *then* $\Sigma((1,2^2),\alpha)$ *is the unique second shortest orbit of* H *on* $V\Gamma\backslash\{\alpha\}$ *, of size* 60 .

Proof. We first check the result for $\ell \le 7$, using 3.3.1. The only point here which is not immediate is that for $\ell = 6$ and $G = A_{12}$, the set $\Sigma((2^3),\alpha)$ splits into two H-orbits of length 60 , since no odd permutation in S_{12} fixes both α and β with $\beta \in \Sigma((2^3),\alpha)$. Thus we take $\ell \ge 8$. Suppose that (a) is false, and choose ℓ minimal such that there is a partition $\rho = (1^{a_1},\ldots,\ell^{a_\ell})$ of ℓ with $\rho \ne (1^\ell),(1^{\ell-2},2^1),(1^{\ell-3},3^1)$ and an H-orbit $\Delta \subseteq \Sigma(\rho,\alpha)$ such that $|\Delta| \le 4\ell(\ell-1)(\ell-2)/3$. As in the proof of 3.3.2 we have $\rho \ne (\ell^1)$, and define i to be minimal such that $a_i \ne 0$. Now define ρ^* as in 3.3.2.

First suppose that $i = 1$, so that $\Delta = \Sigma(\rho,\alpha)$ and $a_1 < \ell - 3$. Then ρ^* is not $(1^{\ell-1})$ or $(1^{\ell-3},2^1)$, so by the minimality of ℓ we have $\sigma_{\ell-1}(\rho^*) \ge 4(\ell-1)(\ell-2)(\ell-3)/3$ (note that $\ell - 1 \ge 7$) . Hence, using (1) in 3.3.2,

$$|\Delta| = \sigma_\ell(\rho) \ge \sigma_{\ell-1}(\rho^*).\ell!/a_1.(\ell-1)!$$

$$\ge 4\ell(\ell-1)(\ell-2)(\ell-3)/3a_1$$

and so $|\Delta| > 4\ell(\ell-1)(\ell-2)/3$ since $a_1 < \ell - 3$; this is a contradiction.

Next assume that $i \ge 3$. Then by (2) of 3.3.2,

$$|\Delta| \ge \sigma_\ell(\rho)/2 \ge 2^{i-2}\ell(\ell-1)\ \ldots\ (\ell-i-1)/ia_i$$

$$\ge 2(\ell-1)(\ell-2)(\ell-3)(\ell-4) > 4\ell(\ell-1)(\ell-2)/3 ,$$

again a contradiction.

Thus $i = 2$, and again by (2),

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$$|\Delta| \geq \sigma_\ell(\rho)/2 \geq \ell(\ell-1)(\ell-2)(\ell-3)/2a_2 \ .$$

Consequently $a_2 \geq 3(\ell-3)/8$. In particular, $a_2 \geq 2$. Now define $\rho^{**} = (2^{a_2-2}, 3^{a_3}, 4^{a_4}, \ldots)$, a partition of $\ell - 4$. By (1) of 3.3.2 applied to ρ^* , we have

$$\sigma_{\ell-2}(\rho^*) = \sigma_{\ell-4}(\rho^{**}).2.(\ell-2)!/2(a_2-1).(\ell-4)!$$

Hence, noting that $\ell - 4 \geq 4$ and using 3.3.2, we have

$$\sigma_{\ell-2}(\rho^*) \geq 2(\ell-2)(\ell-3)(\ell-4)(\ell-5)/2(a_2-1)$$

and so by (1) again, using the fact that $2a_2 \leq \ell$,

$$|\Delta| \geq \sigma_\ell(\rho)/2 \geq \sigma_{\ell-2}(\rho^*).\ell!/2a_2.(\ell-2)!$$

$$\geq (\ell-2)(\ell-3)(\ell-4)(\ell-5).2\ell(\ell-1)/2a_2(2a_2-2)$$

$$\geq 2(\ell-1)(\ell-3)(\ell-4)(\ell-5) > 4\ell(\ell-1)(\ell-2)/3 \ ,$$

a contradiction. This completes the proof.

Now we consider the distance transitive graph Γ . First suppose that $\Gamma_1(\alpha) = \Sigma((1^{\ell-2}, 2^1), \alpha)$. Then it is easily seen that $\Gamma_2(\alpha)$ contains both $\Sigma((1^{\ell-3}, 3^1), \alpha)$ and $\Sigma((1^{\ell-4}, 2^2), \alpha)$, contrary to distance transitivity. Similarly, if $\Gamma_1(\alpha) = \Sigma((1^{\ell-3}, 3^1), \alpha)$ then $\Gamma_2(\alpha)$ contains $\Sigma((1^{\ell-2}, 2^1), \alpha)$ and $\Sigma((1^{\ell-4}, 4^1), \alpha)$, a contradiction. Hence, since by (1.9), $\Gamma_1(\alpha)$ is one of the shortest two H-orbits on $V\Gamma \backslash \{\alpha\}$, it follows from 3.3.3 and 3.3.4 that ℓ is 4,5 or 6.

