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Convergence and applications of reproducing kernels for classes of discrete harmonic functions

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This paper gives convergence properties and applications of the discrete analogs of reproducing kernels for various families of harmonic functions. From these results information is obtained on the solution of interpolation problems, the convergence of the discrete Neumann's function, and the solution to problems involving the discrete biharmonic operator.

1. Introduction

Three Hilbert spaces of harmonic functions are considered, each possessing a reproducing kernel. A discrete analog for each of these reproducing kernels is developed.

One of the reproducing kernels studied is the discrete harmonic kernel of Deeter and Springer [5]. They established convergence on regions bounded by rectangles. Later convergence was established on regions bounded by edges and diagonals of some h-net [9]. Our work extends convergence to regions with curved boundary components. It is shown that the discrete harmonic kernel will converge to the ordinary harmonic kernel under certain conditions on the boundary of the region. Using this result and the known convergence properties of the discrete Green's function, we are able to make some remarks on the convergence of the discrete Neumann's function. Some numerical data were generated and typical results are given

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on the approximation of the Neumann's function for a circular region. The second kernel considered is similar to the discrete harmonic kernel mentioned above.

Another reproducing kernel studied is the discrete analog of the kernel developed by Aronszajn [1] and Zaremba [15] which was used to represent solutions of particular problems involving the biharmonic operator. These problems arise in the theory of elasticity. Convergence of the discrete kernel is established and it is shown that this discrete kernel can be used to represent solutions of analogous discrete biharmonic problems.

The final results deal with the approximation of solutions of interpolation problems in any of the three Hilbert spaces of harmonic functions. We conclude that for small net widths, the solution of an analogous interpolation problem for discrete harmonic functions can be expressed using reproducing kernels, and this solution will converge to the solution of the original problem. This interpolation problem was discussed for arbitrary Hilbert spaces with reproducing kernels in Meschkowski [13]. A development of essentially the same problem for a class of analytic functions appears in Meschkowski [13] and Epstein [7].

2. Some reproducing kernels

Several results from the theory of Hilbert spaces with reproducing kernels are used, all of which may be found in Meschkowski [13] and Aronszajn [1].

Let R be a bounded region with a piecewise smooth boundary. For each h, $0 < h < 1$, an h-net may be formed in the plane by the intersections of the families of lines $x = mh + a$ and $y = mh + b$ where m is an integer and $a + ib$ is a fixed point in R. In this report we only consider values of h such that the h-nets form an increasing collection of subsets of the plane. If z is a point in an h-net, the points $z + h$, $z + ih$, $z - h$, and $z - ih$ are called neighbors of z. Let I be the set of all net points z in R such that all the neighbors of z are also in R. Suppose now that h is sufficiently small so that $I \neq \emptyset$. The set of net points in R which have at least one neighbor in I and at least one neighbor in the complement of R will be denoted by

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B . We call $R = I \cup B$ the discrete region associated with R . We refer to B as the boundary of R and to I as the interior of R .

A function U defined on R is discrete harmonic on I if

$$\Delta U(z) = \frac{1}{h^2}[U(z+h)+U(z+ih)+U(z-h)+U(z-ih)-4U(z)] = 0$$

for every z in I . Let H_1 be the class of all functions U on R which are discrete harmonic on I and which are normalized by

$$h \sum_{z \in B} s(z)U(z) = 0 ,$$

where $s(z)$ is the number of neighbors of z in I . Let U_x and U_y be the partial differences of U defined at $z \in R$ by

$$U_x(z) = \begin{cases} \frac{1}{h}[U(z+h)-U(z)] , & \text{if } z+h \in R \text{ and } z \text{ or } z+h \in I , \\ 0 & , \text{otherwise;} \end{cases}$$

$$U_y(z) = \begin{cases} \frac{1}{h}[U(z+ih)-U(z)] , & \text{if } z+ih \in R \text{ and } z \text{ or } z+ih \in I , \\ 0 & , \text{otherwise.} \end{cases}$$

With the inner product of two functions U and V in H_1 defined by

$$(2.1) \qquad (U, V) = h^2 \sum_{z \in R} [U_x(z)V_x(z)+U_y(z)V_y(z)] ,$$

H_1 is a Hilbert space with a reproducing kernel. This reproducing kernel is called the discrete harmonic kernel and is denoted by $K_1(z, \zeta)$. The norm of a function U in H_1 is defined by $\|U\|^2 = (U, U)$.

