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Minimal vector lattice covers

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We show that each abelian l-group G is a large l-subgroup of a minimal vector lattice V and if G is archimedean then V is unique, in fact, V is the l-subspace of $(G^d)^\wedge$ that is generated by G, where G^d is the divisible hull of G and $(G^d)^\wedge$ is the Dedekind-MacNeille completion of G^d. If G is non-archimedean then V need not be unique, even if G is totally ordered.

Throughout this note group will always mean abelian group. An l-subgroup A of an l-group B is *large* if for each l-ideal $L \neq 0$ of B, $L \cap A \neq 0$ (or equivalently each l-homomorphism of B that is one to one on A is an isomorphism). In this case we shall also call B an *essential extension* of A.

We define U to be a *v-hull* of an l-group G if

i) U is a vector lattice and G is a large l-subgroup of U, and

ii) no proper l-subspace of U contains G.

Note that U contains a copy of G^d and since G^d is divisible and large in U it is dense in U (that is, $0 < u \in U$ implies $0 < g < u$ for some $g \in G^d$). Thus ([*1*], p. 116), the infinite joins and intersections that exist in G agree with those in U.

PROPOSITION. *Each l-group G admits a v-hull. Thus if G is an l-subgroup of a unique minimal vector lattice U then U must be a*

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v-hull of G .

Proof. G is a subdirect sum of o-groups and by the Hahn representation theorem, ([4], p. 59), each o-group can be embedded in a vector lattice of real valued functions. Thus we may assume that G is an l-subgroup of a vector lattice V . Let W be the intersection of all the l-subspaces of V that contain G . Let B be an l-ideal of W that is maximal with respect to $B \cap G = 0$. Then

$$G \cong (B \oplus G)/B \subseteq W/B$$

and this is an essential extension. For if U is an l-ideal of W that contains B and $U/B \cap (B \oplus G)/B = B$, then $B = U \cap (B + G) = B + (U \cap G)$ and so $U \cap G \subseteq B$. Thus $U \cap G = 0$ and $U \supseteq B$ and so $U = B$. Therefore W/B is a v-hull of $(B \oplus G)/B$.

THEOREM. *An archimedean l-group G admits a unique v-hull. This v-hull is (l-isomorphic to) the l-subspace of $(G^d)^\wedge$ that is generated by G and hence it is archimedean.*

Proof. Let U be a v-hull of G .

(1) U is archimedean. For suppose by way of contradiction that $0 < a, b \in U$ and $na < b$ for all integers $n > 0$. Since G is large in U there exists $0 < x \in G$ such that $x < ma$ for some $m > 0$. Since U is minimal,

$$U = \bigvee U(g) \quad \text{for all} \quad g \in G^+ ,$$

where $U(g)$ is the l-ideal of U that is generated by g . Thus $b = b_1 + \ldots + b_t$, where $b_i \in U(g_i)$ for $i = 1, \ldots, t$ and hence

$$b = |b_1 + \ldots + b_t| \leq |b_1| + \ldots + |b_t| .$$

Now there exists $k > 0$ such that $b_i \leq kg_i$ for $i = 1, \ldots, t$. Thus $b \leq k(g_1 + \ldots + g_t) = y \in G$ and hence

$$nx < nma < b \leq y \quad \text{for all} \quad n > 0 ;$$

but this contradicts the fact that G is archimedean.

(2) U is unique. Since U is divisible we may assume that G^d is

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an l-subgroup of U . Thus G^d is dense in U and hence, (see [2]), $(G^d)^\wedge$ is the l-ideal of $U^\wedge$ that is generated by G^d and so $(G^d)^\wedge$ is an l-subspace of $U^\wedge$ that contains G^d . Also since $U^\wedge$ is archimedean it follows that U is an l-subspace of $U^\wedge$, (see [3]). Thus

$$G \subseteq G^d \subseteq U \subseteq (G^d)^\wedge$$

and so U is the l-subspace of $(G^d)^\wedge$ that is generated by G and hence it is unique.

COROLLARY I. *If* U *and* V *are* v*-hulls of an archimedean* l*-group* G *then there exists a unique* l*-isomorphism* π *of* U *onto* V *such that* $g\pi = g$ *for all* $g \in G$.

