Bull. Aust. Math. Soc. **94** (2016), 217–223
doi:10.1017/S0004972716000101

ABSOLUTELY ABNORMAL AND CONTINUED FRACTION NORMAL NUMBERS

JOSEPH VANDEHEY

(Received 1 December 2015; accepted 10 December 2015; first published online 16 March 2016)

Abstract

In this short note, we give a proof, conditional on the generalised Riemann hypothesis, that there exist numbers x which are normal with respect to the continued fraction expansion but not to any base-b expansion. This partially answers a question of Bugeaud.

2010 *Mathematics subject classification*: primary 11K16; secondary 11J70, 11A07.

Keywords and phrases: continued fractions, normal numbers, Artin's conjecture.

1. Introduction

A number $x \in [0,1)$ with base-b expansion $0.a_1a_2a_3\ldots$ is said to be normal to base b if for every finite string of digits $s = (d_1, d_2, \ldots, d_k)$ with $d_i \in \{0,1,2,\ldots,b-1\}$,

$$\lim_{n\to\infty} \frac{A_{s,b}(n;x)}{n} = \frac{1}{b^k}, \tag{1.1}$$

where $A_{s,b}(n;x)$ denotes the number of times s appears in the string $(a_1, a_2, \ldots, a_n)$. Similarly, a number $x \in [0,1)$ with continued fraction expansion

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} = [a_1, a_2, a_3, \ldots], \quad a_i \in \mathbb{N}, \tag{1.2}$$

is said to be normal with respect to the continued fraction expansion (or CF-normal) if for every finite string of positive integer digits $s = (d_1, d_2, \ldots, d_k)$,

$$\lim_{n\to\infty} \frac{B_s(n;x)}{n} = \mu(C_s),$$

where $B_s(n;x)$ denotes the number of times s appears in the string $(a_1, a_2, \ldots, a_n)$, μ denotes the Gauss measure $\mu(A) = \int_A dx/(1+x)\log 2$ and C_s denotes the cylinder set corresponding to s, the set of numbers in $[0,1)$ whose first k continued fraction digits are the string s.

The author acknowledges assistance from the Research and Training Group grant DMS-1344994 funded by the National Science Foundation.

If we let $\mathcal{N}_b \subset [0,1)$ denote the set of base-b normal numbers and let $\mathcal{M} \subset [0,1)$ denote the set of CF-normal numbers, then by standard ergodic techniques all the sets $\mathcal{N}_b$, $b \geq 2$, and $\mathcal{M}$ have full Lebesgue measure. Despite the size of these sets, exhibiting any normal number is quite difficult. It is not known if π is normal to any base b or if it is CF-normal. All explicit examples of normal numbers (such as Champernowne's constant $0.12345678910111213\ldots$ [5], which is normal to base 10, or the Adler–Keane–Smorodinsky CF-normal number [1]) were constructed to be normal. See [3] for a deeper discussion and further examples.

Our interest in this paper is understanding how the sets $\mathcal{N}_b$, $b \geq 2$, and $\mathcal{M}$ relate to one another. Steinhaus asked whether being normal to base b implied normality for all other bases [15]. Maxfield showed that if $b, c \geq 2$ are positive integers such that $b^r = c$ for some rational number r, then $\mathcal{N}_b = \mathcal{N}_c$ [11]. On the other hand, if $b^r = c$ and r is not rational, Cassels and Schmidt showed that $\mathcal{N}_b \setminus \mathcal{N}_c$ contains uncountably many points [4, 14]. In fact, Schmidt showed that given any two sets of positive integers S and T, the set of numbers in $\bigcup_{b\in S} \mathcal{N}_b \setminus \bigcup_{b\in T} \mathcal{N}_b$ is uncountably infinite, unless prevented by Maxfield's theorem.

Even fewer nontrivial examples of constructions of such selectively normal numbers exist. Bailey and Borwein showed that some of the Stoneham constants, which were constructed to be normal to a given base b, are not normal to certain other bases [2]. Martin gave an explicit construction of an irrational number that is not in *any* of the $\mathcal{N}_b$—such a number is said to be absolutely abnormal, as it is not normal to any base [10].

Less is known about CF-normal numbers. Analogous definitions of normality exist for other types of continued fraction expansions besides the regular continued fraction expansion used in (1.2): for some of these expansions, it is known that the set of normal numbers for these expansions equals $\mathcal{M}$ (see, for instance, Kraaikamp and Nakada [7] and the author [16]).

