Glasgow Math. J. **63** (2021) 640–650.
doi:10.1017/S0017089520000415.

FINITE GROUPS WITH THE SAME JOIN GRAPH AS A FINITE NILPOTENT GROUP

ANDREA LUCCHINI

Dipartimento di Matematica "Tullio Levi-Civita", Università degli Studi di Padova, Via Trieste 63, 35121 Padova, Italy
e-mail: lucchini@math.unipd.it

(Received 30 March 2020; revised 22 June 2020; accepted 24 July 2020; first published online 17 August 2020)

Abstract. Given a finite group G, we denote by $\Delta(G)$ the graph whose vertices are the proper subgroups of G and in which two vertices H and K are joined by an edge if and only if $G = \langle H, K\rangle$. We prove that if there exists a finite nilpotent group X with $\Delta(G) \cong \Delta(X)$, then G is supersoluble.

2010 *Mathematics Subject Classification*. 20D60, 20D30

1. Introduction. Let G be a finite group. We define a graph $\Delta(G)$ as follows. The vertices of $\Delta(G)$ are the proper subgroups of G. Two vertices H and K are joined by an edge if G is generated by H and K, that is, $G = \langle H, K\rangle$. This graph was introduced in [**1**] and is called the join graph of G. We have slightly modified the original definition, including in the vertex set the subgroups of G contained in the Frattini subgroup $\mathrm{Frat}(G)$ of G. They correspond to isolated vertices of $\Delta(G)$. In particular, $\Delta(G)$ contains no edge if G is cyclic of prime-power order.

A typical question that arises whenever a graph is associated with a group is the following:

Question 1. How similar are the structures of two finite groups G_1 and G_2 if the graphs $\Delta(G_1)$ and $\Delta(G_2)$ are isomorphic?

We will say that a subgroup H of G is a maximal intersection in G if there exists a family $M_1, \dots, M_t$ of maximal subgroups of G with $H = M_1 \cap \cdots \cap M_t$. Let $\mathcal{M}(G)$ be the subposet of the subgroup lattice of G consisting of G and all the maximal intersections in G. Notice that $\mathcal{M}(G)$ is a lattice in which the meet of two elements H and K coincides with their intersection and their join is the smallest maximal intersection in G containing $\langle H, K\rangle$ (in general $\langle H, K\rangle$ is not a maximal intersection, see the example at the end of Section 2). The maximum element of $\mathcal{M}(G)$ is G, and the minimum element coincides with the Frattini subgroup $\mathrm{Frat}(G)$ of G. The role played by $\mathcal{M}(G)$ in investigating the property of the graph $\Delta(G)$ is clarified by the following proposition.

Proposition 2. *The lattice $\mathcal{M}(G)$ can be completely determined from the knowledge of the graph $\Delta(G)$. In particular, if G_1 and G_2 are finite groups and the graphs $\Delta(G_1)$ and $\Delta(G_2)$ are isomorphic, then also the lattices $\mathcal{M}(G_1)$ and $\mathcal{M}(G_2)$ are isomorphic.*

Notice that the condition $\mathcal{M}(G_1) \cong \mathcal{M}(G_2)$ is necessary but not sufficient to ensure $\Delta(G_1) \cong \Delta(G_2)$. For example, consider $G_1 = A \times \langle x\rangle$ and $G_2 = \mathrm{Sym}(3) \times \langle y\rangle$, where $A \cong C_3 \times C_3$, $\langle x\rangle \cong C_2$ and $\langle y\rangle \cong C_3$. Let a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 be generators for the four different non-trivial proper subgroups of, respectively, A and $\mathrm{Sym}(3)$. The map sending A to $\mathrm{Sym}(3)$ and $\langle a_i, x\rangle$ to $\langle b_i, y\rangle$ for $1 \le i \le 4$ induces an isomorphism between $\mathcal{M}(G_1)$ and $\mathcal{M}(G_2)$; however, all the subgroups of G_1 are maximal intersections, while $\langle (1, 2, 3)y\rangle$ and $\langle (1, 2, 3)y^2\rangle$ are not maximal intersections in G_2. In particular, $\Delta(G_1)$ has 12 vertices and $\Delta(G_2)$ has 14 vertices. So the following variation of Question 1 arises.

