RESTRICTED CHOICES

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A subset consisting of k elements chosen, from n distinct ordered elements, with given restrictions is called a restricted choice. For example, one restriction on the k elements may be that no two consecutive elements appear, while another may be that no two alternate elements appear. Certain restricted choices may be used to obtain solutions to permutation problems ([1, p. 349]; [4]). Each restricted choice corresponds to a "restricted sequence of Bernoulli trials" as described in [1]. In this paper an elementary method of obtaining more general types of restricted choices is given. Some special cases of the restricted choices and restricted Bernoulli trials are presented in the form of examples. Also, an elementary alternative approach to a result by Bizley [3] is given. Here the problem is to find the number of ways of arranging along a straight line balls of different colors with certain restrictions.

Let $X = (x_1, \ldots, x_n)$ be an ordered set of n distinct elements. Let G and S be any non-empty subsets of X.

DEFINITION 1. G is a gap of S if $G = (x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+v}, x_{i+v+1})$, where $x_r \notin S$ for $i < r < i+v+1$ and $x_r \in S$ for $r = i,\ i+v+1$. The length of the gap G is v. Necessarily G contains at least two elements, and $0 \le v \le n-2$.

Example. If S consists of two consecutive elements there is one and only one gap of S, namely S itself; its length is zero.

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DEFINITION 2. Let m, y be integers ≥ 0. A subset M of X is a minus-m, y sequence if,

(a) the length of any gap of M is less than or equal to y,

(b) M has exactly m gaps of length y.

For instance, a minus-$m, 0$ sequence is a subset consisting of $m+1$ consecutive elements. Note that if S consists of k elements then S cannot contain any minus-m, y sequence for $m \geq k$.

DEFINITION 3. Any subset R of X may be written as $R = (x_{i_1}, x_{i_2}, \ldots, x_{i_r})$ where $(i_1, i_2, \ldots, i_r)$ is some subset of $(1, 2, \ldots, n)$ with $i_u < i_v$ for $u < v$. R is said to hold (have, contain) a minus-m, y sequence if for some pair of positive integers $a, b, a+b \leq r$, $(x_{i_a}, x_{i_{a+1}}, \ldots, x_{i_{a+b}})$ is a minus-m, y sequence.

Example. $(x_1 x_2 x_4 x_6)$ holds a minus-2, 1 sequence; in fact $(x_1 x_2 x_4 x_6)$, $(x_2 x_4 x_6)$ are both minus-2, 1 sequences.

Let $A(n, k; w_0, w_1, \ldots, w_r)$ denote the total number of subsets of X such that

(a) each subset consists of k elements,

(b) no subset holds a minus-w_y, y sequence for any $y = 0, 1, \ldots, r$.

Denote the product $\binom{a_2}{a_1}\binom{a_3}{a_2} \cdots \binom{a_n}{a_{n-1}}$ by $\begin{pmatrix} a_n \\ \vdots \\ a_1 \end{pmatrix}$ where $\binom{a_{i+1}}{a_i}$ is the binomial coefficient and $\binom{a_{i+1}}{a_i} = 0$

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whenever $a_i > a_{i+1}$, or $a_i < 0$ for $a_{i+1} \geq 0$; and define $(a_1) \equiv 1$ for $n = 1$. Define

$$A(n,k;0) = \begin{cases} 0 & \text{for } k > 0 \\ 1 & \text{for } k = 0 \end{cases} .$$

Using the method of proof given in [1, lemma 1] we obtain the following result:

$$(1) \qquad A(n,k;w_o, \ldots, w_r) = \sum_{i_1+\ldots+i_{w_o}=k} \binom{i_{w_o}}{\vdots \atop i_1} A(n-k+1, i_{w_o}; t_o, \ldots, t_{r-1})$$

where $t_y = w_{y+1}$ and $r > 0$.

<u>Proof of (1)</u>. Assume $A(n,k;t_o, \ldots, t_{r-1})$ is defined for any $r \geq 1$. Let a dash correspond to an element chosen from X and a dot to an element not chosen. Array n-k dots along a straight line. There are n-k+1 slots formed, including the one before the first dot and the one after the last. For the desired subset, no consecutive sequence of dashes should be of length more than w_o. Thus, divide k dashes into i_{w_o} groups: i_1 groups of w_o dashes each, $i_2 - i_1$ groups of $w_o - 1$ dashes each, ..., $i_{w_o} - i_{w_o-1}$ groups of 1 dash each. These i_{w_o} groups may be arranged along a straight line in $\binom{i_{w_o}}{\vdots \atop i_1}$ ways.