Let $\ell = 4$. Since $(S_2 \ wr \ S_4) \cap A_8 < AGL_3(2) < A_8$, we have $G = S_8$ here, so by 3.3.3(b), $\Gamma_1(\alpha) = \Sigma((2^2), \alpha)$. Then $\Gamma_2(\alpha)$ contains $\Sigma((1^2, 2^1), \alpha)$ and $\Sigma((4^1), \alpha)$, which is false.

Next let $\ell = 5$. By 3.3.3 and 3.3.4, $\Gamma_1(\alpha) = \Sigma((1^1, 2^2), \alpha)$ and we see that $\Gamma_2(\alpha)$ contains $\Sigma((1^3, 2^1), \alpha)$ and $\Sigma((1^1, 4^1), \alpha)$, which is not so.

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Finally, let $\ell = 6$. If $G = S_{12}$ then $\Gamma_1(\alpha) = \Sigma((2^3),\alpha)$ and $\Gamma_2(\alpha)$ contains $\Sigma((1^4,2^1),\alpha)$ and $\Sigma((1^2,4^1),\alpha)$, a contradiction; and if $G = A_{12}$ then by 3.3.4, $|\Gamma_1(\alpha)| = 60$. This is impossible by (1.9), as H has further orbits of sizes 30 and 60.

(3.4) Case (iii) of Lemma 3.1. Here $n = k\ell$ and (k,ℓ) is one of (3,3), (3,4), (4,3) and (5,3) . It is convenient to describe the orbits of H on $V\Gamma$ as follows. We identify $V\Gamma$ with the set of partitions of Ω into ℓ blocks of size k . Let $\alpha = \{A_1,\ldots,A_\ell\} \in V\Gamma$ (with all $|A_i| = k$) . For $\beta = \{B_1,\ldots,B_\ell\} \in V\Gamma$ define M_β to be the $\ell \times \ell$ matrix with (i,j)-entry $|A_i \cap B_j|$. If

$$M = \{M | M \text{ an } \ell \times \ell \text{ matrix over } \mathbb{N} \cup \{0\} \text{ with all row- and column-sums equal to } k\}$$

then $M_\beta \in M$. Define an equivalence relation $\sim$ on M by

$$M_1 \sim M_2 \Leftrightarrow M_2 = PM_1Q \text{ for some } \ell \times \ell \text{ permutation matrices } P,Q .$$

Clearly all the possible choices for M_β (for a given β) are equivalent, and moreover, β_1 and β_2 lie in the same $(S_k\ wr\ S_\ell)$-orbit if and only if $M_{\beta_1} \sim M_{\beta_2}$. For $M \in M$, let $\bar{M}$ be the equivalence class containing M .

Thus the $(S_k\ wr\ S_\ell)$-orbits on $V\Gamma$ are the sets

$$\Sigma(\bar{M}) = \{\beta | M_\beta \in \bar{M}\} .$$

We observe that if M_β has an entry which is at least 2 then there is an odd permutation in $S_{k\ell}$ fixing α and β , and hence $\Sigma(\bar{M}_\beta)$ is also an orbit of $(S_k\ wr\ S_\ell) \cap A_{k\ell}$.

(3.4.1) Case $(k,\ell) = (3,3)$. Here the rank of G on $(G:H)$ is 5 , and the suborbits are $\Sigma(\bar{M}_i)$, $0 \le i \le 4$, with M_i and $|\Sigma(\bar{M}_i)|$ as follows:

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i	M_i	$\lvert\Sigma(\bar{M}_i)\rvert$
0	$\begin{pmatrix} 3 & & \\ & 3 & \\ & & 3 \end{pmatrix}$	1
1	$\begin{pmatrix} 3 & & \\ & 2 & 1 \\ & 1 & 2 \end{pmatrix}$	27
2	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	36
3	$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}$	54
4	$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$	162

Note that $\Sigma(\bar{M}_2)$ is indeed an orbit of $(S_3 \text{ wr } S_3) \cap A_9$: for let

$$\alpha = \{\{123\},\{456\},\{789\}\} \ , \ \beta_2 = \{\{147\},\{258\},\{369\}\} \ .$$

Then $\beta_2 \in \Sigma(\bar{M}_2)$ and the odd permutation $(14)(25)(36)$ fixes both α and β_2 .