QUESTION 3. How similar are the structures of two finite groups G_1 and G_2 if the lattices $\mathcal{M}(G_1)$ and $\mathcal{M}(G_2)$ are isomorphic?

Our aim is to start to investigate Questions 1 and 3, considering the particular case when G_1 is a finite nilpotent group. Notice that if G_1 is a finite nilpotent group and $\Delta(G_1) \cong \Delta(G_2)$, then G_2 is not necessarily nilpotent. For example, if p is an odd prime, C_p is the cyclic group of order p, and D_{2p} is the dihedral group of order $2p$, then the subgroup lattices of $C_p \times C_p$ and D_{2p} are isomorphic and therefore $\Delta(C_p \times C_p) \cong \Delta(D_{2p})$. Our main result is the following.

THEOREM 4. *Let G be a finite group. If there exists a finite nilpotent group X with $\mathcal{M}(G) \cong \mathcal{M}(X)$, then G is supersoluble.*

COROLLARY 5. *Let G be a finite group. If there exists a finite nilpotent group X with $\Delta(G) \cong \Delta(X)$, then G is supersoluble.*

Let $\mathfrak{M}$ be the family of the finite groups G with the property that $\mathcal{M}(G) \cong \mathcal{M}(X)$ for some finite nilpotent group X. In a similar way, let $\mathfrak{D}$ be the family of the finite groups G with the property that $\Delta(G) \cong \Delta(X)$ for some finite nilpotent group X. By Theorem 4, if $G \in \mathfrak{M}$, then G is supersoluble, but there exist supersoluble groups which do not belong to $\mathfrak{M}$ and it is not easy to give a complete characterization of the finite groups in $\mathfrak{M}$ or in $\mathfrak{D}$. We give a solution of this problem in the particular case when G is a finite group with $\mathrm{Frat}(G) = 1$. Recall that a finite group G is called a P-group of G, it is either a non-cyclic elementary abelian group or a semidirect product of an elementary abelian p-group A by a group of prime order $q \ne p$ which induces a non-trivial power automorphism on A (in particular each subgroup of A is normal in G). Some of the properties of P-groups that will be used throughout the paper are highlighted in [**17**, Section 2.2].

PROPOSITION 6. *Let G be a finite group with* $\mathrm{Frat}(G) = 1$. *Then, $G \in \mathfrak{D}$ if and only if G is a direct product of groups with pairwise coprime orders that are either P-groups or elementary abelian p-groups.*

The classification of the Frattini-free groups in $\mathfrak{M}$ is more difficult. First, we need a definition. Let $t \ge 2$ be a positive integer and $p_1, \ldots, p_t$ be prime numbers with the property that p_{i+1} divides $p_i - 1$ for $1 \le i \le t-1$. We denote by $\Lambda(p_1, \ldots, p_t)$ the set of the direct products $H_1 \times \cdots \times H_{t-1}$, where $H_i \cong C_{p_i}^{n_i} \rtimes C_{p_{i+1}}$ is a non-abelian P-group. Moreover, we will denote by $\Lambda^*(p_1, \ldots, p_t)$ the direct products $X \times Y$ with $X \in \Lambda(p_1, \ldots, p_t)$ and $Y \cong C_{p_1}$. Finally, let Λ (respectively Λ^*) be the union of all the families $\Lambda(p_1, \ldots, p_t)$ (respectively, $\Lambda^*(p_1, \ldots, p_t)$), for any possible choice of t and $p_1, \ldots, p_t$.

PROPOSITION 7. *Let G be a finite group with* $\mathrm{Frat}(G) = 1$. *Then, $G \in \mathfrak{M}$ if and only if G is a direct product $H_1 \times \cdots \times H_u$, where the orders of the factors are pairwise coprime and each of the factors is of one of the following types:*

(1) an elementary abelian p-group;
(2) a group in Λ;
(3) a group in Λ^*.

https://doi.org/10.1017/S0017089520000415 Published online by Cambridge University Press

It follows from the previous proposition that $\mathrm{Sym}(3) \times C_2$ is an example (indeed the one of smallest possible order) of a supersoluble group which does not belong to $\mathfrak{M}$.

