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SELFADJOINT METRICS ON ALMOST TANGENT MANIFOLDS WHOSE RIEMANNIAN CONNECTION IS ALMOST TANGENT

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1. Let M be a differentiable manifold of class C^∞, with a given (1, 1) tensor field J of constant rank such that $J^2=\lambda I$ (for some real constant λ). J defines a class of conjugate G-structures on M. For $\lambda>0$, one particular representative structure is an almost product structure. Almost complex structure arises when $\lambda<0$. If the rank of J is maximum and $\lambda=0$, then we obtain an almost tangent structure. In the last two cases the dimension of the manifold is necessarily even. A Riemannian metric S on M is said to be *related* if one of the conjugate structures defined by S has a common subordinate structure with the G-structure defined by J. It is said to be *J-metric* if the orthogonal structure defined by S has a common subordinate structure. On an almost complex manifold a metric is a J-metric if and only if it is Hermitian. A linear connection ∇ on M is a J-connection if

$$\nabla_X J = 0 \tag{1.1}$$

for all vector fields X in M, where ∇_X is the absolute derivation defined by ∇. It is known that the Hermitian connection defined by the Hermitian metric on a Hermitian manifold is almost complex if and only if the fundamental 2-form is closed.

In this paper we shall study a similar problem for the Riemannian connection of a selfadjoint metric (which is a related metric) on an almost tangent manifold. Since the Riemannian connection is symmetric and if

$$\nabla_X J = 0$$

then the Nijenhuis tensor is zero which implies that the almost tangent structure is integrable [2]. Hence a necessary condition for the Riemannian connection to be a J-connection (almost tangent) is that the almost tangent structure is integrable. First we shall find necessary and sufficient conditions for the Riemannian connection of a selfadjoint metric on an integrable almost tangent manifold to be almost tangent and then characterise such metrics on the tangent manifold.

2. A metric S on an almost tangent manifold M is called selfadjoint [1] if

$$(X \cdot JY) = (JX \cdot Y) \tag{2.1}$$

for all vector fields X, Y in M, where $(\cdot)$ is the inner product defined by S.

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The condition (1.1) can be expressed in terms of the following (0, 3) tensor field K defined by

(2.2) $$K(X, Y, Z) = ((\nabla_Y J)Z \cdot X) + ((\nabla_Z J)X \cdot Y) - ((\nabla_X J)Y \cdot Z)$$

where X, Y, Z are vector fields in M.

LEMMA 2.1. *∇ is an almost tangent connection if and only if $K=0$.*

Proof. The condition is clearly necessary. It is sufficient since from (2.2) we have

(2.3) $$2((\nabla_X J)Y \cdot Z) = K(Y, Z, X) + K(Z, X, Y)$$

for all X, Y, Z in M.

LEMMA 2.2. *Using the Riemannian connection of a selfadjoint metric S on an almost tangent manifold we get*

(2.4) $$K(X, Y, Z) = JX(Y \cdot Z) - X(Y \cdot JZ) + ((J[X, Y] - [JX, Y]) \cdot Z) + (([Z, JX] - J[Z, X]) \cdot Y)$$

Proof. Since ∇ is the Riemannian connection

$$2((\nabla_X Y) \cdot Z) = X(Y \cdot Z) + Y(X \cdot Z) - Z(X \cdot Y) + ([X, Y], Z) + ([Z, X] \cdot Y) - ([Y, Z] \cdot X)$$

and since S is selfadjoint

$$(X \cdot JY) = (JX \cdot Y).$$

These two relations together with

$$((\nabla_X J)Y \cdot Z) = ((\nabla_X JY) \cdot Z) - (J(\nabla_X Y) \cdot Z)$$

lead to (2.4).

LEMMA 2.3. [1]. *A metric S on an almost tangent manifold M is selfadjoint if and only if the value of S on every adapted moving frame has the form*

(2.5) $$\begin{bmatrix} T & Q \\ Q & 0 \end{bmatrix}$$

where Q, T are symmetric $n \times n$ matrices and $\det Q \neq 0$.

Using above lemmas and the integrability being a necessary condition we obtain.

THEOREM 2.1. *S is a selfadjoint metric on an integrable almost tangent manifold M of dimension $2n$. If the components of S, relative to the moving frame σ associated with an adapted chart x, are of the form (2.5) then the Riemannian connection ∇ of S is almost tangent if and only if*

(2.6) $$\frac{\partial Q_{ab}}{\partial x^{c+n}} = 0$$

(2.7) $$\frac{\partial T_{ab}}{\partial x^{c+n}} = \frac{\partial Q_{ab}}{\partial x^{c}}$$

$(a, b, c = 1, \ldots, n)$.