In order to obtain the required restrictions we may insert each arrangement of the i_{w_o} groups into the n-k+1 slots in $A(n-k+1, i_{w_o}; t_o, \ldots, t_{r-1})$ ways, where $t_y = w_{y+1}$. Summing over $i_1 + \ldots + i_{w_o} = k$, (1) is obtained. Because

$A(n,k;w) \equiv A(n,k;w,k)$ and $A(n,k;m) \equiv \binom{n}{k}$ for $m \geq k$, it follows from (1) that the number of choices k from n so that no $w+1$ consecutive elements appear in any choice is

$$A(n,k;w) = \sum_{i_1+\ldots+i_w=k} \begin{pmatrix} n-k+1 \\ i_w \\ \vdots \\ i_1 \end{pmatrix}$$

in agreement with [1, Lemma 1]. Therefore

$$\sum_{i_1+\ldots+i_w=k} \begin{pmatrix} i_w \\ \vdots \\ i_1 \end{pmatrix} = \sum_{u=1}^{k} A(k-1,\ k-u;\ w-1) .$$

In the case $w=k$, $\sum_{u=1}^{k} A(k-1,k-u;k-1) = \sum_{u=1}^{k} \binom{k-1}{k-u} = 2^{k-1}$.

Putting $j = k-u$, (1) may be written as,

(2) $A(n,k;\ w_o, w_1, \ldots, w_r) =$

$$\sum_{j=0}^{k-1} A(k-1,j;\ w_o-1)\ A(n-k+1,k-j;\ t_o, \ldots, t_{r-1})$$

where $t_y = w_{y+1}$. For example $A(n,k;1) = \binom{n-k+1}{k}$.

Also, $A(n,k;w)$ is the coefficient of t^k in the generating function

$$\left(\sum_{i=0}^{w} t^i\right)^{n-k+1} = (1-t)^{-(n-k+1)}(1-t^{w+1})^{n-k+1} .$$

Hence we obtain

$$(3) \qquad A(n,k;w) = \sum_{i_1+\dots+i_w=k} \begin{pmatrix} n-k+1 \\ i_w \\ \vdots \\ i_1 \end{pmatrix} = \sum_{s=0} (-1)^s \binom{n-sw-s}{n-k}\binom{n-k+1}{s} .$$

Therefore we have

$$(4) \qquad A(n,k;w_o,\dots,w_r) =$$

$$\sum_{a_o=0} \dots \sum_{a_r=0} \; \sum_{s_o=0} \dots \sum_{s_r=0} (-1)^{s_o+\dots+s_r} \binom{k-a_o}{s_o}\binom{k-a_o-a_1}{s_1} \dots \binom{k-a_o-\dots-a_r}{s_r} \cdot$$

$$\binom{k-1-s_o w_o}{k-1-a_o}\binom{k-1-a_o-s_1 w_1}{k-1-a_o-a_1}\binom{k-1-a_o-a_1-s_2 w_2}{k-1-a_o-a_1-a_2} \dots \binom{k-1-a_o-\dots-a_{r-1}-s_r w_r}{k-1-a_o-\dots-a_r} .$$

$$\binom{n-(r+1)k+r+1+ra_o+(r-1)a_1+\dots+a_{r-1}}{k-a_o-\dots-a_r} .$$

No doubt (4) could be further simplified. Some simpler examples using (2) or (3) are now considered. In each, the number of choices k from n with given restrictions are found. Example 1 is proved as Lemma 3 in [1].