In a similar vein to Steinhaus's question, Bugeaud has asked the following in the above terminology [3, Problem 10.51]: does there exist a $b \geq 2$ such that $\mathcal{M} \setminus \mathcal{N}_b$ is nonempty—that is, do there exist CF-normal numbers which are not normal to a given base b? Furthermore, is $\mathcal{M} \setminus \bigcup_{b=2}^{\infty} \mathcal{N}_b$ nonempty—that is, do there exist CF-normal but absolutely abnormal numbers? As with Cassels and Schmidt, we will show that the answer is yes, but at present we can only offer a conditional yes.

We will in fact prove a slightly stronger result. We say that a number is simply normal to base b if (1.1) holds for all strings s that consist of a single digit.

Theorem 1.1. *On the generalised Riemann hypothesis, the set of numbers that are CF-normal but not even simply normal to any base $b \geq 2$ is dense and uncountable. In particular, $\mathcal{M} \setminus \bigcup_{b=2}^{\infty} \mathcal{N}_b$ is nonempty.*

We will use standard asymptotic notation in this paper. By $f(x) = O(g(x))$, we mean that there exists a constant C such that $|f(x)| \leq C|g(x)|$. By $f(x) = o(g(x))$, we mean that $\lim_{x\to\infty} f(x)/g(x) = 0$.

2. Some additional facts

2.1. On continued fractions. Let $x = [a_1, a_2, a_3, \ldots]$. We let p_n/q_n denote the lowest-terms expression for the truncated expansion $[a_1, a_2, \ldots, a_n]$ and refer to this as the nth convergent. When it is not clear which number q_n refers to, we will state it as $q_n(x)$. The following facts can be found in most standard texts dealing with continued fractions, such as [6].

The q_n satisfy a recurrence relation: with $q_{-1} = 0$ and $q_0 = 1$,

$$q_{n+1} = a_{n+1}q_n + q_{n-1}, \quad n = 0, 1, 2, 3, \ldots.$$

Further, q_n and q_{n+1} are relatively prime for $n \geq 0$. If x is irrational, then its continued fraction expansion is infinite and unique. If x is rational, then its continued fraction expansion is finite and not unique: namely, one expansion is $[a_1, a_2, \ldots, a_n]$ with $a_n \geq 1$, and the other is $[a_1, a_2, \ldots, a_n - 1, 1]$. However, the only finite expansions we will consider are convergents and so this nonuniqueness will not be a concern in this paper.

We can estimate how well the convergents approximate x. In particular,

$$\left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{a_{n+1}q_n^2}. \tag{2.1}$$

Moreover, assuming that x is irrational, $x - p_n/q_n$ is positive if and only if n is even.

If we let $x = [a_1, a_2, \ldots, a_n]$ and $s = (a_1, a_2, \ldots, a_n)$, then the cylinder set C_s consists of the interval between x and $[a_1, a_2, a_3, \ldots, a_n + 1]$. (We will not care about whether end points are included.) Moreover, the length of the cylinder set decays to 0 as n tends to infinity.

2.2. On Artin's conjecture. We reference the following result of Lenstra [8, Theorem 8.3], which examines Artin's conjecture along primes in an arithmetic progression.

Theorem 2.1. *Assume the generalised Riemann hypothesis. Let g, f, a be positive integers and let $P_{g,f,a}$ denote the set of primes p such that g is a primitive root modulo p and $p \equiv a \pmod{f}$. Let d be the discriminant of $\mathbb{Q}(\sqrt{g})$.*

The set $P_{g,f,a}$ is finite if and only if one of the following conditions is true:

(1) *there exists a prime q such that $q | f$, $a \equiv 1 \pmod{q}$ and g is a perfect qth power;*
(2) *we have $d | f$ and $(d/a) = 1$; or*
(3) *we have $d | 3f$, $3 | d$, $((-d/3)/a) = -1$ and g is a perfect cube.*

In the statement of the theorem, (d/a) is the standard Kronecker symbol. Conditional asymptotics on the size of $P_{g,f,a}$ can be found in the work of Moree [12, 13].

From the theorem we get the following simple corollary.

