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PAIRWISE COMPLETE REGULARITY AS A SEPARATION AXIOM

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Abstract

Focussing on complete regularity, we discuss the separation properties of bitopological spaces. The unifying concept is that of separation by a pair of bases $(\mathscr{B}_1, \mathscr{B}_2)$ for the closed sets of a bitopological space $(S, \mathscr{T}_1, \mathscr{T}_2)$. For various separation properties a characterization is presented in terms of separation by a pair of closed bases. This is extended to results concerning pairs of subbases. Here the notion of screening by pairs of subbases plays a central role and the characterization of complete regularity in a natural way fits in between those of regularity and normality. In the key lemma the relation with quasi-proximities is exhibited.

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1. Introduction

The axiom of complete regularity for topological spaces in its usual form, dealing with the existence of enough real-valued functions on a space, looks quite different from the other separation axioms. It is hardly surprising that one has been looking for characterizations of complete regularity which naturally fit in between the axiom of regularity and that of normality. See, for example, [3], [4], [17] and [18]. It seems that the paper [8] has been unnoticed for a long time [2].

A similar characterization of pairwise complete regularity had been presented not long after the notion of bitopological spaces was introduced [7] and

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the corresponding separation axiom was defined [10]. Starting with Steiner's result [18], Seagrove [16] gives an internal characterization of pairwise complete regularity by means of a pairwise normal, pairwise separating pair of families of closed sets.

We shall not repeat here the definitions of the pairwise separation properties, but rather present characterizations of these properties in Sections 2 and 3. This is done in such a way that the characterization of pairwise complete regularity naturally fits in between those of pairwise regularity and pairwise normality. We shall only discuss the pairwise separation properties, leaving to the reader the natural generalization for the non-symmetric case. The unifying concept is that of a pair of bases or subbases for the closed sets of a bitopological space $(X,\mathscr{T}_1,\mathscr{T}_2)$.

Recall that a collection $\mathscr{B}$ of closed subsets of a topological space $(X,\mathscr{T})$ is called a *base for the closed sets of* (X,T), or a *closed base of* $(X,\mathscr{T})$, if each closed set is the intersection of members of $\mathscr{B}$. A collection $\mathscr{S}$ of closed subsets of $(X,\mathscr{T})$ is said to be a *subbase for the closed sets of* $(X,\mathscr{T})$, or a *closed subbase of* $(X,\mathscr{T})$, if the collection $\mathscr{S}'$ of all finite unions of members of $\mathscr{S}$ is a base for the closed sets of $(X,\mathscr{T})$. All bases and subbases considered in this paper are closed bases and closed subbases respectively. If, for $i = 1,2$, $\mathscr{B}_i$ and $\mathscr{S}_i$ are a closed base and a closed subbase of $(X,\mathscr{T}_i)$ respectively, we shall say that $(\mathscr{B}_1,\mathscr{B}_2)$ and $(\mathscr{S}_1,\mathscr{S}_2)$ are a pair of closed bases and closed subbases respectively for the bitopological space $(X,\mathscr{T}_1,\mathscr{T}_2)$.

Now it is to be noted that Steiner's separating family of closed sets is a basis for the closed sets and, analogously, Seagrove's pairwise separating pair is a pair of closed bases for a bitopological space. This enables us to exhibit the strong interrelation between the aforementioned characterizations of complete regularity for the topological case and to extend this to the bitopological case using pairs of closed bases. This is carried out in Section 2.

In Section 3 we generalize the results to subbases. The methods of the proofs employed in this section are an extension of the techniques in [1] and are new even for the topological case.

2. Results for bases

Here we present the characterizations of the pairwise separation properties using pairs of closed bases. The proofs of the propositions are easy and therefore omitted.

DEFINITION 1. Let $(X,\mathscr{T}_1,\mathscr{T}_2)$ be a bitopological space. A pair (C,D) of (disjoint) subsets of X is *separated by the pair of closed bases* $(\mathscr{B}_1,\mathscr{B}_2)$ of $(X,\mathscr{T}_1,\mathscr{T}_2)$ if there exist $E \in \mathscr{B}_1$, $F \in \mathscr{B}_2$ such that $E \cup F = X$ and $D \cap E =$

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$\varnothing = C \cap F$ (and, consequently, $C \subset E$ and $D \subset F$). In this case we also say that the pair (E, F) is a *$(\mathscr{B}_1, \mathscr{B}_2)$-separation of the pair* (C, D).