Example 1. The restriction is that no subset (x_i, x_{i+2}), $i = 1,\dots,n-2$, appears in any choice. Their number is given by

$$A(n,k;2,1) = \sum_{j=0} \begin{pmatrix} n-2k+2+j \\ k-j \\ j \end{pmatrix} .$$

In any toss of a coin denote the head H and the tail by T. The probability that neither H T H nor H H H appear in n tosses of a coin (p being the probability of H in one toss) is

$$\sum_{k=0} A(n,k;\ 2,1)p^{k}\,(1-p)^{n-k}\ .$$

Example 2. The restriction is that x_i, x_{i+2} $(i=1,\ldots,n-2)$ do not both appear in any choice unless x_{i+1} also appears in that same choice. The number is given by

$$A(n,k;k,1) = \sum_{j=0} \binom{k-1}{j}\binom{n-2k+2+j}{k-j}\ .$$

The probability of H T H not appearing in n tosses may be obtained.

Example 3. The restriction is that neither the subset (x_i, x_{i+1}), $i = 1,\ldots,n-1$, nor (x_i, x_{i+3}), $i = 1,\ldots,n-3$, appear in any choice. The number is given by

$$A(n,k;1,k,1) = A(n-k+1,k;k,1) = \sum_{j=0} \binom{k-1}{j}\binom{n-3k+3+j}{k-j}.$$

The probability that neither H H nor H T T H appear in n tosses may be obtained.

Example 4. The restriction is that none of the subsets $(x_i, x_{i+1}), (x_i, x_{i+2}), \ldots, (x_i, x_{i+r-1}), (x_i, x_{i+r}, x_{i+2r}, \ldots, x_{i+sr})$ appear in any choice. The number is given by,

$$A(n,k;\ 1,1,1,\ldots,1_{r-1}, s) = A(n-k+1,k;\ 1,\ldots,1_{r-2}, s)$$
$$= A(n-(r-1)k+r-1,k;s)\ ,$$

where 1_r denotes the r^{th} "1".

Putting $s = 1$, $A(n, k; 1, \ldots, 1_r) = \binom{n-rk+r}{k}$ is the number of choices k from n such that if x_i appears in any choice then none of the elements $x_{i+1}, \ldots, x_{i+r}$ appear in that choice. This is one generalization of $A(n,k;1)$. $A(n,k; 1, \ldots, 1_r)$ may also be obtained from the relation

$$\text{(A)} \quad A(n,k;1,\ldots,1_r) = A(n-1,k;1,\ldots,1_r) + A(n-r-1,k-1;1,\ldots,1_r)$$

with proper initial values. This relation is a simple generalization of the relation for $r = 1$ given by Kaplansky [4] and is obtained by dividing the choices into those not containing x_1 and those containing x_1.

Similarly we obtain the recursive relation

$$\text{(B)} \quad A(n,k;w) = A(n-1,k;w) + A(n-1,k-1;w) - A(n-w-2,k-w-1;w)$$

for $n \geq w+2$. For $n \leq w$,

$$A(n,k;w) = \binom{n}{k} \text{ and } A(w+1,k;w) = \begin{cases} \binom{w+1}{k} & \text{for } k < w+1 \\ 0 & \text{if } k = w+1 . \end{cases}$$

Here the number of choices containing x_1 is given by the last two terms of the relation since we need to subtract from $A(n-1,k-1;w)$ the number of those choices of $k-1$ from $n-1$ each containing the set $(x_2, x_3, \ldots, x_{w+1})$. Since x_{w+2} cannot appear, this number is $A(n-w-2,k-w-1;w)$. The relations (A) and (B) are the same for $r = w = 1$ as $A(n-1,k-1;1) - A(n-3,k-2;1) = A(n-2,k-1;1)$. The numbers $F(n) = \sum_{k=0} A(n,k;1)$ are Fibonacci numbers.

Just as $\binom{n}{k} = \binom{n}{n-k}$ because a one-one correspondence exists between the choices of k and the choices of $n-k$ (in particular "k elements chosen" correspond to "$n-k$ elements not chosen") so does a correspondence in a certain sense exist between restricted choices. To a subset consisting of $n-k$

elements with the restriction that it does not contain any minus-m,y sequence corresponds the subset of k elements not chosen with the "complement" restriction. For example $A(n, n-k;w)$ gives the number of choices k from n such that at least one of the elements $x_i, x_{i+1}, \ldots, x_{i+w}$ belongs to each choice if $(x_i, x_{i+1}, \ldots, x_{i+w})$ is contained in $(x_1, \ldots, x_n)$. From example 4 we see that the probability that neither the sequence T H T nor T H H T nor ... nor T H H ... H_{r-1} T appear in n tosses is $\sum_{k=0} A(n, n-k; 1, \ldots, 1_r)p^k(1-p)^{n-k}$, p being the probability of H in a single toss.