Let H_1 be the completion of the space consisting of the functions u harmonic on R , continuous on $\overline{R}$, with finite Dirichlet integral, and for which

$$\int_{\partial R} u(z)ds = 0 .$$

H_1 is a Hilbert space with the inner product of two functions u and v defined by

(2.2) $$(u, v) = \int_R \int [u_x(z)v_x(z)+u_y(z)v_y(z)]dxdy .$$

The norm of a function u is defined by $\|u\|^2 = (u, u)$. Here u_x and u_y are partial derivatives. The reproducing kernel of H_1 is called the ordinary harmonic kernel and is denoted by $k_1(z, \zeta)$.

A related kernel was discussed by Bergman [2, p. 59]. Let ζ' be some point of R . Let H_2 be the Hilbert space of harmonic functions u on R with finite Dirichlet integral for which $u(\zeta') = 0$; the inner product is defined by (2.2). The reproducing kernel of H_2 , $k_2(z, \zeta)$, is related to that of H_1 by the equation

(2.3) $$k_2(z, \zeta) = k_1(z, \zeta) - k_1(\zeta, \zeta') .$$

If ζ' belongs to I , then the class H_2 of functions U discrete harmonic on I with $U(\zeta') = 0$ is a Hilbert space with inner product (2.1). Its reproducing kernel, $K_2(z, \zeta)$, is related to that of H_1 by the equation

(2.4) $$K_2(z, \zeta) = K_1(z, \zeta) - K_1(\zeta, \zeta') .$$

Thus far we have considered classes of harmonic functions with finite Dirichlet integral and the corresponding classes of discrete harmonic functions. We will now consider the class of harmonic functions with finite square integral. Let H_3 denote the Hilbert space consisting of these functions with the inner product of two functions u and v defined by

$$\langle u, v \rangle = \int_R \int u(z)v(z)dxdy .$$

Note that no normalization is required for this class of functions as was

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necessary in the previous cases.

The norm of a function u will be denoted by $|||u|||$. The Hilbert space H_3 has a reproducing kernel $k_3(z, \zeta)$. For the discrete analog of this reproducing kernel we let H_3 denote the Hilbert space of all discrete harmonic functions on I with the inner product of U and V given by

$$\langle U, V\rangle = h^2 \sum_{z\in R} U(z)V(z) .$$

3. Convergence of the kernel $K_2(z, \zeta)$

Henceforth, we will let ζ and ζ' be fixed points in R belonging to some discrete region R with $\zeta \neq \zeta'$. Suppose h is sufficiently small so that ζ and ζ' are in I. Again we denote by H_2 the class of discrete harmonic functions on I which vanish at ζ'. The function M given by

$$M(z) = \frac{K_2(z,\zeta)}{K_2(\zeta,\zeta)}$$

is the unique function with minimum norm among all functions U in H_2 satisfying $U(\zeta) = 1$. The following theorem establishes convergence of the solution of this discrete minimum problem. The convergence of the kernel $K_2(z, \zeta)$ is obvious since

$$K_2(z, \zeta) = \frac{M(z)}{\|M\|^2} .$$

THEOREM 3.1. *If the Hilbert space* H_2 *has a complete orthonormal system consisting of functions harmonic on* $\overline{R}$ *, then the minimizing function* M *converges uniformly on compact subsets of* R *as* $h \to 0$ *to the unique function* m *in* H_2 *with minimum norm among all* u *in* H_2 *with* $u(\zeta) = 1$. *Furthermore* $\|M\| \to \|m\|$ *as* $h \to 0$.

Proof. Let u' be the solution of the minimum problem in H_2. If q is any linear combination of elements of the complete orthonormal system