Now by (1.9), $\Gamma_1(\alpha)$ is either $\Sigma(\bar{M}_1)$ or $\Sigma(\bar{M}_2)$. First suppose $\Gamma_1(\alpha) = \Sigma(\bar{M}_1)$. Define

$$\beta_1 = \{\{123\},\{457\},\{689\}\} \ , \ \gamma_1 = \{\{457\},\{126\},\{389\}\} \ ,$$
$$\gamma_2 = \{\{457\},\{128\},\{369\}\} \ .$$

Then $\beta_1 \in \Gamma_1(\alpha)$, $\gamma_1 \in \Gamma_1(\beta_1) \cap \Sigma(\bar{M}_3)$ and $\gamma_2 \in \Gamma_1(\beta_1) \cap \Sigma(\bar{M}_4)$, so that $\Gamma_2(\alpha)$ contains $\Sigma(\bar{M}_3)$ and $\Sigma(\bar{M}_4)$, a contradiction. Similarly, if $\Gamma_1(\alpha) = \Sigma(\bar{M}_2)$ we see that $\Gamma_2(\alpha)$ contains $\Sigma(\bar{M}_1)$ and $\Sigma(\bar{M}_4)$, which is again false.

(3.4.2) Case $(k,\ell) = (3,4)$. In this case the rank of G on $(G:H)$ is 12, and the shortest two orbits of H on $V\Gamma\backslash\{\alpha\}$ are $\Sigma(\bar{M}_1)$, $\Sigma(\bar{M}_2)$, of sizes 54, 144 respectively, where

$$M_1 = \begin{pmatrix} 3 & & & \\ & 3 & & \\ & & 2 & 1 \\ & & 1 & 2 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 3 & & & \\ & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & 1 & 1 & 1 \end{pmatrix}.$$

Hence by (1.9), $\Gamma_1(\alpha)$ is $\Sigma(\bar{M}_1)$ or $\Sigma(\bar{M}_2)$. As in (3.4.1), we see that in each case $\Gamma_2(\alpha)$ contains more than one H-orbit, contrary to distance transitivity.

(3.4.3) Case $(k,\ell) = (4,3)$. Here the rank of G on $(G:H)$ is 9 and the shortest two H-orbits on $V\Gamma\backslash\{\alpha\}$ are $\Sigma(\bar{M}_1)$, $\Sigma(\bar{M}_2)$ of sizes 48, 54, where

$$M_1 = \begin{pmatrix} 4 & & \\ & 3 & 1 \\ & 1 & 3 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 4 & & \\ & 2 & 2 \\ & 2 & 2 \end{pmatrix}.$$

We obtain a contradiction as above.

(3.4.4) Case $(k,\ell) = (5,3)$. The rank of G here is 13, and the shortest two H-orbits on $V\Gamma\backslash\{\alpha\}$ are $\Sigma(\bar{M}_1)$, $\Sigma(\bar{M}_2)$ of sizes 75, 250, where

$$M_1 = \begin{pmatrix} 5 & & \\ & 4 & 1 \\ & 1 & 4 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 1 & 0 & 4 \end{pmatrix}.$$

This gives the usual contradiction.

To complete the proof of the theorem in the imprimitive case, it remains to show that $\mathrm{Aut}\,\Gamma \cong S_n$ for Γ as in (1.4) or (1.5). Let Γ be such a graph. Clearly $\mathrm{Aut}\,\Gamma$ contains a subgroup $G \cong S_n$; moreover, it follows from [6] that for $n \geq 8$, G is a maximal subgroup of either $\mathrm{Alt}(V\Gamma)$ or $\mathrm{Sym}(V\Gamma)$ (since $G^{V\Gamma}$ contains the subgroup S_{n-1} acting on $(k-1)$-sets), and hence $G = \mathrm{Aut}\,\Gamma$, as required.

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4. The primitive case

In this section we complete the proof of the theorem by dealing with the case where H^{Ω} is primitive and $H \ngeq A_n$. It is stated in [17, Remark, p341] that for n large enough, H^{Ω} must be 2-transitive. We give a proof here for completeness:

LEMMA 4.1. *The group H is 2-transitive on Ω provided that $n \geq 6$.*

Proof. If $n \leq 11$ the assertion is easily checked using [18]. Thus we take $n \geq 12$. Write n_i for the number of orbits of H on $\Omega^{\{i\}}$. If $n_2 = 1$ then H^{Ω} is 2-transitive: for if not then $|H|$ is odd and hence, by the maximality of H in G, we have $H = AGL_1(p) \cap G$ with $n = p$, a prime. Then the bound (1) below is violated. Now suppose that H is not 2-transitive. Then by (*) in the proof of Lemma 3.1, we have $n_2 = 2$, $n_3 \leq 3$ and $n_4 \leq 4$. We shall obtain a contradiction.

We colour the complete graph on Ω with two colours r, b so that two edges have the same colour if and only if they lie in the same H-orbit on $\Omega^{\{2\}}$. The monochrome subgraphs are both regular and connected and have valency at least 3 (see [14, Lemma 3 and Theorem 5]). By Ramsey's Theorem, there is a monochrome triangle, say of colour r. Since the r-monochrome subgraph is connected, there must also be triangles of type rrb. And since the valency of the b-graph is at least 3, there must also be triangles with at least two sides coloured b. It follows that $n_3 = 3$ and the three types of triangles are rrr, rrb and rbb.

