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THE ASYMPTOTIC BEHAVIOUR OF EQUIDISTANT PERMUTATION ARRAYS

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1. Introduction. An *equidistant permutation array* (EPA) $A(r, \lambda; v)$ is a $v \times r$ array defined on a set V of r symbols such that every row is a permutation of V and any two distinct rows have precisely λ common column entries. Define $R(r, \lambda)$ to be the largest value of v for which there exists an $A(r, \lambda; v)$. Deza [**2**] has shown that

$$R(r, \lambda) \leqq \max \{n^2 + n + 1, \lambda + 2\}$$

where $n = r - \lambda$. Bolton [**1**] has shown that

$$(*) \quad R(r, \lambda) \geqq 2 + \left\lfloor \frac{\lambda}{\lceil \frac{n}{3} \rceil} \right\rfloor .$$

In this paper, we show that equality holds in (*) for $\lambda > \lceil n/3 \rceil (n^2 + n)$. In order to do this we require several more definitions.

An (r, λ)*-design* D is a collection B of subsets (called *blocks*) of a finite set V of elements (called *varieties*) such that any two distinct elements of V are contained in precisely λ common blocks and every variety is contained in exactly r blocks of D. An (r, λ)-design D is said to be *resolvable* or *contain a resolution* R if the blocks of D can be partitioned into classes (called *resolution classes*) such that every variety of D is contained in precisely one block of each resolution class. We say that an (r, λ)-design D is *orthogonally resolvable* (denoted OD (r, λ)) if D contains resolutions R and R' and $R_1, R_2, \ldots, R_r$ and $R_1', R_2', \ldots, R_r'$ are the resolution classes of R and R' respectively such that for all i and j $(1 \leqq i, j \leqq r)$ R_i and R_j' have at most one labelled block in common. (Note: We consider all blocks of D as labelled so that a given subset can occur repeatedly as distinct blocks.)

The following result of Deza, Mullin and Vanstone appeared in [**3**].

THEOREM 1.1. *There exists an* $A(r, \lambda; v)$ *if and only if there exists an* OD(r, λ)*-design having* v *varieties.*

Theorem 1.1 shows the connection between EPAs and (r, λ)-designs. Using results on (r, λ)-designs, we will deduce asymptotic results for $R(r, \lambda)$.

Let D be an (r, λ)-design defined on a set V of v symbols. A block of D is said to be *complete* if it contains all of the varieties. D is said to contain a *complete set of singletons* if D contains v blocks each of size one whose union is V. An (r, λ)-design which contains λ complete blocks is called *trivial*. Let $v_0(r, \lambda)$

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be the smallest positive integer such that if D is any (r, λ)-design on $v > v_0(r, \lambda)$ varieties then D must be trivial. It has been shown [**2**] that

$$v_0(r, \lambda) \leqq \max \{\lambda + 2, n^2 + n + 1\}.$$

For $\lambda > n^2 + n - 1$, Mullin [**5**] has proven that any (r, λ)-design with $v \geqq v_0(r, \lambda)$ varieties only has block sizes $1, v - 1$ and v. Such designs are called *near-trivial*. We will make use of this result in the following section.

2. Main result. Let D be an $\mathrm{OD}(r, \lambda)$-design having $v > n^2 + n + 1$ varieties for $n \geqq 3$ and such that v is a maximum. Since D is an (r, λ)-design, Mullin's result implies that D is near-trivial. Hence, D contains only blocks of size $1, v - 1$, and v. Call the blocks of size $v - 1$ in D the *body* of the design. Clearly, the body of the design can be partitioned into t copies of all $(v - 1)$-subsets of a v-set for some non-negative integer t.

Suppose D has no blocks of size $v - 1$. Then D must be trivial and must contain at least v complete sets of singletons if it is to be orthogonally resolvable. This is impossible since $v > n$. We then deduce that $t \geqq 1$.

Since D is an $\mathrm{OD}(r, \lambda)$-design, it must be resolvable and thus, for each component of the body there must be a complete set of singletons to form v resolution classes of D. (This follows since each block of cardinality $v - 1$ in the body requires a singleton block to form a resolution class). Hence, D must have at least t complete sets of singletons and we easily deduce that $t \leqq \lfloor n/2 \rfloor$ where $\lfloor x \rfloor$ is called the *floor function* of x and denotes the greatest integer less than or equal to x. Denote these t complete sets of singletons by $S_1, S_2, \ldots, S_t$. Suppose D contains s other complete sets of singletons denoted $T_1, T_2, \ldots, T_s$ and these are resolution classes in a resolution R of D.