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for H_2 consisting of functions harmonic on R, then q is harmonic on R. Let O be a region with smooth boundary such that $\overline{R} \subset O$ and q is harmonic on $\overline{O}$. By approximating the solution of the Dirichlet problem on O with the boundary values of q using discrete harmonic functions, we can obtain functions $\{Q : 0 < h < 1\}$ which are discrete harmonic on I and such that Q, Q_x, and Q_y converge uniformly on $\overline{R}$ to q, q_x, and q_y, respectively, as $h \to 0$. For the details of the approximation of q by discrete harmonic functions, we refer to the book by Epstein [6, pp. 199-211]. If q is a function for which $q(\zeta) = 1$ and $q(\zeta') = 0$, then we may assume $Q(\zeta) = 1$ and $Q(\zeta') = 0$. In this case, $\|Q\| \to \|q\|$ as $h \to 0$ and $\|M\| \leq \|Q\|$ for each h. This implies that $\{M : 0 < h < 1\}$ is bounded. Thus $\{M : 0 < h < 1\}$ is equicontinuous and uniformly bounded on compact subsets of R. The proof of this also follows from results in Epstein [6]. Furthermore, any sequence in $\{M : 0 < h < 1\}$ will have a subsequence $\{M_p\}$ such that M_p and its partial differences M_{px} and M_{py} converge uniformly on compact subsets of R to a harmonic function m and its partial derivatives m_x and m_y, respectively.

We will now show that $m = u'$. Let $\{C_i\}$ be an exhaustion of R by compact sets. Let

$$\|M\|_i = h^2 \sum_{x \in R \cap C_i} \left[(M_x(z))^2 + (M_y(z))^2 \right], \tag{3.1}$$

and

$$\|m\|_i = \int_{C_i} \int \left[(m_x(z))^2 + (m_y(z))^2 \right] dx dy . \tag{3.2}$$

Now $\|M_p\|_i \leq \|M_p\|$, which is bounded for all p, and since $\|M_p\|_i \to \|m\|_i$ as $p \to \infty$, we have $\|m\|_i \to \|m\| < \infty$. Since $m(\zeta') = \lim_{p \to \infty} M_p(\zeta') = 0$, $m \in H_2$. Also $m(\zeta) = 1$ and thus $\|u'\| \leq \|m\|$. Let q be any linear combination of elements of the complete orthonormal system of functions harmonic on $\overline{R}$ which has $q(\zeta) = 1$. As discussed earlier, we can construct a sequence $\{Q_p\}$ of discrete harmonic functions such that

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$Q_p(\zeta) = 1$, $Q_p(\zeta') = 0$, and Q_p, Q_{px} , and Q_{py} converge uniformly on $\overline{R}$ to q, q_x , and q_y , respectively. Now $\|Q_p\| \geq \|M_p\|$, which implies that $\|q\| \geq \|m\|$. We conclude that $\|m\| \leq \|u\|$ for all u in H_2 with $u(\zeta) = 1$. Therefore $m = u'$. By the same argument, any other convergent sequence in $\{M : 0 < h < 1\}$ will converge to $u' = m$. Thus M converges uniformly on compact subsets of R to m as $h \to 0$.

It remains to show that $\|M\| \to \|m\|$ as $h \to 0$. Let $\varepsilon > 0$ be given. Let q be a linear combination of elements of the complete orthonormal system of functions harmonic on $\overline{R}$ such that $q(\zeta) = 1$ and $\|q-m\| < \varepsilon/2$. Let $\{Q : 0 < h < 1\}$ be the family of functions described above. There exists an $h_1 > 0$ such that

$$\big|\|Q\|-\|q\|\big| < \varepsilon/2$$

and thus

$$\|M\| \leq \|Q\| < \|m\| + \varepsilon$$

whenever $h < h_1$. Again $\{C_i\}$ denotes an exhaustion of R by compact sets. Let $\|M\|_i$ and $\|m\|_i$ be defined by (3.1) and (3.2). Choose i such that

$$\big|\|m\|_i-\|m\|\big| < \varepsilon/2 .$$

There exists an $h_2 > 0$ such that $h < h_2$ implies

$$\big|\|M\|_i-\|m\|_i\big| < \varepsilon/2 .$$

Thus we have

$$\|M\| \geq \|M\|_i > \|m\|_i - \varepsilon/2 > \|m\| - \varepsilon$$

for all $h < h_2$. If $h < \min\{h_1, h_2\}$, then

$$\big|\|M\|-\|m\|\big| < \varepsilon .$$

Thus $\|M\| \to \|m\|$ as $h \to 0$, which completes the proof of the theorem.

THEOREM 3.2. *If the Hilbert space* H_2 *has a complete orthonormal system of functions harmonic on* $\overline{R}$ *, then* $K_2(z, \zeta)$ *converges to* $k_2(z, \zeta)$

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uniformly in z *on compact subsets of* R *as* $h \to 0$.