Corollary 2.2. *Assume the generalised Riemann hypothesis. Let g, f, a be positive integers such that f is relatively prime to both g and $a - 1$, and $g \geq 2$ is not a perfect square. Then $P_{g,f,a}$ is infinite.*

https://doi.org/10.1017/S0004972716000101 Published online by Cambridge University Press

Proof. Let g' be the largest squarefree divisor of g. First note that, by standard facts about discriminants,

$$d = \begin{cases} g' & \text{if } g' \equiv 1 \pmod 4, \\ 4g' & \text{otherwise,} \end{cases}$$

where here again d is the discriminant of $\mathbb{Q}(\sqrt{g})$. By our assumptions, $g' \geq 2$, so we must have $d \geq 2$ and d shares a prime in common with g.

Since f is relatively prime to $a-1$, then, for any prime $q|f$, we have that $a \not\equiv 1 \pmod q$, so the first condition of Theorem 2.1 fails. Since f is relatively prime to g and d shares a prime in common with g, we cannot have $d|f$ and hence the second condition fails. By a similar argument, if $d|3f$ and $3|d$, then we must have $d = 3$; however, this is impossible since d is either 0 or 1 modulo 4. Hence, the third condition fails as well and so $P_{g,f,a}$ must be infinite. □

3. Proof of the theorem

Consider any $x \in [0, 1)$ that is CF-normal. Let $x = [a_1, a_2, a_3, \ldots]$. Let N be some large even integer and let $\{X_i\}_{i=1}^{\infty}$ be the sequence of strings given by

$$X_1 = (a_1, a_2, \ldots, a_N), \quad X_i = (a_{2^{i-2}N+1}, a_{2^{i-2}N+2}, \ldots, a_{2^{i-1}N}) \quad \text{for } i \geq 2,$$

so that x could be represented as the concatenation of the strings X_i. We will abuse notation slightly and allow ourselves to write $x = [X_1, X_2, X_3, \ldots]$.

Consider a sequence of positive integers $L = \{\ell_1, \ell_2, \ell_3, \ldots\}$ and create a new continued fraction expansion

$$y = [X_1, \ell_1, \ell_2, \ell_3, \ell_4, X_2, \ell_5, \ell_6, \ell_7, \ell_8, X_3, \ell_9, \ell_{10}, \ell_{11}, \ell_{12}, X_4, \ldots] = [a'_1, a'_2, a'_3, \ldots]. \tag{3.1}$$

We will let $q'_n = q_n(y)$ be the denominators of the convergents of y.

It is easy to see that $B_s(n; x)$ will differ from $B_s(n; y)$ by at most $O(\log_2 n) = o(n)$ and thus y will be CF-normal regardless of what L is. However, we claim that we can choose L so that y will be absolutely abnormal. Since x and N were arbitrary and x and y both belong to the cylinder set C_s where $s = (a_1, a_2, \ldots, a_N)$, this would also give us the density statement of the theorem. Moreover, we will show that for any fixed x and N, there are uncountably infinite possibilities for L such that y is absolutely abnormal: if x and N are fixed, then each distinct choice of L gives a distinct value of y, thus giving the uncountability statement of the theorem.

(If we instead wanted to prove a theorem about numbers that are absolutely abnormal and also *not* CF-normal, then we could insert a sufficiently long string of ones after each X_i. The rest of the construction would be unaltered.)

Let us illustrate how we intend to construct an L that will leave y absolutely abnormal. We will assume throughout the rest of this proof that x and N are fixed.

Let $\{b_1, b_2, b_3, \ldots\}$ be a sequence of integers $b_i \geq 2$ which are not perfect powers and such that each such integer occurs infinitely often. For example, we could take $\{2, 2, 3, 2, 3, 5, 2, 3, 5, 6, \ldots\}$. Let $n_i := 2^{i-1}N + 4(i-1)$, so that a'_{n_i} is the digit of y that

https://doi.org/10.1017/S0004972716000101 Published online by Cambridge University Press

occurs at the end of the string X_i. This digit will then be followed by $a'_{n_i+j} = \ell_{4(i-1)+j}$ for $j = 1, 2, 3, 4$. Our choice of $\ell_{4(i-1)+1}$, $\ell_{4(i-1)+2}$ and $\ell_{4(i-1)+3}$ will then be made so that q'_{n_i+3} is a power of b_i; that is, the $(n_i + 3)$th convergent to y will have a finite base-b_i expansion. Let $r_i \in \mathbb{Q}$ denote this $(n_i + 3)$th convergent of y and let k_i denote the number of digits in its finite base-b_i expansion.