An additional restriction is now briefly mentioned. S is called a $(n;a_1, \ldots, a_m)$-subset of $(x_1, \ldots, x_n)$ if

(a) S consists of $a_1 + \ldots + a_m$ subsets, no two having any common elements, and no element of any one of the subsets is consecutive with an element of any other subset.

(b) a_i of the subsets of S consist of i consecutive elements.

It follows that S consists of $\sum_{i=1}^{m} ia_i$ elements. For example, $x_1x_2x_3x_5x_6x_9x_{10}x_{15}$ is a $(n;1, 2, 1)$-subset for $n \geq 15$. Denote by $A(n;a_1, \ldots, a_m;w_1, \ldots, w_r)$ the total number of $(n;a_1, \ldots, a_m)$-subsets of $(x_1, \ldots, x_n)$ such that no subset holds a minus-w_y, y sequence for any $y = 1, \ldots, r$.
Then evidently

$$(5) \qquad A(n;a_1, \ldots, a_m; w_1, \ldots, w_r) = \begin{pmatrix} i_m \\ \vdots \\ i_1 \end{pmatrix} A(n-k+1, i_m; t_o, \ldots, t_{r-1})$$

where $k = \sum_{i=1}^{m} ia_i$, $t_y = w_{y+1}$ and $i_u = \sum_{j=m-u+1}^{m} a_j$, $u = 1, \ldots, m$.

Putting $r = 1$ and $w_1 = \sum_{i=1} ia_i$, the total number of $(n; a_1, \ldots, a_m)$-subsets is obtained.

Using appropriate modification in the proof of (1) a corresponding result may be obtained for circular restricted choices i.e., where x_1 and x_n are consecutive. As this result is quite involved we give here only one particular case. Denote by $A^c(n, k; w)$ the number of choices of k from n objects arranged in a circle so that no $w+1$ consecutive objects appear in any choice. Then

$$A^c(n,k;w) = \sum_{a_1+\ldots+a_w=k} \left[\begin{pmatrix} n-k-1 \\ a_w-1 \\ \vdots \\ a_1-1 \end{pmatrix}(w+1) + \begin{pmatrix} n-k-1 \\ a_w-1 \\ \vdots \\ a_2-1 \\ a_1 \end{pmatrix} w + \ldots + \begin{pmatrix} n-k-1 \\ a_w \\ \vdots \\ a_1 \end{pmatrix}\right]$$

To obtain this formula we note that in using the proof of (1) we consider only $n-k$ slots. Let the slot before the first dot and after the last dot be marked X. Then the i^{th} term $(i=1, \ldots, w)$ in the above expression is the total number of restricted choices such that only one group of $w-i+1$ dashes is inserted in the slot X and the rest elsewhere. The last term is the total number of choices such that no group is inserted in slot X. Further, any insertion of a group of r dashes into the slot X gives rise to $r+1$ choices, since these dashes may represent any one of the subsets $(x_1, x_2, \ldots, x_r)$, $(x_n, x_1, x_2, \ldots, x_{r-1})$, $\ldots$, $(x_{n-r+1}, \ldots, x_n)$. When $w = 1$,

$$A^c(n,k;1) = 2\binom{n-k-1}{k-1} + \binom{n-k-1}{k} = \frac{n}{n-k}\binom{n-k}{k}$$

in agreement with [3, Lemma 2].

We now give a self-contained elementary solution to a permutation problem which has been solved in 1963 by Bizley [3]. The problem is to find the number of ways of arranging $a_1+\ldots+a_n$ balls of n distinct colors, a_i of color A_i along a straight line so that no j_i+1 balls of color A_i are consecutive. This number is denoted by $g(a_1, j_1; a_2, j_2; \ldots; a_n, j_n)$. The special symmetrical case $g(a_1, 1; a_2, 1; \ldots; a_n, 1)$, denoted by $f(a_1, a_2, \ldots, a_n)$, (i.e., the number of arrangements such that no two consecutive balls of the same color appear) is obtained first.