4. Convergence of the kernel $K_1(z, \zeta)$

The Hilbert space H_1 will have a complete orthonormal system of functions harmonic on $\overline{R}$ if and only if the same is true for the space H_2 . By equations (2.3) and (2.4), $\|K_2(z, \zeta)\| = \|K_1(z, \zeta)\|$ for $0 < h < 1$ and $\|k_2(z, \zeta)\| = \|k_1(z, \zeta)\|$. These observations lead to the next theorem.

THEOREM 4.1. *If* H_1 *has a complete orthonormal system of functions harmonic on* $\overline{R}$ *, then* $K_1(z, \zeta)$ *converges to* $k_1(z, \zeta)$ *uniformly in* z *on compact subsets of* R *as* $h \to 0$.

Proof. Note that $\|K_2(z, \zeta)\| = 1/\|M\|$ and $\|k_2(z, \zeta)\| = 1/\|m\|$. Furthermore, $\|K_1(z, \zeta)\|^2 = K_1(\zeta, \zeta)$ and $\|k_1(z, \zeta)\|^2 = k_1(\zeta, \zeta)$. Since $\|M\| \to \|m\|$ as $h \to 0$, $K_1(\zeta, \zeta) \to k_1(\zeta, \zeta)$ as $h \to 0$. Therefore $K_1(\zeta, \zeta') = K_1(\zeta, \zeta) - K_2(\zeta, \zeta) \to k_1(\zeta, \zeta) - k_2(\zeta, \zeta) = k_1(\zeta, \zeta')$ as $h \to 0$. We conclude that $K_1(z, \zeta) = K_2(z, \zeta) - K_1(\zeta, \zeta')$ converges to $k_2(z, \zeta) - k_1(\zeta, \zeta') = k_1(z, \zeta)$ uniformly in z on compact subsets of R as $h \to 0$.

The next corollary follows from known results on when the space H_1 will have a complete orthonormal system of functions harmonic on $\overline{R}$ (see [*13*]). In the first case the system can be taken as polynomials in x and y . In the second case it can be taken as rational functions with one singular point in each component of the complement of $\overline{R}$.

COROLLARY 4.2. *If* R *is either bounded by a simple closed contour or is a finitely connected region bounded by analytic curves, then* $K_1(z, \zeta)$ *converges to* $k_1(z, \zeta)$ *uniformly in* z *on compact subsets of* R *as* $h \to 0$.

5. Convergence of the discrete Neumann's function

For $0 < h < 1$, the discrete Green's function for R is defined to

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be the function $G(z, \zeta)$ which vanishes on B, the boundary of R, and which is a discrete harmonic function of z on I except at $z = \zeta$ where

$$\Delta G(z, \zeta) = -\frac{1}{h^2}.$$

For each z in B, let $s(z)$ denote the number of neighbors of z in I. Let z_i, $i = 1, \ldots, s(z)$, be the neighbors of z in I. The discrete Neumann's function for R is the function $N(z, \zeta)$ satisfying the conditions:

(i) $s(z)N(z, \zeta) - \sum_{i=1}^{s(z)} N(z_i, \zeta) = -\frac{s(z)}{\sum_{z \in B} s(x)}$ for all z in B;

(ii) $N(z, \zeta)$ is a discrete harmonic function of z except at $z = \zeta$ where

$$\Delta N(z, \zeta) = -\frac{1}{h^2},$$

and

(iii) $N(z, \zeta)$ is normalized by the condition

$$h \sum_{z \in B} s(z)N(z, \zeta) = 0.$$

It was shown by Deeter and Springer [5, p. 421] that

$$K_1(z, \zeta) = N(z, \zeta) - G(z, \zeta).$$

Under the hypothesis of Corollary 4.2, there exists an ordinary Green's function and Neumann's function for the region R. If $g(z, \zeta)$ is the Green's function and $n(z, \zeta)$ is the Neumann's function for R, then $k_1(z, \zeta) = n(z, \zeta) - g(z, \zeta)$. It is well known that as $h \to 0$, $G(z, \zeta)$ converges to $g(z, \zeta)$ uniformly in z on any compact subset of R which does not contain ζ. Combining this result with Corollary 4.2, the following theorem is proved.