DEFINITION 2 (cf. [13, Definition 6.9]). A bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is *pairwise base-R_0 with respect to the pair of closed bases* $(\mathscr{B}_1, \mathscr{B}_2)$ if for every $C \in \mathscr{B}_i$ and each $x \notin C$, there is a $D \in \mathscr{B}_j$ such that $x \in D$ and $C \cap D = \varnothing$, $i, j \in \{1, 2\}$ and $i \neq j$.

PROPOSITION 1. *A bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise R_0 if and only if there exists a pair $(\mathscr{B}_1, \mathscr{B}_2)$ of closed bases such that $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise base-R_0 with respect to $(\mathscr{B}_1, \mathscr{B}_2)$.*

In [16] there is defined the notion of a pairwise separating pair of families of closed sets. The next proposition provides an alternative definition.

PROPOSITION 2. *Suppose $(X, \mathscr{T}_1, \mathscr{T}_2)$ is a bitopological space. Suppose $\mathscr{F}$ is a family of $\mathscr{T}_1$-closed sets and $\mathscr{G}$ is a family of $\mathscr{T}_2$-closed sets. Then the pair $(\mathscr{F}, \mathscr{G})$ is a pairwise separating pair of families of closed sets if and only if it is a pair of closed bases such that $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise base-R_0 with respect to $(\mathscr{F}, \mathscr{G})$.*

DEFINITION 3 (cf. [15]). A bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is *pairwise base-R_1 with respect to the pair of closed bases* $(\mathscr{B}_1, \mathscr{B}_2)$ if for every pair of distinct points $x, y \in X$ the following properties (i) and (ii) hold.

(i) Whenever there is a $C \in \mathscr{B}_1$ such that $x \notin C$ and $y \in C$, there exists a $(\mathscr{B}_1, \mathscr{B}_2)$-separation of the pair $(\{y\}, \{x\})$.
(ii) Whenever there is a $D \in \mathscr{B}_2$ such that $x \notin D$ and $y \in D$, there exists a $(\mathscr{B}_1, \mathscr{B}_2)$-separation of the pair $(\{x\}, \{y\})$.

DEFINITION 4 (cf. [7]). A bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is *pairwise base-Hausdorff with respect to the pair of closed bases* $(\mathscr{B}_1, \mathscr{B}_2)$ if for any two distinct points $x, y \in X$ there exist a $(\mathscr{B}_1, \mathscr{B}_2)$-separation of the pair $(\{x\}, \{y\})$ as well as a $(\mathscr{B}_1, \mathscr{B}_2)$-separation of the pair $(\{y\}, \{x\})$.

PROPOSITION 3. *A bitopological space is pairwise R_1 [respectively pairwise Hausdorff] if and only if it is pairwise base-R_1 [respectively pairwise base-Hausdorff] with respect to every pair of closed bases.*

DEFINITION 5 (cf. [7]). A bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is *pairwise base-regular with respect to the pair of closed bases* $(\mathscr{B}_1, \mathscr{B}_2)$ if for each pair $(C, \{x\})$ with $x \notin C \in \mathscr{B}_1$ as well as for each pair $(\{y\}, D)$ with $y \notin D \in \mathscr{B}_2$ there are $(\mathscr{B}_1, \mathscr{B}_2)$-separations.

PROPOSITION 4. *A bitopological space $(X,\mathcal{T}_1,\mathcal{T}_2)$ is pairwise regular if and only if there exists a pair $(\mathcal{B}_1,\mathcal{B}_2)$ of bases such that $(X,\mathcal{T}_1,\mathcal{T}_2)$ is pairwise base-regular with respect to $(\mathcal{B}_1,\mathcal{B}_2)$.*

It is to be observed that a pairwise regular [respectively pairwise R_0] bitopological space $(X,\mathcal{T}_1,\mathcal{T}_2)$ may have pairs of bases $(\mathcal{B}_1,\mathcal{B}_2)$ such that $(X,\mathcal{T}_1,\mathcal{T}_2)$ fails to be pairwise base-regular [respectively pairwise base-R_0] with respect to $(\mathcal{B}_1,\mathcal{B}_2)$. In view of Proposition 3 this is in contrast with the situation for pairwise Hausdorff and pairwise R_1. Suppose, for example, that X is a set consisting of three points and that both $\mathcal{T}_1$ and $\mathcal{T}_2$ are the discrete topology. If, for $i = 1,2$, $\mathcal{B}_i$ is the base consisting of the two-point subsets of X, then $(X,\mathcal{T}_1,\mathcal{T}_2)$ is not pairwise base-regular, and a fortiori not pairwise base-R_0 with respect to $(\mathcal{B}_1,\mathcal{B}_2)$. This example is essentially from [4]. In the same vein various examples from [4] can be adapted to the bitopological case.