For $\omega \in \Omega$ and $c \in \{r,b\}$, set

$$\Sigma_c(\omega) = \{\beta \mid \{\omega,\beta\} \text{ has colour } c\}, \text{ and } v_c = |\Sigma_c(\omega)| .$$

Then $\Omega = \{\omega\} \cup \Sigma_r(\omega) \cup \Sigma_b(\omega)$. Since there are no bbb-triangles, counting the b-edges between $\Sigma_b(\omega)$ and $\Sigma_r(\omega)$ gives $v_b(v_b-1) \leq v_r v_b$. Hence $v_r \geq v_b - 1$, whence in fact $v_r \geq v_b$ (see [14, Theorem 3]). Also $v_b(v_b-1) \geq v_r$.

Suppose first that $v_r = v_b$. Now H_ω is 2-homogeneous on $\Sigma_b(\omega)$ since H is transitive on bbr-triangles. Also H_ω cannot be 2-transitive

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on $\Sigma_b(\omega)$ by [14, Theorem 4]. Hence by [14, Theorem 1], $|H_\omega|$ is odd and so $|H|$ is odd since $n = 2v_r + 1$. As in the first paragraph of this proof, this is impossible by the bound (1) below.

Hence $v_r > v_b$. We next consider tetrahedra, that is, subgraphs of the complete graph on Ω isomorphic to K_4. Firstly, since $v_b \geq 3$ and there are no bbb-triangles, there is an rrr-triangle in $\Sigma_b(\omega)$, so there is a tetrahedron with three b-edges and an rrr-triangle. Secondly, since $v_r \geq 6$, by Ramsey's Theorem there are rrr-triangles in $\Sigma_r(\omega)$, so there is a monochrome r-tetrahedron. Thirdly, there is a tetrahedron with three r- and three b-edges but no rrr-triangle: for let $\beta \in \Sigma_b(\omega)$, $\gamma \in \Sigma_b(\beta)$ with $\gamma \neq \omega$ (so that $\gamma \in \Sigma_r(\omega)$); then $\Sigma_b(\omega) \nsubseteq \Sigma_b(\gamma)$, so we can choose $\delta \in \Sigma_b(\omega) \cap \Sigma_r(\gamma)$ - then $\{\omega,\beta,\gamma,\delta\}$ is the required tetrahedron. Fourthly, there is a tetrahedron with only one or two non-incident b-edges: for if $\beta \in \Sigma_b(\omega)$, there are two points $\gamma,\delta \in \Sigma_r(\omega) \cap \Sigma_r(\beta)$ (since as $v_r > v_b$, there are at least two r-edges from β into $\Sigma_r(\omega)$). Finally, there is a tetrahedron with a total of two b-edges, these being incident: for if $\beta \in \Sigma_r(\omega)$ then as $v_r > v_b$, there are points $\gamma,\delta \in \Sigma_r(\beta) \cap \Sigma_b(\omega)$.

All these five tetrahedra are of distinct types and so must lie in distinct H-orbits, contradicting the fact that $n_4 \leq 4$. This completes the proof.

Now by (1.10) we have

(1) $$|G : H| \leq t_n(G),$$

where $t_n(G) = \sum \chi(1)$, the sum being over all irreducible characters χ of G. By a theorem of Schur, we have

(2) $$t_n(S_n) = |\{g \in S_n \mid g^2 = 1\}|$$

(see [17, p.341]). By [9, p.66], if λ is a partition of n and λ' the conjugate partition, with $\lambda \neq \lambda'$, then $\chi^\lambda_{A_n} = \chi^{\lambda'}_{A_n}$ is an irreducible character of A_n; and if $\lambda = \lambda'$ then $\chi^\lambda_{A_n}$ is a sum of two distinct irreducible characters $\chi^\lambda_1\ \chi^\lambda_2$ of A_n of equal degrees. Hence

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(3) $$t_n(A_n) = (t_n(S_n) + \sum_{\lambda=\lambda'} \chi^\lambda(1))/2 .$$

Note that of course $t_n(A_n) < t_n(S_n)$. For a given value of n , the degrees $\chi^\lambda(1)$ for $\lambda = \lambda'$ can be calculated using [9, 2.3.15].

Now $|H| < 4^n$ by [15] and hence, as in [17, p.342], the bound (1) forces $n \le 60$. Also H^Ω is 2-transitive by 4.1, so it is either contained in an affine group or it is one of the groups listed in [2, 5.3]. Using (1), (2) and (3), we see that in fact n is at most 12 .

