On the Composition of Simultaneous Differential Systems of the First Order

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§ 1. Consider the system of n first order linear differential equations:

$$\frac{du_i}{dx} + \sum_{k=1}^{n} g_{ik}(x)\, u_k = 0 \qquad (i = 1, 2, \ldots . n),$$

together with the n boundary conditions

$$\sum_{k=1}^{n} a_{ik}\, u_k(a) + \sum_{k=1}^{n} b_{ik}\, u_k(b) = 0 \qquad (i = 1, 2, \ldots . n),$$

where a_{ij}, b_{ij} are constants and where we assume for simplicity of notation that $g_{ii} \equiv 0$.

We write this system of equations and boundary conditions in matrix notation thus:

$$\begin{bmatrix} d/dx & g_{12} & .. & g_{1n} \\ g_{21} & d/dx & .. & g_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ g_{n1} & g_{n2} & .. & d/dx \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ .. \\ u_n \end{bmatrix} = 0,$$

$$\begin{bmatrix} a_{11} & a_{12} & .. & a_{1n} \\ a_{21} & a_{22} & .. & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & .. & a_{nn} \end{bmatrix} \begin{bmatrix} u_1(a) \\ u_2(a) \\ \ldots \\ u_n(a) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & .. & b_{1n} \\ b_{21} & b_{22} & .. & b_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ b_{n1} & b_{n2} & .. & b_{nn} \end{bmatrix} \begin{bmatrix} u_1(b) \\ u_2(b) \\ \ldots \\ u_n(b) \end{bmatrix} = 0,$$

or with an obvious notation

$$PU = 0$$
$$W_a\, U(a) + W_b\, U(b) = 0. \tag{1}$$

It is the product of operator systems of the form

$$P$$
$$W_a + W_b \tag{2}$$

and

$$Q$$
$$Z_a + Z_b \tag{3}$$

that will be considered in this paper. It will be shown in § 2 that if we define the product of two such systems in the same manner as

J. M. Whittaker[1] defined it in the case $n = 1$, then the Green's matrix of the product system is the product by composition of the second kind of the Green's matrices of the component systems. In § 3 the product of a system into its adjoint system will be considered and it will be shown that the differential equations of the second order which appear in the product system can all be obtained from a quadratic form in the elements of the component systems in the same way that the Lagrange's equations in dynamics are obtained from the energy function. Lastly a relation is obtained between the solution of the non-homogeneous product system and the solutions of the component systems, when $n = 1$ and the order is no longer restricted to be the first.

§ 2. The Green's matrix for the system (1) is defined by Birkhoff[2] to be the matrix

$$G(x, \xi) \equiv \begin{bmatrix} G_{11}(x, \xi), & \dots, & G_{1n}(x, \xi) \\ G_{21}(x, \xi), & \dots, & G_{2n}(x, \xi) \\ \dots & \dots & \dots \\ G_{n1}(x, \xi), & \dots, & G_{nn}(x, \xi) \end{bmatrix},$$

where $G_{ij}(x, \xi)$, $i \neq j$ are continuous functions with continuous first derivatives, while $G_{ii}(x, \xi)$ each have only one discontinuity, a jump of amount 1, at $x = \xi$, *i.e.*

$$G(x, x - 0) - G(x, x + 0) = I,$$

where I is the unit matrix. Further, each column of the above matrix satisfies the system of equations except at $x = \xi$, and satisfies the boundary conditions identically in ξ.

Let $L(x, \xi)$, $M(x, \xi)$ be the Green's matrices thus defined for the systems (2) and (3) respectively; then the Green's function of

$$\begin{aligned} PQU &= 0, \\ Z_a U(a) + Z_b\, U(b) &= 0, \\ [W_a Q]\, U(a) + [W_b Q]\, U(b) &= 0, \end{aligned} \tag{4}$$

is given by

$$G(x, \xi) = \int_a^b M(x, \eta)\, L(\eta, \xi)\, d\eta. \tag{5}$$

For since

$$G_{ij}(x, \xi) = \int_a^b \sum_{k=1}^{n} M_{ik}(x, \eta)\, L_{kj}(\eta, \xi)\, d\eta$$

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[1] *Proc. Edin. Math. Soc.*, 3 (1932), 10; Dr Whittaker considers the case when the operator is of any order.