DEFINITION 6 (cf. [7]). A bitopological space $(X,\mathcal{T}_1,\mathcal{T}_2)$ is *pairwise base-normal with respect to the pair of closed bases* $(\mathcal{B}_1,\mathcal{B}_2)$ if for each $C \in \mathcal{B}_1$ and each $D \in \mathcal{B}_2$ with $C \cap D = \varnothing$ there exists a $(\mathcal{B}_1,\mathcal{B}_2)$-separation of the pair (C,D); that is, there exist $E \in \mathcal{B}_1$ and $F \in \mathcal{B}_2$ such that $C \subset E$, $D \subset F$, $E \cup F = X$ and $C \cap F = \varnothing = D \cap E$.

PROPOSITION 5. *A bitopological space $(X,\mathcal{T}_1,\mathcal{T}_2)$ is pairwise normal if and only if it is pairwise base-normal with respect to the pair of all closed sets in $\mathcal{T}_1$ on the one hand and all closed sets in $\mathcal{T}_2$ on the other hand.*

Now to see how complete regularity fits in, let us first recall the definitions. As usual, $\mathcal{L}$ and $\mathcal{R}$ denote the left hand and the right hand topology on $\mathbf{R}$; the families of all closed sets of $\mathcal{L}$ and $\mathcal{R}$ are $\{[c,\infty) | c \in \mathbf{R}\} \cup \{\varnothing, \mathbf{R}\}$ and $\{(-\infty,d] | d \in \mathbf{R}\} \cup \{\varnothing,\mathbf{R}\}$ respectively.

DEFINITION 7 [10]. A bitopological space $(X,\mathcal{T}_1,\mathcal{T}_2)$ is *pairwise completely regular* if (i) and (ii) hold.

(i) For every $\mathcal{T}_1$-closed set C and each $x \notin C$, there is a pairwise continuous real-valued function

$$f: (X,\mathcal{T}_1,\mathcal{T}_2) \to (\mathbf{R},\mathcal{L},\mathcal{R}) \quad \text{such that } f(C) = 1 \text{ and } f(x) = 0.$$

(ii) For every $\mathcal{T}_2$-closed set D and each $y \notin D$, there is a pairwise continuous real-valued function

$$g: (X,\mathcal{T}_1,\mathcal{T}_2) \to (\mathbf{R},\mathcal{L},\mathcal{R}) \quad \text{such that } g(D) = 0 \text{ and } g(y) = 1.$$

THEOREM 1. *A bitopological space $(X,\mathscr{T}_1,\mathscr{T}_2)$ is pairwise completely regular if and only if there is a pair $(\mathscr{B}_1,\mathscr{B}_2)$ of closed bases such that*

(i) *$(X,\mathscr{T}_1,\mathscr{T}_2)$ is pairwise base-R_0 with respect to $(\mathscr{B}_1,\mathscr{B}_2)$,*
(ii) *$(X,\mathscr{T}_1,\mathscr{T}_2)$ is pairwise base-normal with respect to $(\mathscr{B}_1,\mathscr{B}_2)$.*

PROOF. Using Proposition 2 one can see that Theorem 1 is essentially [16, Theorem 3.1]. For later use we observe that the proof of Theorem 1 boils down to constructing for any non-empty $R \in \mathscr{B}_1$ and any non-empty $S \in \mathscr{B}_2$ with $R \cap S = \varnothing$ a pairwise continuous function $f\colon (X,\mathscr{T}_1,\mathscr{T}_2) \to (\mathbf{R},\mathscr{L},\mathscr{R})$ with $f(R) = 1$ and $f(S) = 0$. This construction is very similar to the proof of the Urysohn lemma in [6] (cf. [7, Theorem 2.7] and [16, Theorem 3.1]). Suppose R and S are as described. Let D be the set of all numbers in $[0,1]$ of the form $p \cdot 2^{-q}$, where q is a positive integer and p is an integer with $0 \le p \le 2^q$. Let $S_0 = S$, $R_0 = X$, $S_1 = X$ and $R_1 = R$. For $t \in D \cap (0,1)$ write $t = (2m+1)2^{-n}$ and, inductively on n, let (R_t, S_t) be a $(\mathscr{B}_1,\mathscr{B}_2)$-separation of $R_{(2m+2)2^{-n}}$ and $S_{2m \cdot 2^{-n}}$. This can be done, because $(X,\mathscr{T}_1,\mathscr{T}_2)$ is pairwise base-normal with respect to $(\mathscr{B}_1,\mathscr{B}_2)$. Now the rest of the construction is straightforward.