THEOREM 5.1. *If R is bounded by a simple closed contour or is a finitely connected region bounded by analytic curves, then as $h \to 0$, $N(z, \zeta)$ converges to $n(z, \zeta)$ uniformly in z on any compact subset of*

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R *which does not contain* ζ .

From the discussion of the discrete Green's function in Forsythe and Wasow [8, pp. 314-318] it is known that if the boundary of the region is sufficiently smooth, then $G(z, \zeta) - g(z, \zeta) = O(h)$ uniformly on the region provided z is bounded away from ζ by a constant multiple of $h^{1/2}$. For regions bounded by edges and diagonals of some net, Huddleston [9] obtained an error of the order $O(h^2)$ on compact subsets not containing ζ .

Results so far on the discrete Neumann's function indicate convergence may be much slower. The discretization error given by Deeter and Springer [5] and Huddleston [9] is $O(h \log h)$. We note that there are some differences between the associated discrete regions and the definitions of the discrete Green's and Neumann's function in [5], [8], and [9].

In an attempt to shed some light on the rate of convergence of the discrete Neumann's function, and hence the discrete harmonic kernel, the following computations were made. The region was taken to be the interior of the unit circle. The h-nets contained the origin and the values of h used were $h = 1/4$, $1/8$, and $1/16$. For the continuous function we have

$$n(z, 0) = -\frac{1}{2\pi} \log|z| .$$

The discrete function $N(z, 0)$ was calculated by solving the system of equations which arise from the definition in Section 5. Since the continuous and discrete functions are both symmetric with respect to the x and y axes and the line $y = x$, only values of z with $0 \le y \le x$ are considered. The computed discrete functions and the continuous function are compared at points of the $1/4$-net in Table 1. This table also contains the values obtained by extrapolation to $h = 0$ [8, p. 307] using the values of the discrete functions for $h = 1/8$ and $h = 1/16$. Table 2 presents a comparison of the continuous and discrete function along the nonnegative real axis when $h = 1/16$.

All computations were executed on a UNIVAC 1106 computer using gaussian elimination to solve the system of equations. Some calculations were made using $\zeta \neq 0$ and the observed error was approximately the same. Since there is considerable error in our approximation even after extra-

TABLE 1

$N(z, 0)$ and $n(z, 0)$ at some points in $|z| < 1$

z	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$h = \frac{1}{16}$	extrapolated value	$n(z, 0)$
0	.45400	.57665	.69383	.81101	$+\infty$
$\frac{1}{4}$	.20400	.21333	.21688	.22043	.22064
$\frac{1}{2}$	.09400	.10075	.10642	.11209	.11032
$\frac{3}{4}$	.04400	.04043	.04462	.04881	.04579
$\frac{1}{4} + \frac{1}{4}i$	.13400	.15191	.16012	.16833	.16548
$\frac{1}{2} + \frac{1}{4}i$	.06400	.08007	.08741	.09475	.09256
$\frac{3}{4} + \frac{1}{4}i$	.01400	.02738	.03354	.03970	.03740
$\frac{1}{2} + \frac{1}{2}i$	.01400	.03850	.04772	.05694	.05516
$\frac{3}{4} + \frac{1}{2}i$	.03600	−.00432	.00682	.01796	.01652

TABLE 2

$N(z, 0)$ and $n(z, 0)$ where $h = 1/16$ and $0 \le z < 1$

z	$N(z, 0)$	$n(z, 0)$	z	$N(z, 0)$	$n(z, 0)$
0	.69383	$+\infty$	$\frac{1}{2}$	.10642	.11032
$\frac{1}{16}$	.44383	.44126	$\frac{9}{16}$	.08803	.09157
$\frac{1}{8}$	.33045	.33095	$\frac{5}{8}$	.07181	.07480
$\frac{3}{16}$	.26356	.26642	$\frac{11}{16}$	.05742	.05963
$\frac{1}{4}$	.21688	.22064	$\frac{3}{4}$	.04462	.04579
$\frac{5}{16}$	.18103	.18512	$\frac{13}{16}$	.03329	.03305
$\frac{3}{8}$	.15192	.15611	$\frac{7}{8}$	.02333	.02125
$\frac{7}{16}$	.12746	.13158	$\frac{15}{16}$	.01471	.01027

polation (a maximum of 0.00385 in Table 1), the possibility of using smaller values of h along with some iterative method arises. However, the coefficient matrix is not diagonally dominant and it is unclear whether the standard iterative methods for solving linear systems will converge in this case. An alternative method for improving accuracy might be the inclusion of certain points on the unit circle in the discrete region. The preceding development would have to be modified accordingly.