Notice that our proof of Theorem 4 uses the classification of the finite simple groups. Theorem 4 is invoked in the proof of Proposition 7, which therefore in turn depends on the classification. On the contrary, Proposition 6 can be directly proved without using Theorem 4 and the classification of the finite simple groups. Indeed, it turns out that if $G \in \mathfrak{D}$ and $\mathrm{Frat}(G) = 1$, then G has the same subgroup lattice as a finite abelian group, and the groups with this property have been classified by Baer [**3**]. However, we are not able to deduce Corollary 5 from Proposition 6, so also our proof of this result depends on the classification. To avoid the use of the classification in the proof of Corollary 5, one should give a positive answer to the following question that we leave open.

Question 8. Does $\Delta(G_1) \cong \Delta(G_2)$ imply $\Delta(G_1/\mathrm{Frat}(G_1)) \cong \Delta(G_2/\mathrm{Frat}(G_2))$?

The obstacle in dealing with this question is that it is not clear whether and how one can deduce which vertices of the graph $\Delta(G)$ correspond to subgroups of G containing $\mathrm{Frat}(G)$.

2. Preliminary results.

Denote by $\mathcal{N}_G(X)$ the neighborhood of the vertex X in the graph $\Delta(G)$. We define an equivalence relation $\equiv_G$ by the rules $X \equiv_G Y$ if and only if $\mathcal{N}_G(X) = \mathcal{N}_G(Y)$. If $X \le G$, let $\tilde{X}$ be the intersection of the maximal subgroups of G containing X (setting $\tilde{G} = G$).

Lemma 9. *$\mathcal{N}_G(X) \subseteq \mathcal{N}_G(Y)$ if and only if $\tilde{X} \le \tilde{Y}$. In particular, $X \equiv_G Y$ if and only if $\tilde{X} = \tilde{Y}$.*

Proof. Assume $\mathcal{N}_G(X) \subseteq \mathcal{N}_G(Y)$ and let M be a maximal subgroup of G. If $Y \le M$, then $\langle Y, M\rangle \ne G$, so $M \notin \mathcal{N}_G(Y)$. It follows that $M \notin \mathcal{N}_G(X)$, that is, $\langle X, M\rangle \ne G$. This implies $X \le M$. It follows that $\tilde{X} \le \tilde{Y}$. Conversely, assume $\tilde{X} \le \tilde{Y}$, or equivalently that every maximal subgroup of G containing Y contains also X. If $Z \notin \mathcal{N}_G(Y)$, then $\langle Y, Z\rangle \le M$ for some maximal subgroup M of G. It follows $\langle X, Z\rangle \le M$ and consequently $Z \notin \mathcal{N}_G(X)$. □

Proof of Proposition 2. Notice that if $X \le G$, then $\tilde{X}$ is a maximal intersection in G, and if X is itself a maximal intersection, then $\tilde{X} = X$. So, by Lemma 9, the map sending the equivalence class containing X to $\tilde{X}$ induces a bijection from the set of the equivalence classes to the set of the maximal intersections in G. Moreover, if $X_1, X_2 \in \mathcal{M}(G)$, then $X_1 \le X_2$ if and only if $\mathcal{N}_G(X_1) \subseteq \mathcal{N}_G(X_2)$. □

We conclude this section with an example showing that if $X_1, X_2 \in \mathcal{M}(G)$, then it is not necessarily true that $\langle X_1, X_2\rangle \in \mathcal{M}(G)$. Let $\mathbb{F}$ be the field with three elements, and let $C = \langle -1\rangle$ be the multiplicative group of $\mathbb{F}$. Let $V = \mathbb{F}^3$ be a 3-dimensional vector space over $\mathbb{F}$ and let $\sigma = (1, 2, 3) \in \mathrm{Sym}(3)$. The wreath product $H = C \wr \langle\sigma\rangle$ has an irreducible action on V defined as follows: if $v = (f_1, f_2, f_3) \in V$ and $h = (c_1, c_2, c_3)\sigma^i \in H$, then $v^h = (f_{1\sigma^{-i}}c_{1\sigma^{-i}}, f_{2\sigma^{-i}}c_{2\sigma^{-i}}, f_{3\sigma^{-i}}c_{3\sigma^{-i}})$. Consider the semidirect product $G = V \rtimes H$ and let $v = (1, -1, 0) \in V$. Since H and H^v are two maximal subgroups of G, $K := H \cap H^v = C_H(v) = \{(1, 1, z) \mid z \in C\} \cong C_2$ is a maximal intersection in G. Since $G/V \cong H$ and $\mathrm{Frat}(H) = 1$, V is also a maximal intersection in G. However, VK is not a maximal intersection in G. Indeed, if X is a maximal intersection in G containing V, then $X = VY$ with Y a maximal intersection in H. But $H \cong C_2 \times \mathrm{Alt}(4)$ and the unique subgroup of order 2 of H that can be obtained as an intersection of maximal subgroups is $\{(z, z, z) \mid z \in C\}$.