The next theorem, which is easily seen to be equivalent to Theorem 1, shows how naturally pairwise complete regularity fits between pairwise regularity and pairwise normality.

THEOREM 2. *A bitopological space $(X,\mathscr{T}_1,\mathscr{T}_2)$ is pairwise completely regular if and only if there is a pair $(\mathscr{B}_1,\mathscr{B}_2)$ of closed bases such that*

(i) *$(X,\mathscr{T}_1,\mathscr{T}_2)$ is pairwise base-regular with respect to $(\mathscr{B}_1,\mathscr{B}_2)$,*
(ii) *$(X,\mathscr{T}_1,\mathscr{T}_2)$ is pairwise base-normal with respect to $(\mathscr{B}_1,\mathscr{B}_2)$.*

It is to be noted that in Theorem 1 "pairwise base-R_0" cannot be replaced by "pairwise base-R_1". See [4, Example 3].

3. Results for subbases

In this section we extend the results of the previous section to subbases. In general it can be said that the proofs of the theorems are somewhat more delicate.

The first step is the generalization of the notion of separation.

DEFINITION 8. Let $(X,\mathscr{T}_1,\mathscr{T}_2)$ be a bitopological space. A pair (C_1,C_2) of (disjoint) subsets of X is *screened by the pair $(\mathscr{S}_1,\mathscr{S}_2)$ of closed subbases* of $(X,\mathscr{T}_1,\mathscr{T}_2)$ if there exists a finite subfamily $\mathscr{F}_i$ of $\mathscr{S}_i$, $i = 1,2$, such that

$\mathscr{F}_1 \cup \mathscr{F}_2$ covers X and $C_i \cap G = \varnothing$ for all $G \in \mathscr{F}_j$, $i, j \in \{1,2\}$ and $i \neq j$. We also say that $(\mathscr{F}_1, \mathscr{F}_2)$ is an *$(\mathscr{S}_1, \mathscr{S}_2)$-screening of the pair* (C_1, C_2).

It is to be observed that in the situation described in Definition 8 every member of $\mathscr{F}_1 \cup \mathscr{F}_2$ meets at most one of the sets C_1 and C_2.

The generalizations of the results concerning the R_0, R_1 and Hausdorff separation axioms are straightforward.

DEFINITION 9. A bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is *pairwise subbase-R_0 with respect to the pair of closed subbases* $(\mathscr{S}_1, \mathscr{S}_2)$ if for each $S \in \mathscr{S}_i$ and every $x \notin S$ there exists a $T \in \mathscr{S}_j$ such that $x \in T$ and $S \cap T = \varnothing$ for $i, j = 1, 2$ and $i \neq j$.

In the topological case, for $\mathscr{S}_1 = \mathscr{S}_2$, the subbase $\mathscr{S}_1$ is said to be a *T_1-subbase* if the space is pairwise subbase-R_0 with respect to the pair $(\mathscr{S}_1, \mathscr{S}_2)$; T_1-subbases arise in a natural way in compactification theory (see, for example, [12] and [19]).

PROPOSITION 6. *A bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise R_0 if and only if there exists a pair $(\mathscr{S}_1, \mathscr{S}_2)$ of subbases such that $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise subbase-R_0 with respect to $(\mathscr{S}_1, \mathscr{S}_2)$.*

DEFINITION 10. A bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is *pairwise subbase-R_1 with respect to the pair of closed subbases* $(\mathscr{S}_1, \mathscr{S}_2)$ if for every pair $(C, \{x\})$ with $x \notin C \in \mathscr{S}_1$ as well as for every pair $(\{y\}, D)$ with $y \notin D \in \mathscr{S}_2$, there are $(\mathscr{S}_1, \mathscr{S}_2)$-screenings of all pairs $(\{w\}, \{x\})$ and $(\{y\}, \{z\})$ where $w \in C$ and $z \in D$.

DEFINITION 11. A bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is *pairwise subbase-Hausdorff with respect to the pair of closed subbases* $(\mathscr{S}_1, \mathscr{S}_2)$ if for every two distinct points $x, y \in X$ there is an $(\mathscr{S}_1, \mathscr{S}_2)$-screening of the pair $(\{x\}, \{y\})$ as well as an $(\mathscr{S}_1, \mathscr{S}_2)$-screening of the pair $(\{y\}, \{x\})$.

PROPOSITION 7. *A bitopological space is pairwise R_1 [respectively pairwise Hausdorff] if and only if it is pairwise subbase-R_1 [respectively pairwise subbase-Hausdorff] with respect to every pair of closed subbases.*

DEFINITION 12. A bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is *pairwise subbase-regular with respect to the pair of closed subbases* $(\mathscr{S}_1, \mathscr{S}_2)$ if for every pair $(C, \{x\})$ with $x \notin C \in \mathscr{S}_1$ as well as for every pair $(\{y\}, D)$ with $y \notin D \in \mathscr{S}_2$ there are $(\mathscr{S}_1, \mathscr{S}_2)$-screenings.