Each component of the body of the design and each complete set of singletons contributes one to n. However a complete block contributes zero to n. Thus, $n = 2t + s$.

LEMMA 2.1. *For $S_1, S_2, \ldots, S_t$, T_1, $T_2, \ldots, T_s$ and D as defined above we have $s \leqq t$.*

Proof. Since D is an $\mathrm{OD}(r, \lambda)$-design, it contains a second resolution R' which is orthogonal to R. R' must contain $T_1', T_2', \ldots, T_s'$ complete sets of singletons and $S_1', S_2', \ldots, S_t'$ complete sets of singletons associated with the body of D.

Consider T_i', $1 \leqq i \leqq s$. T_i' can contain at most s blocks from $T_1, T_2, \ldots, T_s$ (at most one from each). Thus T_i' contains at least $v - s$ singletons from $S_1, S_2, \ldots, S_t$. For this to be possible, for all i, $1 \leqq i \leqq s$,

$$s(v - s) \leqq tv.$$

This implies

$$(1) \quad s \leqq \frac{v - \sqrt{v^2 - 4tv}}{2} \quad \text{or}$$

$$(2) \quad s \geqq \frac{v + \sqrt{v^2 - 4tv}}{2}$$

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Since $t \leqq \lfloor n/2 \rfloor$, it is easy to see that

$$\frac{v - \sqrt{v^2 - 4tv}}{2} < t + 1.$$

Moreover, it readily follows that

$$\left\lfloor \frac{v - \sqrt{v^2 - 4tv}}{2} \right\rfloor = t.$$

Since $n = 2t + s$, and $t \geqq 1$,

$$s < n.$$

Thus, if (2) is true

$$\frac{v + \sqrt{v^2 - 4vt}}{2} < n$$

which is impossible since $v > n^2 + n + 1$. This completes the proof of the lemma.

Now, if we let $S_1' = T_1$, $S_2' = T_2, \ldots, S_s' = T_s$, $S_{s+1}' = S_1, \ldots, S_t' = S_{t-s}$, $T_1' = S_{t-s+1}, \ldots, T_s' = S_t$ then it is easily seen that R and R' are orthogonal resolutions. Since $t \geqq s$ and $t + s \geqq 2$, the above is always possible.

By Lemma 2.1, we have

$$n = 2t + s \leqq 3t$$

which implies that $t \geqq n/3$. Since t must be an integer

$$t \geqq \lceil n/3 \rceil$$

where $\lceil x \rceil$ is called the *roof function* of x and means the least integer greater than or equal to x. It now follows that

(3) $\lceil n/3 \rceil \leqq t \leqq \lfloor n/2 \rfloor$.

If D contains c complete blocks then

(4) $$r = t(v-1) + n - t + c \quad \text{and} \\ \lambda = t(v-2) + c$$

where $c < t$. The restriction that $c < t$ follows from the fact v is maximum. If $c > t$ then it is possible to construct an $\mathrm{OD}(r, \lambda)$-design having more than v varieties.

Since $r = n + \lambda$, (4) becomes

(5) $v - 2 = (\lambda - c)/t = \lfloor \lambda/t \rfloor$.

Since v is a maximum, (5) implies that t must be a minimum and from (3) we get that $t = \lceil n/3 \rceil$. Therefore,

(6) $$v = 2 + \left\lfloor \frac{\lambda}{\left\lceil \frac{n}{3} \right\rceil} \right\rfloor.$$

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But (6) is true whenever $v > n^2 + n + 1$ which implies that

$$\lambda > \lceil n/3 \rceil (n^2 + n).$$

This proves the following theorem.

THEOREM 2.1. $R(r, \lambda) = 2 + \left\lfloor \dfrac{\lambda}{\lceil \frac{n}{3} \rceil} \right\rfloor$

whenever $\lambda > \lceil n/3 \rceil (n^2 + n)$.

3. Conclusion. Theorem 2.1 provides an asymptotic evaluation of $R(r, \lambda)$. Thus, for any value of n, there are only a finite number of values of $R(r, \lambda)$ to determine. This appears to be a difficult problem. For some of the known results in this area, the reader is referred to [**4**; **6**; and **7**].

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