Proof. This is true for G^d and for $(G^d)^\wedge$.

COROLLARY II. *If* G *is a subdirect sum of reals then its* v*-hull is contained in this sum of reals.*

Proof. If $G \subseteq \prod R_i$, then $(G^d)^\wedge \subseteq \prod R_i$, (see [2]).

If A is an l-subgroup of an l-group B then B is an *a-extension* of A if $L \to L \cap A$ is a one to one mapping of the l-ideals of B onto the l-ideals of A (or equivalently for each $0 < b \in B$ there exist $0 < a \in A$ and a positive integer n such that $a < nb$ and $b < na$).

PROPOSITION. *Each* v*-hull of an* o*-group* G *is an* o*-group, but it need not be an* a*-extension. An* o*-group* G *admits a* v*-hull that is an* a*-extension, but it need not be unique.*

Proof. If H is an essential extension of G , $0 < a, b \in H$ and $a \wedge b = 0$ then there is a positive integer n and $0 < x, y \in G$ such that $na > x$ and $nb > y$. Thus $0 = n(a \wedge b) = na \wedge nb \geq x \wedge y \geq 0$ which contradicts the fact that G is an o-group. Thus H is totally ordered and so each v-hull of G is an o-group.

Let $U = R \oplus R \oplus R$ lexicographically ordered from the left and let G be the subgroup of U generated by $(0, 0, 1)$, $(\pi, 1, 0)$ and $(1, \pi, 0)$. Then each subspace N of U that contains G must contain

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$\pi(\pi, 1, 0) - (1, \pi, 0) = (\pi^2-1, 0, 0)$ and hence $N = U$. Therefore U is a v-hull of G but clearly not an a-extension since U has two proper convex subgroups, but G has only one.

REMARKS. Note that

$$G \cong (\text{the subgroup of } R \text{ generated by } \pi \text{ and } 1) \oplus R$$

and so $R \oplus R$ is a v-hull and an a-extension of G. If we let $U = R \oplus R$ lexicographically ordered from the left and let G be the subgroup of U generated by $(\pi, 1)$ and $(1, \pi)$ then U is a minimal vector lattice containing G but it is not an essential extension and so not a v-hull. Also, G but not U is archimedean.

Now by Hahn's representation theorem each o-group G can be embedded in a vector lattice V that is an a-extension of G and hence the intersection of all the subspaces of V that contain G is a v-hull and an a-extension of G.

Finally let $V = \prod_{i=1}^{\infty} R_i$ lexicographically ordered from the left and let f be a group isomorphism of R onto $\prod_{i=2}^{\infty} R_i$ such that $f(1) = (1, 0, 0, \ldots)$ and in general $f(x) = \big(f_2(x), f_3(x), \ldots\big)$. Define

$$(x_1, x_2, \ldots)\tau = \big(x_1, x_2+f_2(x_1), x_3+f_3(x_1), \ldots\big) .$$

Then τ is an o-automorphism of V. For each $x \in V$ and $r \in R$ define

$$r*(x\tau) = (rx)\tau .$$

Then $(V, *)$ is a vector lattice. Note that

$$\begin{aligned} r*\big(x, f_2(x), f_3(x), \ldots\big) &= r*\big((x, 0, 0, \ldots)\tau\big) \\ &= (rx, 0, 0, \ldots)\tau \\ &= \big(rx, f_2(rx), f_3(rx), \ldots\big) . \end{aligned}$$

In particular, for $x = 1$ we have

$$r*(1, 1, \ldots) = \big(r, f_2(r), f_3(r), \ldots\big) .$$

Thus $(V, *)$ is a v-hull and an a-extension of the l-group $G = \sum_{i=1}^{\infty} R_i$ and G is also a vector lattice with respect to the natural scalar multiplication and so G is its own v-hull. But G and V are not o-isomorphic since V is a-closed (that is, admits no proper a-extensions) and G is not.

REMARK. If the chain of convex subgroups of an o-group G satisfies the DCC then the Hahn group corresponding to G is the unique v-hull of G that is also an a-extension.

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