The following theorem deals with a subbase characterization of pairwise regularity. Because every separation is a screening, the "only if" part follows from Proposition 4.

THEOREM 3. *A bitopological space $(X,\mathcal{T}_1,\mathcal{T}_2)$ is pairwise regular if and only if there exists a pair $(\mathcal{S}_1,\mathcal{S}_2)$ of closed subbases such that $(X,\mathcal{T}_1,\mathcal{T}_2)$ is pairwise subbase-regular with respect to $(\mathcal{S}_1,\mathcal{S}_2)$.*

PROOF. As has just been observed, only the "if" part requires a proof. Let C be a $\mathcal{T}_1$-closed set and $x \notin C$. As $\mathcal{S}_1$ is a closed subbase, there are $S_1,\ldots,S_n$ in $\mathcal{S}_1$ such that $C \subset S_1 \cup \cdots \cup S_n$ and $x \notin S_1 \cup \cdots \cup S_n$. Now for each $i \in \{1,\ldots,n\}$ there is an $(\mathcal{S}_1,\mathcal{S}_2)$-screening $(\mathcal{F}_{i1},\mathcal{F}_{i2})$ of $(S_i,\{x\})$. Write $E_i = \bigcup \mathcal{F}_{i1}$ and $F_i = \bigcup \mathcal{F}_{i2}$, $i = 1,\ldots,n$. Then for $i = 1,\ldots,n$, E_i is closed in $\mathcal{T}_1$, F_i is closed in $\mathcal{T}_2$, $E_i \cap \{x\} = \varnothing$, $F_i \cap S_i = \varnothing$ and $E_i \cup F_i = X$.

Now let $E = \bigcup\{E_i | i = 1,\ldots,n\}$ and $F = \bigcap\{F_i | i = 1,\ldots,n\}$. It follows that E is closed in $\mathcal{T}_1$, F is closed in $\mathcal{T}_2$, $x \notin E$, $F \cap C = \varnothing$ and $E \cup F = X$. The other case, $y \notin D$ and D a $\mathcal{T}_2$-closed set, is treated similarly.

DEFINITION 13. A bitopological space $(X,\mathcal{T}_1,\mathcal{T}_2)$ is *pairwise subbase-normal with respect to the pair of closed subbases* $(\mathcal{S}_1,\mathcal{S}_2)$ if for each $C \in \mathcal{S}_1$ and $D \in \mathcal{S}_2$ with $C \cap D = \varnothing$, there exists an $(\mathcal{S}_1,\mathcal{S}_2)$-screening of the pair (C,D).

Now we come to the main result of the paper. Also for the topological case, $(\mathcal{S}_1 = \mathcal{S}_2)$, the method of the proof is new.

THEOREM 4. *A bitopological space $(X,\mathcal{T}_1,\mathcal{T}_2)$ is pairwise completely regular if and only if there are closed subbases $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively such that*

(i) $(X,\mathcal{T}_1,\mathcal{T}_2)$ is pairwise subbase-R_0 with respect to $(\mathcal{S}_1,\mathcal{S}_2)$,
(ii) $(X,\mathcal{T}_1,\mathcal{T}_2)$ is pairwise subbase-normal with respect to $(\mathcal{S}_1,\mathcal{S}_2)$.

As in the base case there is the following variant of Theorem 4.

THEOREM 5. *A bitopological space $(X,\mathcal{T}_1,\mathcal{T}_2)$ is pairwise completely regular if and only if there are closed subbases $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively such that*

(i) $(X,\mathcal{T}_1,\mathcal{T}_2)$ is pairwise subbase-regular with respect to $(\mathcal{S}_1,\mathcal{S}_2)$,
(ii) $(X,\mathcal{T}_1,\mathcal{T}_2)$ is pairwise subbase-normal with respect to $(\mathcal{S}_1,\mathcal{S}_2)$.