(4.2) Case $n = 12$. The 2-transitive groups which are maximal in A_{12} or S_{12} are $L_2(11).2$ and M_{12} (see [2], or [18]). If $H = L_2(11).2$ then $G = S_{12}$ and (1) gives a contradiction. Hence $H = M_{12}$ and $G = A_{12}$. We sketch a description of the action of G on $\Sigma = (G:H)$, and refer the reader to [5] for further details. The group G has rank 4 on Σ (see [4,p.91]) and, taking $\alpha = H \in \Sigma$, the orbits of H on $\Sigma \backslash \{\alpha\}$ are

$$\Sigma_1 = \{Hx | x \text{ a 3-cycle in } A_{12}\} , \quad \text{of size } 440 ,$$
$$\Sigma_2 = \{Hx | x \text{ a } 2^2\text{-element in } A_{12}\} , \text{ of size } 495 ,$$
$$\Sigma_3 = \{Hx | x \text{ a 5-cycle in } A_{12}\} , \quad \text{of size } 1584 ;$$

for clearly H fixes these sets, and hence they are H-orbits since the rank of G is 4 . We may pick 3-cycles a,b,c and 2^2-elements d,e,f in A_{12} such that ab is a 2^2-element, ac and df are 5-cycles and de is a 3-cycle.

Now by (1.9), $\Gamma_1(\alpha)$ is Σ_1 or Σ_2 . If $\Gamma_1(\alpha) = \Sigma_1$ then $\beta = Ha \in \Gamma_1(\alpha)$ and $Hab \in \Gamma_1(\beta) \cap \Sigma_2$, $Hac \in \Gamma_1(\beta) \cap \Sigma_3$, contrary to the distance transitivity of Γ . And if $\Gamma_1(\alpha) = \Sigma_2$ then $\gamma = Hd \in \Gamma_1(\alpha)$ and $Hde \in \Gamma_1(\gamma) \cap \Sigma_1$, $Hdf \in \Gamma_1(\gamma) \cap \Sigma_3$, again a contradiction.

(4.3) Case $n = 11$. Here, by (1) and [18], we have $H = M_{11}$ and $G = A_{11}$. By [4, p.75] the rank of G on $\Delta = (G:H)$ is 5 . Taking M_{12}, A_{12} to act naturally on the set $\{1,\ldots,12\}$, we regard H,G as the stabilizers in M_{12}, A_{12} of the point 1. There is a G-isomorphism $Hx \to M_{12}x$ $(x \in A_{11})$ between Δ and $\Sigma = (A_{12} : M_{12})$, and we identify Δ and Σ via this isomorphism.

Since H fixes the set $\{Hx \mid x$ a 3-cycle in $A_{11}\}$ of size 330, we see that the set Σ_1 described above in (4.2) must break up into two H-orbits, of sizes 110 and 330. Thus if $\alpha = H \in \Delta$, the orbits of H on $\Delta \backslash \{\alpha\}$ are

$\Delta_1 = \{M_{12}y \mid y$ a 3-cycle in A_{12} involving $1\}$, of size 110,

$\Delta_2 = \{Hx \mid x$ a 3-cycle in $A_{11}\}$, of size 330,

$\Delta_3 = \{Hx \mid x$ a 2^2-element in $A_{11}\}$, of size 495,

$\Delta_4 = \{Hx \mid x$ a 5-cycle in $A_{11}\}$, of size 1584.

By (1.9), $\Gamma_1(\alpha)$ is Δ_1 or Δ_2. If $\Gamma_1(\alpha) = \Delta_2$ we obtain a contradiction as in (4.2). Thus let $\Gamma_1(\alpha) = \Delta_1$. Pick $\beta = M_{12}(1,2,3)$. Then β corresponds to the coset $M_{11}g(1,2,3)$, where $g \in M_{12}$ and $g(1,2,3)$ fixes 1 (so that $1^g = 3$). Since M_{12} is 5-transitive on $\{1,\dots,12\}$, we may choose $h,k \in M_{12}$ such that $1^h = 4$, $2^h = 5$, $3^h = 3$ and $1^k = 4$, $2^k = 5$, $3^k = 2$. Let $\beta_1 = M_{11}h(1,2,4)$, $\beta_2 = M_{11}k(1,2,4)$. Then $\beta_1, \beta_2 \in \Gamma_1(\alpha)$. Applying the elements $h(1,2,4)$ and $k(1,2,4)$ of A_{11} to the edge between α and β, we see that β_1 is joined to $\beta h(1,2,4)$ and β_2 is joined to $\beta k(1,2,4)$. Now

$$\beta h(1,2,4) = M_{11}g(1,2,3)h(1,2,4) = M_{11}gh(4,5,3)(1,2,4),$$
$$= M_{11}gh(1,2,4,5,3),$$
$$\beta k(1,2,4) = M_{11}gk(4,5,2)(1,2,4) = M_{11}gk(1,2)(4,5).$$

Since $gh, gk \in M_{12}$, we see that $\beta h(1,2,4)$, $\beta k(1,2,4)$ correspond to the cosets $M_{12}(1,2,4,5,3)$, $M_{12}(1,2)(4,5)$, and hence $\beta h(1,2,4) \in \Delta_4$, $\beta k(1,2,4) \in \Delta_3$. This means that $\Gamma_2(\alpha)$ contains Δ_3 and Δ_4, a contradiction.