Since we have assumed that N is even, $n_i + 3$ is always odd and, hence, for any choice of L, $y - r_i$ is negative. Consider r_i in its nonterminating base-b_i expansion, that is, the one that ends on the digit $b_i - 1$ repeating. In particular, all digits after the k_ith digit are equal to $b_i - 1$. By (2.1), if ℓ_{4i} is sufficiently large—in particular, if ℓ_{4i} exceeds $b_i^{k_i^2}$—then the base-b_i expansions of y and r_i only start to differ after the (k_i^2)th place. (Since $y < r_i$, this was why we considered the nonterminating expansion for r_i.)

Suppose that we can find L such that all of the r_i are rationals whose denominator is a power of b_i and that ℓ_{4i} is always sufficiently large, as defined in the previous paragraph. Then, for any base $b \geq 2$ that is not a perfect power, there are infinitely many b_i such that $b_i = b$. For each such i, we know that at most k_i of the first k_i^2 digits in the base-b expansion of y differ from $b - 1$. Thus, if we let $s = b - 1$, then

$$\limsup_{n\to\infty} \frac{A_{s,b}(n; y)}{n} = 1.$$

So, y cannot be simply normal to base b for any such base.

For the remaining bases, we note that any perfect power $c \geq 2$ can be represented as a power of a nonperfect power $b \geq 2$. Suppose that $c = b^r$ with $r \in \mathbb{N}$. The digit $c - 1$ will appear in the nth place of the base-c expansion of a number if and only if the string $(b - 1, b - 1, \dots, b - 1)$ of r digits equal to $b - 1$ appears starting in the $((n - 1)r + 1)$th place of the base-b expansion of the number. Thus, for any i such that $b_i = b$, at most $\lceil k_i/r\rceil$ of the first $\lfloor k_i^2/r\rfloor$ digits in the base-c expansion will not equal $c - 1$. Since there are infinitely many such i, y is not simply normal to base c.

Thus, y is not simply normal for any base $b \geq 2$.

It remains to show that we can find not just one, but uncountably many L with the desired properties. We will do this by showing that regardless of what $\{\ell_1, \ell_2, \dots, \ell_{4(i-1)}\}$ are, we can choose $\ell_{4(i-1)+1}, \ell_{4(i-1)+2}, \ell_{4(i-1)+3}$ so that r_i has a denominator that is a power of b_i. Thus, we can construct uncountably many L with the desired properties by first finding a desired ℓ_1, ℓ_2, ℓ_3, then noting that we have countably many possibilities for ℓ_4, namely all the values that are sufficiently large; and, for each choice of ℓ_4, we can find a desired ℓ_5, ℓ_6, ℓ_7, and then have countably many possibilities for ℓ_8; and so on.

Fix i for now, and suppose that $\ell_1, \ell_2, \dots, \ell_{4(i-1)}$ have already been chosen. We will let q'_n refer to the denominator of the rational number $[a'_1, a'_2, \dots, a'_n]$, where the a'_j are as in (3.1), presuming we have chosen the values of enough ℓ_j for this definition to make sense.

Since q'_n and q'_{n+1} are always relatively prime for any n, we know that for any prime p, there exists an $\ell \in \mathbb{N}$ such that

$$\ell q'_{n_i} + q'_{n_i-1} \not\equiv 0 \pmod{p}.$$

https://doi.org/10.1017/S0004972716000101 Published online by Cambridge University Press

By the Chinese remainder theorem, we can thus find an $\ell \in \mathbb{N}$ such that $\ell q'_{n_i} + q'_{n_i-1}$ is relatively prime to both b_i and $q'_{n_i} - 1$. We let $\ell_{4(i-1)+1} = \ell$, so that q'_{n_i+1} is now relatively prime to b_i and $q'_{n_i} - 1$.

By our assumption that the generalised Riemann hypothesis is true, together with Corollary 2.2 and our choice of q'_{n_i+1}, there exist infinitely many primes congruent to q'_{n_i} modulo q'_{n_i+1} for which b_i is a primitive root. We choose $\ell_{4(i-1)+2} \in \mathbb{N}$, so that $q'_{n_i+2} = \ell_{4(i-1)+2} q'_{n_i+1} + q'_{n_i}$ is one of these primes.