PROOF OF THEOREM 4. As has already been observed the "only if" part requires no proof. For the proof of the "if" part we introduce the following notations. For $i = 1,2$, $\mathcal{S}_i^*$ is the collection of all finite unions of all finite intersections of $\mathcal{S}_i$. Let C and D be any subsets of X. We write $(C,D) \in \Delta$ and say that (C,D) is *disjoint modulo* $(\mathcal{S}_1,\mathcal{S}_2)$ if there are $S \in \mathcal{S}_1$ and $T \in \mathcal{S}_2$ such that $C \subset S$, $D \subset T$ and $S \cap T = \varnothing$. We write $(C,D) \in \Sigma$ and say that

(C,D) is *discernible modulo* $(\mathscr{S}_1,\mathscr{S}_2)$ if there exist C_i, $i = 1,\ldots,k$, and D_j, $j = 1,\ldots,l$, such that $C = \bigcup\{C_i|i = 1,\ldots,k\}$, $D = \bigcup\{D_j|j = 1,\ldots,l\}$ and $(C_i,D_j) \in \Delta$, $i = 1,\ldots,k$, $j = 1,\ldots,l$. It is to be observed that for every $S \in \mathscr{S}_1$ and every $x \notin S$, $(S,\{x\}) \in \Delta$, because of the pairwise subbase R_0-property. It follows that for every G which is closed in $\mathscr{T}_1$ and every $x \notin G$, $(G,\{x\}) \in \Sigma$. Similarly, $(\{y\},H) \in \Sigma$ for every H which is closed in $\mathscr{T}_2$ and every $y \notin H$. Thus in order to prove (i) and (ii) of Definition 7 it is sufficient to construct for any non-empty $R \subset X$ and any non-empty $S \subset X$ with $(R,S) \in \Sigma$ a pairwise continuous function $f\colon (X,\mathscr{T}_1,\mathscr{T}_2) \to (\mathbf{R},\mathscr{L},\mathscr{R})$ with $f(R) = 1$ and $f(S) = 0$. The construction follows the pattern which has been outlined in the proof of Theorem 1.

Let R and S be as mentioned and let $D \subset \mathbf{R}$ be as in the proof of Theorem 1. Let $S_0 = S$, $R_0 = X$, $S_1 = X$ and $R_1 = R$. For $t \in D \cap (0,1)$ write $t = (2m+1)2^{-n}$ and, inductively on n, let (R_t,S_t) be a pair of sets such that

$$R_{(2m+2)\cdot 2^{-n}} \subset R_t \in \mathscr{S}_1^*, \quad S_{2m\cdot 2^{-n}} \subset S_t \in \mathscr{S}_2^*, \quad R_t \cup S_t = X \quad \text{and}$$
$$(R_t, S_{2m\cdot 2^{-n}}) \in \Sigma, \quad (R_{(2m+2)\cdot 2^{-n}}, S_t) \in \Sigma.$$

The existence of the pair (R_t,S_t) follows from the lemma below. It is to be noted that $\mathscr{S}_1^*$ and $\mathscr{S}_2^*$ are closed bases for $\mathscr{T}_1$ and $\mathscr{T}_2$ respectively.

LEMMA. *For $i = 1,2$, let $\mathscr{S}_i^*$ be the collection of all finite unions of all finite intersections of $\mathscr{S}_i$. Suppose the relation Σ is defined as in the proof of Theorem 4. Suppose $(C,D) \in \Sigma$. Then there are subsets G and H of X such that $C \subset G \in \mathscr{S}_1^*$, $D \subset H \in \mathscr{S}_2^*$, $G \cup H = X$, $(G,D) \in \Sigma$ and $(C,H) \in \Sigma$.*

PROOF. As $(C,D) \in \Sigma$, there are C_i, $i = 1,\ldots,k$, and D_j, $j = 1,\ldots,l$, such that $(C_i,D_j) \in \Delta$ for $i = 1,\ldots,k$, $j = 1,\ldots,l$ (The relation Δ has been defined in the beginning of the proof of Theorem 4.) Then for $i = 1,\ldots,k$, $j = 1,\ldots,l$, select $S_{ij} \in \mathscr{S}_1$ and $T_{ij} \in \mathscr{S}_2$ such that $C_i \subset S_{ij}$, $D_j \subset T_{ij}$ and $S_{ij} \cap T_{ij} = \varnothing$. Also let $\mathscr{V}_{ij}$ be an $(\mathscr{S}_1,\mathscr{S}_2)$-screening of (S_{ij},T_{ij}). Now write $\mathscr{V} = \bigwedge\{\mathscr{V}_{ij}|i = 1,\ldots,k;\ j = 1,\ldots,l\}$, the collection of all sets of the form $\bigcap\{V_{ij}|i = 1,\ldots,k,\ j = 1,\ldots,l;\ V_{ij} \in \mathscr{V}_{ij}\}$.