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Proof. From lemma 2.2 it follows that the value of the tensor field K on σ is zero if and only if, for $a, b, c=1,\ldots,n$

$$J\frac{\partial}{\partial x^c}\left(\frac{\partial}{\partial x^a}\cdot\frac{\partial}{\partial x^{b+n}}\right)=0$$

and

$$J\frac{\partial}{\partial x^c}\left(\frac{\partial}{\partial x^a}\cdot\frac{\partial}{\partial x^b}\right)=\frac{\partial}{\partial x^c}\left(\frac{\partial}{\partial x^a}\cdot\frac{\partial}{\partial x^{b+n}}\right).$$

Therefore the tensor field K is zero if and only if (2.6) and (2.7) hold.

The tangent manifold $T\mathcal{M}$ of a differentiable manifold $\mathcal{M}$ admits an integrable almost tangent structure in a natural way. Let Π be the natural projection $\Pi: T\mathcal{M}\to\mathcal{M}$ which takes a vector at the point $m\in\mathcal{M}$ to the point m. Corresponding to any chart x on a neighbourhood U of a point $m\in\mathcal{M}$ we can define a standard chart on $\Pi^{-1}U$ which we denote by (x, y). Let g be a Riemannian metric on $\mathcal{M}$ with components g_{ab} relative to the chart x defined on a coordinate neighbourhood U. Then the complete lift g^c of g to $T\mathcal{M}$ with components

$$\begin{bmatrix}\frac{\partial g_{ab}}{\partial x^d}y^d & g_{ab}\\ g_{ab} & 0\end{bmatrix} \tag{2.8}$$

on $\Pi^{-1}U$ relative to the chart (x, y) is a selfadjoint metric on $T\mathcal{M}$ [2, 3]. Let h be a symmetric $(0, 2)$ tensor field on $\mathcal{M}$ whose components relative to a chart x are h_{ab}. The vertical lift h^v of h is a symmetric $(0, 2)$ tensor field on $T\mathcal{M}$ with components

$$\begin{bmatrix}h_{ab} & 0\\ 0 & 0\end{bmatrix} \tag{2.9}$$

relative to the standard chart (x, y) on $T\mathcal{M}$ [3].

THEOREM 2.2. *Let S be a selfadjoint metric on $T\mathcal{M}$. A necessary and sufficient condition for the Riemannian connection ∇ of S to be almost tangent is that*

$$S = g^c+h^v \tag{2.10}$$

where g is a Riemannian metric on $\mathcal{M}$ and h is a symmetric $(0, 2)$ *tensor field on $\mathcal{M}$.*

Proof. If S is of the form (2.10) then the value of S on the natural moving frame associated with a chart (x, y) on $T\mathcal{M}$ is

$$\begin{bmatrix}\frac{\partial g_{ab}}{\partial x^c}y^c+h_{ab} & g_{ab}\\ g_{ab} & 0\end{bmatrix}$$

It is easy to see that the conditions of theorem 2.1 are satisfied. Hence the Riemannian connection of S is almost tangent.

Suppose that S is a selfadjoint metric on $T\mathcal{M}$ whose Riemannian connection ∇ is almost tangent. Let (x, y), $(\bar{x}, \bar{y})$ be charts on $T\mathcal{M}$ with intersecting domains and σ, $\bar{\sigma}$ the associated moving frames. If

$$\text{So } \sigma = \begin{bmatrix} T & Q \\ Q & 0 \end{bmatrix} \quad \text{and} \quad \text{So } \bar{\sigma} = \begin{bmatrix} \bar{T} & \bar{Q} \\ \bar{Q} & 0 \end{bmatrix}$$

then

$$\left.\begin{aligned} \bar{Q} &= A'QA \\ \bar{T} &= A'TA+B'QA+A'QB \end{aligned}\right] \tag{2.11}$$

where

$$A = \left[\frac{\partial x^a}{\partial \bar{x}^b}\right], \qquad B = \left[\frac{\partial^2 x^a}{\partial \bar{x}^b\, \partial \bar{x}^c}\, \bar{y}^c\right],$$

and A' denotes the transpose of the matrix A. The conditions of theorem 2.1 imply that Q_{ab} are functions of x's only and that

$$T_{ab} = \frac{\partial Q_{ab}}{\partial x^c}\, y^c + H_{ab}(x^1, \ldots, x^n)$$

There exist functions g_{ab} and h_{ab} on $\mathcal{M}$ such that $g_{ab} \circ \Pi = Q_{ab}$, $h_{ab} \circ \Pi = H_{ab}$. Using equations (2.11) it can be shown that functions g_{ab}, h_{ab} are components of symmetric (0, 2) tensor fields. Since Q is non-singular so is $g = [g_{ab}]$. Therefore g is a Riemannian metric on $\mathcal{M}$. The value of S on σ is of the required form (2.10).

3. **Remark.** The previous technique can be applied to other G-structures. For example suppose that S is a positive-definite almost product metric on an integrable almost product manifold. The Riemannian connection of S is almost product if and only if its components relative to any adapted chart have the form

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

where A depends on $x_1, \ldots, x_r$ and B depends on $x_{r+1}, \ldots, x_n$.

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