(4.4) Case $n = 10$. Here, by [18], we have either $H = M_{10}$, $G = A_{10}$ or $H = P\Gamma L_2(9)$, $G = S_{10}$.

(4.4.1) Suppose first that $H = M_{10}$, $G = A_{10}$. We claim that 1_H^G is not multiplicity-free, a contradiction. Write $X = A_{12}$, $M = M_{12}$, acting on $\{1,\ldots,12\}$. Since the rank of X on $(X:M)$ is 4 and the rank of X_1 on M_1 is 5, by the Schur branching law ([9, 2.4.3]), there is an irreducible constituent χ^λ of 1_M^X with $\lambda = (1^{a_1},\ldots,12^{a_{12}})$ such that at least two of the a_i, say a_r and a_s, are non-zero. Using the Schur branching rule again twice, we see that the character χ^μ of G appears in 1_H^G with multiplicity at least 2, where μ is the partition of 10 obtained from λ by decreasing one part of size r and one part of size s both by 1. Hence 1_H^G is not multiplicity-free, as claimed.

(4.4.2) Now let $H = P\Gamma L_2(9)$, $G = S_{10}$. With X,M as above, we may take $G = X_{\{1,2\}}$ and $H = M \cap G$. There is a G-isomorphism between $(G:H)$ and $(X:M)$, and we identify these sets via this isomorphism. Calculation shows that the orbit Σ_1 of X as in (4.2) splits into the three H-orbits

$\Sigma_{11} = \{Mx \mid x$ a 3-cycle involving 1 and 2$\}$, of size 20,

$\Sigma_{12} = \{Mx \mid x$ a 3-cycle involving one point of $\{1,2\}\}$, of size 180,

$\Sigma_{13} = \{Mx \mid x$ a 3-cycle in $G\}$, of size 240.

The orbit Σ_2 splits into three H-orbits

$\Sigma_{21} = \{Mx \mid x = (1,2)(a,b)\}$, of size 45,

$\Sigma_{22} = \{Mx \mid x = (1,a)(2,b)\}$, of size 90,

$\Sigma_{23} = \{Mx \mid x = (i,a)(b,c)$ with $i \in \{1,2\}\}$, of size 360,

where a,b,c range over triples of distinct elements of $\{3,\ldots,12\}$. The orbit Σ_3 splits into three H-orbits of sizes 144, 720, 720.

Thus by (1.9), $\Gamma_1(\alpha)$ is Σ_{11} or Σ_{21}. If $\Gamma_1(\alpha) = \Sigma_{11}$ then $\Gamma_2(\alpha)$ contains $M(1,2,3)(1,2,4)$ and $M(1,2,3)(1,4,2)$, and hence contains

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Σ_{22} and Σ_{12}, which is a contradiction. We obtain a similar contradiction if $\Gamma_1(\alpha) = \Sigma_{21}$.

(4.5) Case $n = 9$. By [18], (H,G) is $(P\Gamma L_2(8), A_9)$, $(ASL_2(3), A_9)$ or $(AGL_2(3), S_9)$. In the first case G has rank 3 on $(G:H)$, giving the graphs Σ_{120}, $\bar{\Sigma}_{120}$ under (1.6). In fact A_9 here is contained in the larger rank 3 group $O_8^+(2)$ of degree 120 (see [4, p.85]); since by [11], the only group lying between $O_8^+(2)$ and A_{120} is the 2-transitive group $Sp_8(2)$, we have $\mathrm{Aut}(\Sigma_{120}) \cong O_8^+(2)$, as claimed in the theorem.

(4.5.1) Now let $H = ASL_2(3)$, $G = A_9$. We claim that 1_H^G is not multiplicity-free here. By [13, Appendix] the permutation character of S_9 on $(S_9 : AGL_2(3))$ contains both $\chi^{(4^2,1)}$ and $\chi^{(3,2^3)}$. Since these restrict to the same character of A_9, our claim follows.