The following elementary remark is used several times throughout the paper.

LEMMA 10. *If a finite Frattini-free nilpotent group X contains t maximal subgroups that intersect trivially, then $|X|$ is a product of at most t (not necessarily distinct) primes.*

3. Proof of Theorem 4. Recall that the Möbius function μ_G is defined on the subgroup lattice of G as $\mu_G(G)=1$ and $\mu_G(H)=-\sum_{H<K}\mu_G(K)$ for any $H<G$. If $H\le G$ cannot be expressed as an intersection of maximal subgroups of G, then $\mu_G(H)=0$ (see [12, Theorem 2.3]), so for every $H\in\mathcal{M}(G)$, the value $\mu_G(H)$ can be completely determined from the knowledge of the lattice $\mathcal{M}(G)$. The following result could be easily deduced from [15, Theorem 2.6]. We prefer to give a direct proof.

PROPOSITION 11. *Let G be a finite soluble group. For every irreducible G-module V define $q(V)=|\operatorname{End}_G(V)|$, set $\theta(V)=0$ if V is a trivial G-module, and $\theta(V)=1$ otherwise, and let $\delta(V)$ be the number of chief factors G-isomorphic to V and complemented in an arbitrary chief series of G. Let $\mathcal{V}(G)$ be the set of irreducible G-modules V with $\delta(V)\neq 0$. Then*

$$\mu_G(1)=\begin{cases}\prod_{V\in\mathcal{V}(G)}(-1)^{\delta(V)}|V|^{\theta(V)\delta(V)}q(V)^{\binom{\delta(V)}{2}} & \text{if } \prod_{V\in\mathcal{V}(G)}|V|^{\delta(V)}=|G|,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. We prove the statement by induction on the order of G. Let N be a minimal normal subgroup of G. By [13, Lemma 3.1]

$$\mu_G(1)=\mu_{G/N}(1)\sum_{K\in\mathcal{K}}\mu_G(K),$$

denoting by $\mathcal{K}$ the set of all subgroups of G which complement N. If $\mathcal{K}=\varnothing$, then N is a non-complemented chief factor of G and $\mu_G(1)=0$. Moreover in this case, $\prod_{V\in\mathcal{V}(G)}|V|^{\delta(V)}\le|G|/|N|<|G|$. In any case, since N is a minimal normal subgroup of G and G is soluble, if $K\in\mathcal{K}$, then K is a maximal subgroup of G and consequently $\mu_G(K)=-1$. Thus, $\mu_G(1)=-\mu_{G/N}(1)\cdot c$, where c is the number of complements of N in G. To conclude it suffices to notice that, by [10, Satz 3], $c=|N|^{\theta(N)}q(N)^{\delta(N)-1}$. □

COROLLARY 12. *If $X\cong C_{p_1}^{m_1}\times\cdots\times C_{p_t}^{m_t}$, then $\mu_X(1)=(-1)^{m_1}p_1^{\binom{m_1}{2}}\cdots(-1)^{m_t}p_t^{\binom{m_t}{2}}$.*

LEMMA 13. *Let G be a finite group and assume $G\in\mathfrak{M}$. If N is a normal subgroup of G containing* $\operatorname{Frat}(G)$*, then*

(1) $\mu_G(N)\neq 0$;
(2) *N is a maximal-intersection in G*;
(3) $\operatorname{Frat}(G/N)=1$;
(4) $G/N\in\mathfrak{M}$.