Furthermore we write
$\mathscr{E} = \{V \in \mathscr{V}|(V,D_j) \in \Delta \text{ for } j = 1,\ldots,l\}$,
$\mathscr{F} = \{V \in \mathscr{V}|(C_i,V) \in \Delta \text{ for } i = 1,\ldots,k\}$,
$E = \bigcup\{V|V \in \mathscr{E}\}$ and $F = \bigcup\{V|V \in \mathscr{F}\}$.

First we show that $\mathscr{V} = \mathscr{E} \cup \mathscr{F}$ by deriving a contradiction from the assumption $\mathscr{V} \neq \mathscr{E} \cup \mathscr{F}$.

Assume $V \in \mathscr{V}$ and $V \notin \mathscr{E} \cup \mathscr{F}$. Then for some $i \in \{1,\ldots,k\}$ and for some $j \in \{1,\ldots,l\}$, we have $(C_i,V) \notin \Delta$ and $(V,D_j) \notin \Delta$. Then a fortiori $(S_{ij},V) \notin \Delta$ and $(V,T_{ij}) \notin \Delta$. Choose any V_{ij} such that $V \subset V_{ij} \in \mathscr{V}_{ij}$. It follows that $S_{ij} \cap V_{ij} \neq \varnothing$ and $V_{ij} \cap T_{ij} \neq \varnothing$. This contradicts the fact that $\mathscr{V}_{ij}$ is a screening of (S_{ij}, T_{ij}). Thus we get $E \cup F = X$, $(E, D) \in \Sigma$ and $(C, F) \in \Sigma$ (and, consequently, $C \subset E$ and $D \subset F$). Now we proceed as follows. As $(E, D) \in \Sigma$, there exist E'_i, $i = 1, \ldots, m$, and D'_j, $j = 1, \ldots, n$, such that $E = \bigcup\{E'_i | i = 1, \ldots, m\}$, $D = \bigcup\{D'_j | j = 1, \ldots, n\}$ and $(E'_i, D'_j) \in \Delta$ for all i and j. For $i = 1, \ldots, m$ and $j = 1, \ldots, n$ choose S'_{ij} and T'_{ij} such that $E'_i \subset S'_{ij} \in \mathscr{S}_1$, $D'_j \subset T'_{ij} \in \mathscr{S}_2$ and $S'_{ij} \cap T'_{ij} = \varnothing$. For $i = 1, \ldots, m$ write $G_i = \bigcap\{S'_{ij} | j = 1, \ldots, n\}$. Then $G_i \in \mathscr{S}_1^*$ and $(G_i, D'_j) \in \Delta$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Now let $G = \bigcup\{G_i | i = 1, \ldots, m\}$. Then $E \subset G \in \mathscr{S}_1^*$ and $(G, D) \in \Sigma$. In a similar way, F can be enlarged to a set $H \in \mathscr{S}_2^*$ such that $(C, H) \in \Sigma$. Obviously $G \cup H = X$.

The properties of Σ, derived thus far, suggest that Σ is related to a not necessarily symmetric proximity relation on X. That is the content of the following proposition. We refer to Lane's paper [11] for the definition of *quasi-proximity*. For ideas along these lines in the topological case see [1] and [5].

PROPOSITION 8. *Suppose $(X, \mathscr{T}_1, \mathscr{T}_2)$ is a bitopological space. Suppose $\mathscr{S}_1$ and $\mathscr{S}_2$ are subbases of $\mathscr{T}_1$ and $\mathscr{T}_2$ respectively such that $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise subbase-regular as well as pairwise subbase-normal with respect to $(\mathscr{S}_1, \mathscr{S}_2)$. We assume that $\{\varnothing, X\} \subset \mathscr{S}_1 \cap \mathscr{S}_2$. Suppose the relation Σ is defined as in the proof of Theorem 4. Suppose the relation δ on the power set of X is defined as follows: $(A, B) \in \delta$ if and only if $(B, A) \notin \Sigma$. Then*

(i) *δ is a quasi-proximity on X,*
(ii) *the corresponding bitopological space for (X, δ) is $(X, \mathscr{T}_1, \mathscr{T}_2)$*

PROOF. The major part of the proof directly follows from the properties of Σ which have been established already. The lemma, for example, can be restated as follows.

If $(A, B) \notin \delta$, then there are $B_1 \in \mathscr{S}_1^*$ and $A_1 \in \mathscr{S}_2^*$ such that $A_1 \cup B_1 = X$, $(A_1, B) \notin \delta$, $(A, B_1) \notin \delta$. This is one of the axioms for a quasi-proximity δ. From the assumption $\{\varnothing, X\} \subset \mathscr{S}_1 \cap \mathscr{S}_2$ it may be deduced that $(X, \varnothing) \notin \delta$ and $(\varnothing, X) \notin \delta$. Obviously, if $A \cap B \neq \varnothing$, then $(A, B) \in \delta$.