(4.5.2) To complete (4.5), let $H = AGL_2(3)$, $G = S_9$. Let P be the affine plane corresponding to H, and for distinct $a,b \in \Omega$, let $\ell(a,b)$ be the line in P containing a and b. We describe the orbits of H on $(G:H)\backslash\{H\}$. First, we have the H-orbit

$$\Phi_1 = \{Hx \mid x \text{ a 2-cycle in } G\}, \text{ of size } 36 .$$

The set $\{Hx \mid x$ a 3-cycle in $G\}$ splits into the two H-orbits

$$\Phi_2 = \{H(abc) \mid c \in \ell(a,b)\} \text{ of size } 8 ,$$

$$\Phi_3 = \{H(abc) \mid c \notin \ell(a,b)\} \text{ of size } 144 .$$

The set $\{Hx \mid x$ a 2^2-element of $G\}$ splits into three H-orbits: writing $x = (ab)(cd)$, we have $\Phi_1 = \{Hx \mid \ell(a,b) \cap \ell(c,d) = \emptyset\}$. For the the other two orbits Φ_4, Φ_5, write $\{e\} = \ell(a,b) \cap \ell(c,d)$. Then

$$\Phi_4 = \{Hx \mid e \notin \{a,b,c,d\}\} \text{ of size } 27 ,$$

$$\Phi_5 = \{Hx \mid e \in \{a,b,c,d\}\} \text{ of size } 216 .$$

In this fashion it can be seen that there are precisely three further H-

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orbits, of sizes 48, 144 and 216 .

We now argue with the parameters k_i, b_i, c_i of the distance transitive graph Γ , as defined in [19]. Here $k_i = |\Gamma_i(\alpha)|$, and if $d(\alpha,\beta) = i$, then the number of vertices adjacent to β and at distance $i - 1$ or $i + 1$ from α is c_i or b_i , respectively. From [19], we have

(4) $$k_1 > b_1 \geq \ldots \geq b_{d-1} \text{ and } 1 = c_1 \leq c_2 \leq \ldots \leq c_d .$$

Some consequences of this for the k_i are given in [16, 1.1].

Now by (1.9), k_1 is 8 or 27 . If $k_1 = 8$ then k_2 must be 48 , so by [16, 1.1] we must have $k_9 = 27$, $k_8 = 36$ and $k_7 \in \{144,216\}$. This forces $b_7 = 1$ and $b_8 > 1$, contradicting (4). Hence $k_1 = 27$. By [16, 1.1], k_2 is 36 or 48 . First let $k_2 = 48$. Then $(b_1,c_2) = (16,9)$. Now k_3 is 144 or 216 . If $k_3 = 144$ then (b_2,c_3) must be (6,18) or (3,9) (using (4)); and k_4 is 144 or 216, so (b_3,c_4) is (a,a) or $(3a,2a)$ for some integer a . Neither of these is possible by (4). Hence $k_3 = 216$. Then (b_2,c_3) is $(9a,2a)$ for some integer a , whence by (4) we have $b_1 = 16 \geq 9a$ and $c_2 = 9 \leq 2a$, an impossibility.

Thus $k_2 = 36$ and so $(b_1,c_2) = (4a,3a)$ for some integer a . Then k_3 is 48 or 144 . If $k_3 = 144$ then $(b_2,c_3) = (4b,b)$ for some b , and so by (4), we have $4a \geq 4b$, $3a \leq b$, a contradiction. Hence $k_3 = 48$ and so k_4 is 144 or 216 . In both cases (4) is again violated. Thus no distance transitive graph arises in (4.5.2).

(4.6) Case $n = 8$. If $G = A_8$ then H must be $AGL_3(2)$ by [18] (since $L_2(7)$ of degree 8 is contained in $AGL_3(2)$) . But then G is 2-transitive on $(G:H)$, so Γ is the complete graph $K_{15} = J(15,1)$. Hence we may take $G = S_8$, and by [18], $H = L_2(7).2$. We consider H as $L_3(2).2$ embedded in $G = L_4(2).2$, with $V = V_4(2)$ the natural module for

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G' . Then H is the stabilizer of a pair $\alpha = \{U_0, W_0\}$ of subspaces of V satisfying $V = U_0 \oplus W_0$, $\dim U_0 = 1$, $\dim W_0 = 3$. Regarding $V\Gamma$ as the set of all such pairs $\{U, W\}$ of subspaces, the orbits of H on $V\Gamma \backslash \{\alpha\}$ are

$$\Delta_1 = \{\{U,W\} \mid U = U_0 \text{ or } W = W_0\} \backslash \{\alpha\} \text{, of size } 14 \text{ ,}$$

$$\Delta_2 = \{\{U,W\} \mid U_0 \le W \text{ , } U \le W_0\} \text{ , of size } 28 \text{ ,}$$

$$\Delta_3 = \{\{U,W\} \mid U_0 \le W \text{ , } U \nleq W_0 \text{ or } U_0 \nleq W \text{ , } U \le W_0\} \text{ , of size } 56 \text{ ,}$$

$$\Delta_4 = \{\{U,W\} \mid U_0 \nleq W \text{ and } U \nleq W_0\} \text{ , of size } 21 \text{ .}$$

Thus by (1.9), $\Gamma_1(\alpha)$ is Δ_1 or Δ_4 . If $\Gamma_1(\alpha) = \Delta_1$ it is easily seen that $\Gamma_2(\alpha)$ contains $\Delta_3 \cup \Delta_4$; and if $\Gamma_1(\alpha) = \Delta_4$ then $\Gamma_2(\alpha)$ contains $\Delta_1 \cup \Delta_2$. This contradicts the distance transitivity of Γ .