[2] Birkhoff and Langer, *Proc. Amer. Acad. Arts & Sc.*, 58 (1923), 59.

we see that $G_{ij}(x, \xi)$ is continuous. It is also seen from the nature of the discontinuities of $L_{ik}(x, \xi)$ and $M_{ik}(x, \xi)$ that G_{ij} has a continuous first derivative unless $x = \xi$ where G_{ii} has a jump of magnitude unity. Further,

$$QG(x, \xi) = \int_a^b \begin{bmatrix} d/dx & q_{12} & \dots & q_{1n} \\ q_{21} & d/dx & \dots & q_{2n} \\ \dots & \dots & \dots & \dots \\ q_{n1} & q_{n2} & \dots & d/dx \end{bmatrix} \begin{bmatrix} M_{11}(x, \eta) & \dots & M_{1n}(x, \eta) \\ M_{21}(x, \eta) & \dots & M_{2n}(x, \eta) \\ \dots & \dots & \dots \\ M_{n1}(x, \eta) & \dots & M_{nn}(x, \eta) \end{bmatrix}$$

$$\times \begin{bmatrix} L_{11}(\eta, \xi) & \dots & L_{1n}(\eta, \xi) \\ L_{21}(\eta, \xi) & \dots & L_{2n}(\eta, \xi) \\ \dots & \dots & \dots \\ L_{n1}(\eta, \xi) & \dots & L_{nn}(\eta, \xi) \end{bmatrix} d\eta$$

$$= \begin{bmatrix} L_{11}(x, \xi) & \dots & L_{1n}(x, \xi) \\ \dots & \dots & \dots \\ L_{n1}(x, \xi) & \dots & L_{nn}(x, \xi) \end{bmatrix},$$

since each column of $M(x, \xi)$ satisfies the system Q, the only discontinuities being along the principal diagonal; hence

$$PQG(x, \xi) = 0,$$

and $G(x, \xi)$ satisfies the boundary conditions of (4), and so is the Green's matrix for the product system.

§ 3. The adjoint to the system (1) is given by[1]

$$\begin{bmatrix} -d/dx & g_{21} & \dots & g_{n1} \\ g_{12} & -d/dx & \dots & g_{n2} \\ \dots & \dots & \dots & \dots \\ g_{1n} & g_{2n} & \dots & -d/dx \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix} = 0,$$

or

$$P' Y = 0$$

together with the boundary conditions

$$[y_1(a) \dots y_n(a)]\, W_a^{-1} + [y_1(b) \dots y_n(b)]\, W_b^{-1} = 0.$$

Then we state the following result which can easily be verified: *let*

$$Q(u) = \sum_{k=1}^{n} u_k'^2 + \sum_{i,r,k=1}^{n} g_{ik}\, g_{rk}\, u_i\, u_r - \sum_{i,r=1}^{n} g_{ri}\, u'_i\, u_r + \sum_{i,k=1}^{n} g_{ik}^2\, u_i^2,$$

where $g_{ss} = 0$ *and in the second summation* $i \neq r$; *then the differential equations of the system*

$$PP'Y = 0$$

[1] Birkhoff and Langer, *loc. cit.*, 64.

are given by

$$\frac{d}{dx}\left\{\frac{\partial Q}{\partial u_i'}\right\} - \frac{\partial Q}{\partial u_i} = 0 \qquad (i = 1, 2, \ldots . n).$$

This incidentally proves that *this system is self-adjoint.*[1]

§4. Let

$$\begin{aligned} &Py_1 = 0 \\ &[A_i y_1]_a + [B_i y_1]_b = 0 \qquad (i = 1, 2 \ldots . p), \end{aligned} \tag{6}$$

where P is a linear operator of order p, be an incompatible system. The solution of

$$\begin{aligned} &Qz = f(x) \\ &[C_i z]_a + [D_i z]_b = 0 \qquad (i = 1, 2, \ldots . 2), \end{aligned}$$

when the corresponding homogeneous system is incompatible, is given by

$$z(x) = \int_a^b M(x, \xi)\, f(\xi)\, d\xi.$$

Then we state the following result: *the solution of the product system*

$$\begin{aligned} &PQ\, u(x) = f(x) \\ &[A_i Qu]_a + [B_i Qu]_b = 0 \\ &[C_i u]_a \quad + [D_i u]_b \quad = 0 \end{aligned}$$

is given by

$$u(x) = \int_a^b M(x, \eta)\, y_1(\eta)\, d\eta$$

in terms of solutions of the component systems; where

$$y_1(x) = \int_a^b L(x, \zeta)\, f(\zeta)\, d\zeta$$

is the solution of the system

$$\begin{aligned} &Py_1 = f(x) \\ &[A_i y_1]_a + [B_i y_1]_b = 0 \qquad (i = 1, 2, \ldots, p). \end{aligned}$$

[1] See Hilbert, *Gött. Nach.* (1906), 474.

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