In order to show that δ is a quasi-proximity, what remains to prove is that for all subsets A, B and C of X, $(A, B \cup C) \in \delta$ if and only if $(A, B) \in \delta$ or $(A, C) \in \delta$ and $(A \cup B, C) \in \delta$ if and only if $(A, C) \in \delta$ or $(B, C) \in \delta$.

As these two statements are similar, we only prove the first. That is, we show that its contraposition holds true: $(B \cup C, A) \in \Sigma$ if and only if $(B, A) \in \Sigma$ and $(C, A) \in \Sigma$.

If $(B \cup C, A) \in \Sigma$, then there are D_i, $i = 1, \ldots, k$, and A_j, $j = 1, \ldots, l$, such that $B \cup C = \bigcup\{D_i | i = 1, \ldots, k\}$, $A = \bigcup\{A_j | j = 1, \ldots, l\}$ and $(D_i, A_j) \in \Delta$ for $i = 1, \ldots, k$, $j = 1, \ldots, l$. Then $(D_i \cap B, A_j) \in \Delta$ for $i = 1, \ldots, k$, $j = 1, \ldots, l$.

As $B = \bigcup\{D_i \cap B | i = 1, \ldots, k\}$, we get $(B, A) \in \Sigma$. Similarly, $(C, A) \in \Sigma$.

Conversely, if $(B, A) \in \Sigma$ and $(C, A) \in \Sigma$, then there are B_i, $i = 1, \ldots, k$, A_j, $j = 1, \ldots, l$, C_m, $m = 1, \ldots, p$, and A'_n, $n = 1, \ldots, q$, such that

$$A = \bigcup\{A_j | j = 1, \ldots, l\} = \bigcup\{A'_n | n = 1, \ldots, q\},$$
$$B = \bigcup\{B_i | i = 1, \ldots, k\},$$
$$C = \bigcup\{C_m | m = 1, \ldots, p\},$$
$$(B_i, A_j) \in \Delta \quad \text{for } i = 1, \ldots, k,\ j = 1, \ldots, l,$$

and

$$(C_m, A'_n) \in \Delta \quad \text{for } m = 1, \ldots, p,\ n = 1, \ldots, q.$$

Then $(B_i, A_j \cap A'_n) \in \Delta$ for $i = 1, \ldots, k$, $j = 1, \ldots, l$, $n = 1, \ldots, q$, and $(C_m, A_j \cap A'_n) \in \Delta$ for $m = 1, \ldots, p$, $j = 1, \ldots, l$, $n = 1, \ldots, q$.

As $A = \bigcup\{A_j \cap A'_n | j = 1, \ldots, l,\ n = 1, \ldots, q\}$ we see that $(B \cup C, A) \in \Sigma$. Thus statement (1) has been proved.

The corresponding bitopological space $(X, \mathcal{T}_1', \mathcal{T}_2')$ for (X, δ) is defined by the closure operators

$$\mathrm{cl}_{\mathcal{T}_1'} A = \{x \in X | (\{x\}, A) \in \delta\}$$

and

$$\mathrm{cl}_{\mathcal{T}_2'} A = \{x \in X | (A, \{x\}) \in \delta\}.$$

We shall show that $\mathcal{T}_1 = \mathcal{T}_1'$. That $\mathcal{T}_2 = \mathcal{T}_2'$ can be proved in a similar way. Let G be a $\mathcal{T}_1$-closed subset of X. If $x \notin G$, then $(G, \{x\}) \in \Sigma$, as we have seen in the proof of Theorem 4. It follows that $(\{x\}, G) \notin \delta$ and $x \notin \mathrm{cl}_{\mathcal{T}_1'} G$. We have proved that G is $\mathcal{T}_1'$-closed.

Conversely, let G be a $\mathcal{T}_1'$-closed subset of X. If $x \notin G$, then $(\{x\}, G) \notin \delta$, that is, $(G, \{x\}) \in \Sigma$. From the lemma it follows that there is a $G' \in \mathcal{S}_1^*$ such that $G \subset G'$ and $x \notin G'$. We see that G is the intersection of members of $\mathcal{S}_1^*$. Thus G is $\mathcal{T}_1$-closed.

It is to be observed that the "if" part of Theorem 4 also can be derived from Proposition 8 above and [11, Theorem 3.3]. With the topological case in mind it can hardly be surprising that pairwise complete regularity of a space can be proved in various ways. The main feature of this paper is that with a technical device, as presented in the lemma, the original Urysohn approach can be used.

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