The formula for $f(a_1, a_2, \ldots, a_n)$ given by Bizley is simpler for actual computation. Our beginning is the same as Bizley's.

We have $a_1+\ldots+a_n$ balls of n distinct colors, a_i of color A_i. The a_i balls of color A_i may be divided into t_i groups (with at least one ball in a group) and these groups may be arranged along a straight line. The total number of ways of doing both these things is $\binom{a_i-1}{t_i-1}$.

Suppose $2 \le r \le n-2$, and $a_1+\ldots+a_{r-1}$ balls are arranged along a straight line without restriction. Then $a_1+\ldots+a_{r-1}+1$ slots are formed, including one before and one after all the balls. Let c_i be the total number of pairs of adjacent balls of color A_i, forming c_i slots in this arrangement. The t_r groups of the a_r balls of color A_r may be inserted into the $a_1+\ldots+a_{r-1}+1$ slots, with at most one group inserted into any one slot, in

$$\sum_{j_1=0} \ldots \sum_{j_{r-1}=0} \binom{c_1}{j_1} \cdots \binom{c_{r-1}}{j_{r-1}} \binom{a_1+\ldots+a_{r-1}+1-c_1-\ldots-c_{r-1}}{t_r-j_1-\ldots-j_{r-1}}$$

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ways. Having done this the t_{r+1} groups of a_{r+1} balls may be inserted in

$$\sum_{s_1=0} \cdots \sum_{s_r=0} \binom{c_1 - j_1}{s_1} \cdots \binom{c_{r-1} - j_{r-1}}{s_{r-1}} \binom{a_r - t_r}{s_r} \cdot$$

$$\binom{a_1 + \ldots + a_{r-1} + 1 - c_1 - \ldots - c_{r-1} + j_1 + \ldots + j_{r-1} + t_r}{t_{r+1} - s_1 - \ldots - s_r} \text{ ways.}$$

Note that the t_r groups give rise to $a_r - t_r$ pairs of adjacent A_r colored balls.

For $r = 2$, the t_2 groups may be inserted in

$$\sum_{i_1=0} \binom{a_1 - 1}{i_1} \binom{2}{t_2 - i_1}$$ ways. Inductively the t_r groups for $r = 3, \ldots, n-1$ may be inserted. In inserting the a_n balls of color A_n, all slots formed by 2 adjacent balls of one color need to be filled and $t_n = a_n$. Hence, if $r+1 = n$ then $s_m = c_m - j_m$ for $m = 1, \ldots, n-2$ and $s_{n-1} = a_{n-1} - t_{n-1}$.

It follows that

(6) $f(a_1, \ldots, a_n) =$

$$\sum_{t_2=1}^{a_2} \cdots \sum_{t_{n-1}=1}^{a_{n-1}} \prod_{u=2}^{n-1} \binom{a_u - 1}{t_u - 1} \sum_{i_1=0} \binom{a_1 - 1}{i_1} \binom{2}{t_2 - i_1} \sum_{i_2=0} \sum_{i_3=0}$$

$$\binom{a_1 - 1 - i_1}{i_2} \binom{a_2 - t_2}{i_3} \binom{t_2 - i_1 + 2}{t_3 - i_2 - i_3} \sum_{i_4=0} \sum_{i_5=0} \sum_{i_6=0} \binom{a_1 - 1 - i_1 - i_2}{i_4}$$

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$$\binom{a_2-t_2-i_3}{i_5}\binom{a_3-t_3}{i_6}\binom{t_2+t_3+i_1+i_2+i_3+2}{t_4-i_4-i_5-i_6} \ldots \Sigma\Sigma \ldots \Sigma_{i_{\frac{(n-1)(n-2)}{2}}}(\ldots)\ldots(\ldots)$$

$$\binom{t_2+\ldots+t_{n-1}+i_1+\ldots+i_{\frac{(n-1)(n-2)}{2}}+2}{a_1+\ldots+a_{n-1}-a_n+1}.$$