Now, since q'_{n_i+2} is a prime for which b_i is a primitive root and since q'_{n_i+1} and q'_{n_i+2} are relatively prime, the following equation has a solution:

$$q'_{n_i+1} \equiv b_i^k \pmod{q'_{n_i+2}}.$$

We may suppose, without loss of generality, that k is a sufficiently large integer so that $b_i^k > 2q'_{n_i+2}$. Then we choose $\ell_{4(i-1)+3}$ so that $b_i^k = q_{n_i+3} = \ell_{4(i-1)+3} q_{n_i+2} + q_{n_i+1}$. But this shows that q_{n_i+3} is a power of b_i as desired and the proof is complete.

4. Further questions

It would be nice if we could give an unconditional proof of Theorem 1.1. It would be especially nice if such a proof came as a result of a proof of the generalised Riemann hypothesis, but, given the difficulty of such a problem, we propose the following question, which could lead to an unconditional result. Let $b \geq 2$ be fixed. For any rational number $p/q = [a_1, a_2, \ldots, a_n]$, does there exist a finite string $s = (a'_1, a'_2, \ldots, a'_m)$ of length $m = o(n)$ such that the rational number $p'/q' = [a_1, a_2, \ldots, a_n, a'_1, a'_2, \ldots, a'_m]$ has a denominator that is a power of b? Can one replace $o(n)$ with $O(1)$?

We showed, assuming the generalised Riemann hypothesis, that $\mathcal{M} \setminus \bigcup_{b=2}^{\infty} \mathcal{N}_b$ is uncountable and dense. Mance communicated the following question: what is the Hausdorff dimension of this set? Mance has studied the dimensional analysis of such difference sets in other cases, most notably for Q-Cantor series, and found that these difference sets often have full Hausdorff dimension [9].

Acknowledgements

The author would like to thank Kevin Ford, Greg Martin and Paul Pollack for their helpful comments.

References

[1] R. Adler, M. Keane and M. Smorodinsky, 'A construction of a normal number for the continued fraction transformation', *J. Number Theory* **13**(1) (1981), 95–105.

[2] D. H. Bailey and J. M. Borwein, 'Nonnormality of Stoneham constants', *Ramanujan J.* **29**(1–3) (2012), 409–422.

[3] Y. Bugeaud, *Distribution Modulo One and Diophantine Approximation*, Cambridge Tracts in Mathematics, 193 (Cambridge University Press, Cambridge, 2012).

[4] J. W. S. Cassels, 'On a problem of Steinhaus about normal numbers', *Colloq. Math.* **7** (1959), 95–101.

[5] D. G. Champernowne, 'The construction of decimals normal in the scale of ten', *J. Lond. Math. Soc. (2)* **8** (1933), 254–260.
[6] K. Dajani and C. Kraaikamp, *Ergodic Theory of Numbers*, Carus Mathematical Monographs, 29 (Mathematical Association of America, Washington, DC, 2002).
[7] C. Kraaikamp and H. Nakada, 'On normal numbers for continued fractions', *Ergod. Th. & Dynam. Sys.* **20**(5) (2000), 1405–1421.
[8] H. W. Lenstra Jr, 'On Artin's conjecture and Euclid's algorithm in global fields', *Invent. Math.* **42** (1977), 201–224.
[9] B. Mance, 'Number theoretic applications of a class of Cantor series fractal functions. I', *Acta Math. Hungar.* **144**(2) (2014), 449–493.
[10] G. Martin, 'Absolutely abnormal numbers', *Amer. Math. Monthly* **108**(8) (2001), 746–754.
[11] J. E. Maxfield, 'Normal k-tuples', *Pacific J. Math.* **3** (1953), 189–196.
[12] P. Moree, 'On primes in arithmetic progression having a prescribed primitive root', *J. Number Theory* **78**(1) (1999), 85–98.
[13] P. Moree, 'On primes in arithmetic progression having a prescribed primitive root. II', *Funct. Approx. Comment. Math.* **39**(1) (2008), 133–144.
[14] W. M. Schmidt, 'Über die Normalität von Zahlen zu verschiedenen Basen', *Acta Arith.* **7** (1961–1962), 299–309.
[15] H. Steinhaus, 'Problem 144', in: *The New Scottish Book: Wrocław 1946–1958*, http://www.wmi.uni.wroc.pl/New_Scottish_Book.
[16] J. Vandehey, 'On the joint normality of certain digit expansions', Preprint, 2014, arXiv:1408.0435.

JOSEPH VANDEHEY, Department of Mathematics, University of Georgia, Athens, GA 30602, USA
e-mail: vandehey@uga.edu