(4.7) Case $n = 7$. If $G = A_7$ then $H = L_2(7)$ by [18], and G is 2-transitive on $(G:H)$, so $\Gamma = K_{15}$. Thus we take $G = S_7$, whence by [18], $H = F_{42}$, a Frobenius group of order 42 . By [13, Appendix], G has rank 7 on $(G:H)$. Elementary calculation shows that the orbit sizes of H on $(G:H)$ are 1,7,14,14,21,21,42 . Hence by [16, 1.1] we have $k_1 = 7$, $k_2 = 14$ and $k_3 = 21$. Since H is 2-transitive on $\Gamma_1(\alpha)$ we have $b_1 = 6$, and hence $c_2 = 3$, $b_2 = 3$ and $c_3 = 2$. But then $c_2 > c_3$, contradicting the inequalities (4).

(4.8) Case $n = 6$. By [18], H is $L_2(5)$ or $L_2(5).2$ and $|G:H| = 6$; moreover G is 2-transitive on $(G:H)$, so Γ is K_6 .

(4.9) Case $n = 5$. Here $|G:H| = 6$ and Γ is again K_6 .

This completes the proof of the theorem.

5. Final remarks on Aut A_6

To conclude, we complete the proof of the corollary to the theorem by dealing with the case where $G \le \text{Aut}\, A_6$ and $G \nleq S_6$ (in the notation of the statement of the corollary). Let $\alpha \in V\Gamma$ and $H = G_\alpha$. From [4] we see that $|G:H|$ is 10,36 or 45 . If $|G:H| = 10$ then G is 2-transitive on $V\Gamma$, so $\Gamma = K_{10}$. Thus we suppose that $|G:H|$ is 36 or 45 .

First let $|G:H| = 36$, so that $H \cap A_6 = D_{10}$. It is easy to see that the permutation character of A_6 on $(A_6:D_{10})$ is

$$1 + \chi^{(5,1)} + \chi^{(3^2)} + \chi^{(4,2)} + \chi_1^{(3,2,1)} + \chi_2^{(3,2,1)} ,$$

a sum of irreducible characters of A_6 of degrees 1,5,5,9,8,8 . Thus (see [*4*, p.5]) if $G = \text{Aut } A_6$ then $1_H^G = 1 + \chi_9 + \chi_{10} + \chi_{16}$, where χ_i is an irreducible character of G of degree i . Then G has rank 4 on $(G:H)$, and the subdegrees must be 1,5,10,20 . Hence by (1.9), k_1 is 5 or 10 . If $k_1 = 5$ then Γ is the distance transitive graph Σ_{36} as in (1.7) (see [*1*, p.153]); and if $k_1 = 10$ then Γ is the graph obtained by joining vertices at distance 3 in Σ_{36} , which is easily seen not to be distance transitive. And if $G < \text{Aut } A_6$ then the subdegrees of G on $(G:H)$ are either 1,5,10,20 or 1,5,10,10,10 ; in the first case Γ is Σ_{36} again, and in the second $k_1 = 5$ by [*16*, 1.1], whence $\Gamma \cong \Sigma_{36}$ and G is not distance transitive on Γ , a contradiction. Thus in all cases $\Gamma \cong \Sigma_{36}$. Finally we remark that $\text{Aut } \Sigma_{36} \cong \text{Aut } A_6$, as is well known (see [*1*, p.153]).

Now let $|G:H| = 45$. The permutation character of A_6 on the cosets of $H \cap A_6 = D_8$ is

$$1 + \chi^{(5,1)} + \chi^{(2^3)} + 2\chi^{(4,2)} + \chi_1^{(3,2,1)} + \chi_2^{(3,2,1)} ,$$

a sum of characters of degrees $1,5,5,2 \times 9,8,8$. If $G = \text{Aut } A_6$ then G has rank 5 on $(G:H)$, and hence has subdegrees 1,4,8,16,16 . Then by (1.9), k_1 is 4 or 8 . If $k_1 = 4$ then Γ is Σ_{45} as in (1.8), the line graph of the 8-cage (see [*20*, Chapter 8]); and if $k_1 = 8$ then Γ is the graph obtained from Σ_{45} by joining vertices at distance 2 ,

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which is easily seen not to be distance transitive. If $G < \mathrm{Aut}\, A_6$ then G has subdegrees 1,4,8,8,8,16 , so Γ is not distance transitive by [16, 1.1]. Finally, it is well known that $\mathrm{Aut}\, \Sigma_{45} \cong \mathrm{Aut}\, A_6$.

Final Remark. While this paper was in preparation we received a preprint from A. Ivanov [8] which contains some of our results. Much of our work was done several years ago, and our paper is independent of his.

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