In the trivial case, $f(a_1, a_2) = \binom{2}{a_1-a_2+1}$. However, the derivations by Bizley are more elegant. Using his formula we obtain

$$f(a_1, \ldots, a_n) = \sum_{r_1=1}^{a_1} \sum_{r_2=1}^{a_2} \ldots \sum_{r_n=1}^{a_n} (-1)^{\Sigma a - \Sigma r} \binom{a_1-1}{r_1-1}\binom{a_2-1}{r_2-1} \ldots$$

$$\binom{a_n-1}{r_n-1} \frac{\left(\sum_{i=1}^{n} r_i\right)!}{r_1!\, r_2! \ldots r_n!}.$$

Evidently, in the general case

(7) $g(a_1, j_1; \ldots; a_n, j_n)$

$$= \Sigma \begin{pmatrix} \alpha_{j_1} \\ \vdots \\ \alpha_1 \end{pmatrix} \begin{pmatrix} \beta_{j_2} \\ \vdots \\ \beta_1 \end{pmatrix} \ldots \begin{pmatrix} \lambda_{j_n} \\ \vdots \\ \lambda_1 \end{pmatrix} f(\alpha_{j_1}, \beta_{j_2}, \ldots, \lambda_{j_n}),$$

(the sum taken over $\alpha_1+\ldots+\alpha_{j_1} = a_1; \ldots; \lambda_1+\ldots+\lambda_{j_n} = a_n$)

$$= \Sigma\, A(a_1-1, a_1-r_1; j_1-1) \ldots A(a_n-1, a_n-r_n; j_n-1)\, f(r_1, \ldots, r_n)$$

(the sum taken over $r_1 = 1, \ldots, a_1; \ldots; r_n = 1, \ldots, a_n$)

$$= \sum \prod_{i=1}^{n} (-1)^{s_i} \binom{a_i - 1 - s_i j_i}{r_i - 1} \binom{r_i}{s_i} f(r_1, r_2, \ldots, r_n)$$

(the sum taken over

$$r_1 = 1, \ldots, a_1; \ldots; r_n = 1, \ldots, a_n; \; s_1 = 0, 1, \ldots, r_1; \ldots; s_n = 0, 1, \ldots, r_n)$$

using (3).

Checking for $g(a_1, a_1; a_2, 1; a_3, 1)$ we obtain, using (6) and (7),

$$g(a_1, a_1; a_2, 1; a_3, 1) = \sum_{r=1}^{a_1} \binom{a_1 - 1}{r-1} f(r, a_2, a_3)$$

$$= \sum_{t=1}^{a_2} \binom{a_2 - 1}{t-1} \sum_{i=0}^{t} \binom{a_1 - 1}{i} \binom{2}{t-i} \sum_{r=1}^{a_1} \binom{a_1 - 1 - i}{a_1 - r} \binom{t+i+2}{r+a_2-a_3+1}$$

$$= \sum_{t=1}^{a_2} \binom{a_2 - 1}{t-1} \binom{a_1 + 1}{t} \binom{a_1 + t + 1}{t + a_3 - a_2},$$

in agreement with [2] wherein we need multiply the answer by $a_1!\, a_2!\, a_3!$ since the colored balls are replaced by distinguishable objects of a kind.

By dropping the summation signs from (7) we obtain stronger restriction for the arrangements:

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$$(8)\quad \begin{pmatrix} {}_1h_{j_1} \\ \vdots \\ {}_1h_1 \end{pmatrix} \begin{pmatrix} {}_2h_{j_2} \\ \vdots \\ {}_2h_1 \end{pmatrix} \cdots \begin{pmatrix} {}_nh_{j_n} \\ \vdots \\ {}_nh_1 \end{pmatrix} f({}_1h_{j_1}, {}_2h_{j_2}, \ldots, {}_nh_{j_n}),$$

where ${}_ih_1 + {}_ih_2 + \ldots + {}_ih_{j_i} = a_i$ for $i = 1, \ldots, n$, gives the number of arrangements along a straight line of the $a_1 + \ldots + a_n$ balls such that exactly ${}_ih_{r+1} - {}_ih_r$ maximal runs of $j_i - r$ consecutive balls for $r = 1, 2, \ldots, j_i - 1$ of color A_i and exactly ${}_ih_1$ maximal runs of 1 ball of color A_i appear.