6. Convergence and applications of the kernel $K_3(z, \zeta)$

The proof of the convergence of $K_3(z, \zeta)$ is similar to the proof for $K_2(z, \zeta)$ in Section 3. As before, we begin by considering the unique function M of minimum norm among all functions in H_3 assuming the value 1 at ζ. Appealing to the work of Laasonnen [10], if C is any compact subset of R there is a constant K, independent of h, such that

$$\max_{z \in C} |M(z)| \leq K \|M\| .$$

This inequality, together with the assumption that H_3 has a complete orthonormal system of functions harmonic on $\bar{R}$, implies that $\{M : 0 < h < 1\}$ is uniformly bounded on compact subsets of R. Hence, from Verblunsky [14], $\{M : 0 < h < 1\}$ is equicontinuous on compact subsets of R and any sequence will have a subsequence which converges uniformly on compact subsets of R. An application of the results stated thus far, and an examination of the proof of Theorem 3.1, establishes convergence of the function M to the solution of the analogous minimum problem in H_3. The convergence of the kernel $K_3(z, \zeta)$ to $k_3(z, \zeta)$ follows directly.

THEOREM 6.1. *If H_3 has a complete orthonormal system of functions harmonic on $\bar{R}$, then $K_3(z, \zeta)$ converges to $k_3(z, \zeta)$ uniformly in z on compact subsets of R as $h \to 0$.*

The hypothesis of this theorem will be satisfied, as with H_1 and H_2, if the region R is bounded by a simple closed contour or is a multiply connected region bounded by analytic curves.

The techniques used to establish convergence of $K_3(z, \zeta)$ do not lend themselves to an analysis of the rate of convergence. Although it would not be difficult to calculate $K_3(z, \zeta)$ for particular associated discrete regions, we could conclude little about the error since no easily computable expression is known for the function $k_3(z, \zeta)$ even for simple regions.

It was noted by Aronszajn [1] and Zaremba [15] that the reproducing kernel $k_3(z, \zeta)$ could be used to obtain integral representations of solutions of particular problems involving the biharmonic operator. We consider analogous problems for the discrete biharmonic operator Δ^2 defined by $\Delta^2 U = \Delta(\Delta U)$. This operator was discussed by Courant, Friedrichs and Lewy [4], who also developed the following discrete Green's formula which we now state. If U and V are functions defined on the discrete region R with interior I and boundary B, then

$$(6.1)\quad h^2 \sum_{z\in I} [U(z)\Delta V(z)-V(z)\Delta U(z)] = h \sum_{z\in B} [U(z)V_n(z)-V(z)U_n(z)] ,$$

where

$$U_n(z) = \frac{1}{h}\left[s(z)U(z) - \sum_{i=1}^{s(z)} U(z_i)\right]$$

with z_i, $i = 1, \ldots, s(z)$, the neighbors of z in I.

In order for the quantity $\Delta^2 U(z)$ to be defined at a point z, it is necessary that U be defined at each point which is a neighbor of z and also at each point which is the neighbor of a neighbor of z. Let R' be the set of points in the h-net which belong to R or have a neighbor in R. The boundary points of R' will be denoted by B'. Now $\Delta^2 U$ is defined at every point of I when U is defined on R'.

The kernel $K_3(z, \zeta)$ and the Green's function $G(z, \zeta)$ give the solution to the problem of finding a function U satisfying:

(i) $U(z) = U_n(z) = 0$ for $z \in B'$,

(ii) $\Delta^2 U(z) = \Phi(z)$ for $z \in I$,

where Φ is a given function defined on I . Suppose Ψ is a function such that $\Delta\Psi = \Phi$ on I . Then the function U which solves the above problem is given by

$$(6.2) \quad U(z) = -h^2 \sum_{z' \in R} G(z, z')\left[\Psi(z') - h^2 \sum_{z'' \in R} K_3(z', z'')\Psi(z'')\right] .$$

To verify this equation we note that if U solves the above problem, then $\Delta U - \Psi$ is discrete harmonic on I , which implies

$$h^2 \sum_{z'' \in R} K_3(z', z'')[(\Delta U - \Psi)(z'')] = \Delta U(z') - \Psi(z') ,$$

by the reproducing property of the kernel. Thus

$$\Psi(z') - h^2 \sum_{z'' \in R} K_3(z, z'')\Psi(z'') = \Delta U(z') - h^2 \sum_{z'' \in R} K_3(z', z'')\Delta U(z'') .$$

With this substitution and an application of (6.1) together with (i), the right hand side of (6.2) becomes

$$-h^2 \sum_{z' \in R} G(z, z')\Delta U(z') .$$

A second application of (6.1) reduces this expression to $U(z)$.

One possible choice for the function Ψ appearing in (6.2) is given by

$$\Psi(z) = h^2 \sum_{z' \in I} L(z'-z)\Phi(z')$$

where L is the "free space" discrete Green's function of McCrea and Whipple [12]. The function L is defined at every point of the h-net in the plane and satisfies the equation

$$L(z) = \begin{cases} 0 & \text{for } z \neq 0 , \\ \dfrac{1}{h^2} & \text{for } z = 0 . \end{cases}$$

Estimates for the function L , along with known integral representations, are found in the paper by Mangad [11].

Now if we let the function Φ in (ii) be defined by

$$\Phi(z) = \begin{cases} 0 & \text{for } z \neq \zeta , \\ \frac{1}{h^2} & \text{for } z = \zeta , \end{cases}$$

then the solution to (i) and (ii) becomes the discrete analog of the biharmonic Green's function for the region R. If we let $\Psi(z)$ in (6.2) be $-G(z, \zeta)$, then the discrete biharmonic Green's function $G_2(z, \zeta)$ satisfies

$$G_2(z, \zeta) = h^2 \sum_{z' \in R} G(z, z')\left[G(z', \zeta) - h^2 \sum_{z'' \in R} G(z'', \zeta)K_3(z', z'')\right] .$$

7. Applications to interpolation problems

In this last section we will let H denote one of the Hilbert spaces of harmonic functions H_1, H_2, or H_3 defined in Section 2. The space H will be the analogous space of discrete harmonic functions. Inner products will be $(\cdot, \cdot)$ and norms $\|\cdot\|$.

Let $z_1, z_2, \ldots, z_n$ be distinct points (different from ζ' in the case of H_2) belonging to the region R and let $w_1, w_2, \ldots, w_n$ be real numbers. By considering linear combinations of the harmonic polynomials $\operatorname{Re}(z^k)$, $k = 0, 1, \ldots, n$, and $\operatorname{Im}(z^k)$, $k = 1, 2, \ldots, n$, (suitably normalized in the case of H_1 and H_2) it is possible to construct a harmonic function p in H such that $p(z_j) = w_j$ for $j = 1, 2, \ldots, n$. Since the set of all functions u in H with $u(z_j) = w_j$, $j = 1, 2, \ldots, n$, is a nonempty closed convex subset of H, this set will contain a unique function with minimum norm.

THEOREM 7.1. *In the Hilbert space* H, *the unique function* u *with minimum norm satisfying* $u(z_j) = w_j$, $j = 1, 2, \ldots, n$, *has the form*

$$u(z) = \sum_{i=1}^{n} a_i k(z, z_i) .$$

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The determinant

$$\mathcal{D} = \begin{vmatrix} k(z_1, z_1) & \dots & k(z_1, z_n) \\ \vdots & & \vdots \\ k(z_n, z_1) & \dots & k(z_n, z_n) \end{vmatrix}$$

does not vanish and the constants a_i *are determined by the system of equations*

$$(7.1) \qquad \sum_{i=1}^{n} a_i k(z_j, z_i) = w_j , \quad j = 1, 2, \dots, n .$$

Proof. We follow the proof of a similar result in Epstein [7, p. 24] for a Hilbert space of analytic functions. Let S denote the closed subspace of linear combinations of $k(z, z_1), k(z, z_2), \dots, k(z, z_n)$. Let u be the solution of the stated minimum problem. Now u can be uniquely expressed in the form $u = v + w$, $v \in S$, $w \in S^{\perp}$, where $S^{\perp}$ is the orthogonal complement of S. Since w belongs to $S^{\perp}$ and $k(z, z_i)$ and v belong to S, we see that $w(z_i) = (w(z), k(z, z_i)) = 0$ for $i = 1, 2, \dots, n$, and $\|u\|^2 = \|v\|^2 + \|w\|^2$. Thus $\|v\| \le \|u\|$ and $v(z_i) = w_i$. Since u is the unique solution of the minimum problem, this implies that $u = v$ and hence u belongs to S. The function u can be written as