The problem of the balls may be restated in terms of restricted choices since $g(a_1, j_1; a_2, j_2; \ldots; a_n, j_n)$ gives the number of ways of dividing $a_1 + \ldots + a_n$ distinct ordered elements into n disjoint subsets such that the i^{th} $(i = 1, \ldots, n)$ subset contains a_i elements of which no $j_i + 1$ elements are consecutive.

Finally, a variation of the above problem of the balls is mentioned.

We are given $r + a_1 + a_2 + \ldots + a_n$ balls of $n+1$ different colors, r of color R and a_i of color A_i for $i = 1, \ldots, n$. Denote by $h(r; a_1, j_1; a_2, j_2; \ldots; a_n, j_n)$ the number of ways of arranging all the balls along a straight line such that

(a) no $j_i + 1$ balls of color A_i $(i = 1, \ldots, n)$ are consecutive in any arrangement and

(b) no ball of color A_i is adjacent to a ball of color A_j for $i \neq j$ and $i, j = 1, \ldots, n$.

Placing the r balls of color R along a straight line hence forming $r+1$ slots and then inserting the other balls into the slots in the appropriate manner we obtain

(9) $\quad h(r; a_1, j_1; \ldots; a_n, j_n) =$

$$\sum_{\alpha_1+\ldots+\alpha_{j_1}=a_1} \; \sum_{\beta_1+\ldots+\beta_{j_2}=a_2} \cdots \sum_{\lambda_1+\ldots+\lambda_{j_n}=a_n} \begin{pmatrix} \alpha_{j_1} \\ \vdots \\ \alpha_2 \\ \alpha_1 \end{pmatrix} \begin{pmatrix} \beta_{j_2} \\ \vdots \\ \beta_2 \\ \beta_1 \end{pmatrix} \cdots \begin{pmatrix} \lambda_{j_n} \\ \vdots \\ \lambda_2 \\ \lambda_1 \end{pmatrix} \cdot$$

$$\binom{r+1}{\alpha_{j_1}} \binom{r+1-\alpha_{j_i}}{\beta_{j_2}} \binom{r+1-\alpha_{j_1}-\beta_{j_2}}{\gamma_{j_3}} \cdots \binom{r+1-\alpha_{j_1}-\beta_{j_2}-\ldots-\kappa_{j_{n-1}}}{\lambda_{j_n}}$$

$$= \sum_{u_1=1} \cdots \sum_{u_n=1} \; \sum_{v_1=0} \cdots \sum_{v_n=0} (-1)^{v_1+\ldots+v_n} \prod_{i=1}^{n} \prod_{k=1}^{n} \binom{a_i - 1 - v_i j_i}{u_i - 1} \binom{u_k}{v_k} \cdot$$

$$\frac{(r+1)!}{(u_1)!\,(u_2)!\ldots(u_n)!\,(r+1-\sum_{x=1}^{n} u_x)!}$$

[Although we do not state one here, a more involved formula containing the further restriction that no y balls of color R are adjacent may be given.]

To find the number of arrangements of the balls with restriction (b) only, we put $j_i = a_i$ $(i = 1, \ldots, n)$ and obtain

(10) $\quad h(r; a_1, a_1; \ldots; a_n, a_n) =$

$$\sum_{u_1=1}^{a_1} \cdots \sum_{u_n=1}^{a_n} \prod_{i=1}^{n} \binom{a_i - 1}{u_i - 1} \cdot \frac{(r+1)!}{(u_1)!\,(u_2)!\ldots(u_n)!\,(r+1-\sum_{x=1}^{n} u_x)!}$$

Replacing the balls of a certain color by distinct objects of a certain kind, we see that the number of ways of arranging along a straight line $r+a_1+a_2+\ldots+a_n$ distinct objects of $n+1$ different

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kinds, r of kind R and a_i of kind A_i $(i = 1, \ldots, n)$ such that no object of kind A_i is adjacent to one of kind A_j for $i \neq j$ is equal to $(r)!(a_1)!(a_2)! \ldots (a_n)!\, h(r; a_1, a_1; \ldots; a_n, a_n)$. For example, using the special case $r = 44$, $a_1 = a_2 = 4$ the probability that at least one Jack and one Queen are adjacent in a shuffled deck of ordinary playing cards is found to be .4863. [This is problem E1713. Amer. Math. Monthly Vol. 71 (63) p. 793.]

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