$$u(z) = \sum_{i=1}^{n} a_i k(z, z_i) .$$

The constants a_i must satisfy the system of equations (7.1) since this is equivalent to $u(z_j) = w_j$, $j = 1, 2, \dots, n$. The determinant $\mathcal{D}$ cannot vanish, for otherwise the rows of the matrix would be linearly dependent, which asserts that for certain choices of the w_i our minimum problem would be unsolvable. This contradicts the remarks preceding this theorem. Thus the constants a_i are determined by the system of equations.

The problem of solving an analogous minimal interpolation problem in the space H is quite different. If $z_1, z_2, \dots, z_n$ belong to R, then

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it is possible that for certain choices of $w_1, w_2, \ldots, w_n$ there is no function U in H satisfying $U(z_i) = w_i$, $i = 1, 2, \ldots, n$. Thus the problem would be unsolvable. There is a unique solution, for any choice of the w_i, if the determinant

$$D = \begin{vmatrix} K(z_1, z_1) & \ldots & K(z_1, z_n) \\ \vdots & & \vdots \\ K(z_n, z_1) & \ldots & K(z_n, z_n) \end{vmatrix}$$

does not vanish, for then there will be a unique function U of the form

$$U(z) = \sum_{i=1}^{n} A_i K(z, z_i)$$

satisfying $U(z_i) = w_i$, $i = 1, 2, \ldots, n$. As in the continuous case this function solves the minimum problem.

THEOREM 7.2. *Suppose* H *has a complete orthonormal system of functions harmonic on* $\bar{R}$. *Let* $z_1, z_2, \ldots, z_n$ *be points of* R *belonging to some h-net and let* $w_1, w_2, \ldots, w_n$ *be real numbers. If* h *is sufficiently small, there is a unique function* U *in* H *with minimum norm satisfying* $U(z_i) = w_i$, $i = 1, 2, \ldots, n$.. *Furthermore, as* $h \to 0$, *the function* U *converges uniformly on compact subsets of* R *to the unique function* u *in* H *with minimum norm satisfying* $u(z_i) = w_i$, $i = 1, 2, \ldots, n$.

Proof. From the previous results on the convergence of discrete harmonic kernels (Theorems 3.2, 4.1, and 6.1)

$$K(z_i, z_j) \to k(z_i, z_j)$$

as $h \to 0$ for $1 \le i$, $j \le n$. Thus $\lim_{h\to 0} D = \mathcal{D} \neq 0$ which implies that $D \neq 0$ for all h sufficiently small. Therefore the minimum problem in H has a unique solution. The solution U can be expressed in the form

$$U(z) = \sum_{i=1}^{n} A_i K(z, z_i)$$

with the A_i satisfying the system

$$\sum_{i=1}^{n} A_i K(z_j, z_i) = w_j , \quad j = 1, 2, \ldots, n .$$

Using Cramer's rule to solve for the A_i, we see that the solution U can be written as

$$U(z) = -\frac{1}{D}\begin{vmatrix} 0 & K(z, z_1) & \ldots & K(z, z_n) \\ w_1 & K(z_1, z_1) & \ldots & K(z_1, z_n) \\ \vdots & \vdots & & \vdots \\ w_n & K(z_n, z_1) & \ldots & K(z_n, z_n) \end{vmatrix} .$$

The function U converges uniformly on compact subsets of R to a function u defined by

$$u(z) = -\frac{1}{D}\begin{vmatrix} 0 & k(z, z_1) & \ldots & k(z, z_n) \\ w_1 & k(z_1, z_1) & \ldots & k(z_1, z_n) \\ \vdots & \vdots & & \vdots \\ w_n & k(z_n, z_1) & \ldots & k(z_n, z_n) \end{vmatrix} .$$

Upon solving for the coefficients in (7.1) we recognize u as the solution to our minimum problem in H.

It can now be concluded that the minimal interpolation problem in H can be solved provided h is small enough, and this solution approximates the solution of the analogous problem in H. The work of Chalmers [3] on reproducing kernels and minimum problems in subspaces of Hilbert spaces motivated the results in this section.

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