Proof. Since $G\in\mathfrak{M}$, there exists a finite nilpotent group with $\mathcal{M}(G)\cong\mathcal{M}(X)$. We have $\mathcal{M}(G/\operatorname{Frat}(G))\cong\mathcal{M}(G)\cong\mathcal{M}(X)\cong\mathcal{M}(X/\operatorname{Frat}(X))$, and this implies $\mu_{X/\operatorname{Frat}(X)}(1)=\mu_{G/\operatorname{Frat}(G)}(1)$. By Corollary 12, $\mu_{X/\operatorname{Frat}(X)}(1)\neq 0$ and therefore $\mu_{G/\operatorname{Frat}(G)}(1)\neq 0$. If N is a normal subgroup of G containing $\operatorname{Frat}(G)$, then we deduce from [13, Lemma 3.1] that $\mu_G(N)=\mu_{G/N}(1)$ divides $\mu_{G/\operatorname{Frat}(G)}(1)$. As a consequence, $\mu_G(N)\neq 0$ and N is a maximal intersection in G. This implies in particular $\operatorname{Frat}(G/N)=1$. Finally, there exists $Y\le X$ such that $\mathcal{M}(G/N)\cong\mathcal{M}(X/Y)$, so $G/N\in\mathfrak{M}$. □

https://doi.org/10.1017/S0017089520000415 Published online by Cambridge University Press

LEMMA 14. *Let H be a finite supersoluble group and V a faithful irreducible H-module. Consider the semidirect product $G = V \rtimes H$. Suppose that there exists a finite nilpotent group X with $\mathcal{M}(G) \cong \mathcal{M}(X)$. Then V is cyclic of prime order.*

Proof. Since $\mathcal{M}(X) \cong \mathcal{M}(X/\operatorname{Frat}(X))$, we may assume $\operatorname{Frat}(X) = 1$. There exist v and w in V such that $C_H(v) \cap C_H(w) = 1$ (see [19, Theorem A]). This implies that H, H^v, H^w are maximal subgroups of G with trivial intersection. But then also X must contain three maximal subgroups with trivial intersection, and consequently, by Lemma 10, $|X|$ is the product of at most three (not necessarily distinct) primes. Suppose $|V| = p^a$, with p a prime and $a \geq 2$. Since $\operatorname{Frat}(X) = 1$, it follows from Corollary 12 that $\mu_X(1) \neq 0$. Moreover, by Proposition 11, $\mu_X(1) = \mu_G(1)$ is divisible by p^a. By Corollary 12, this is possible only if $X \cong C_p \times C_p \times C_p$ and $\mu_X(1) = \mu_G(1) = -p^3$. By Proposition 11, $|V|$ divides $\mu_G(1)$ so V is a p-group. By Lemma 13, $V \in \mathcal{M}(G)$. Since V is a minimal element in $\mathcal{M}(G)$, it follows that $\mathcal{M}(H) \cong \mathcal{M}(G/V) \cong \mathcal{M}(C_p \times C_p)$ and therefore, by Corollary 12, $\mu_H(1) = p$. Moreover 2 is the maximal length of a chain in $\mathcal{M}(H)$ and $\operatorname{Frat}(H) = 1$ by Lemma 13. So H is a supersoluble group in which the intersection of any pair of maximal subgroups is trivial. This implies that $|H|$ is the product of two primes, say p_1 and p_2, and we may assume that H has a normal subgroup of order p_1. By Proposition 11, $\mu_H(1) = 1$ if H is cyclic, $\mu_H(1) = p_1$ otherwise. Since $\mu_H(1) = p$, it follows that $O_p(H) \neq 1$, in contradiction with the fact that V is a faithful irreducible H-module of p-power order. □

LEMMA 15. *If G is a finite almost simple group, then there exist maximal subgroups $M_1, \ldots, M_t$ of G, with $t \leq 5$, with the property that $M_1 \cap \cdots \cap M_t = 1$.*

Proof. The result follows from [5, Theorem 1], except when $S = \operatorname{soc}(G)$ is an alternating group or a classical group and all the primitive actions of G are of standard type. If $\operatorname{soc}(G)$ is of alternating type, then the result follows from [7, Corollaries 1.4, 1.5, Remark 1.6] (see also [9, Lemma 2] and its proof). In the case of classical groups, we are done if we are able to build up a non-standard action by taking primitive actions with stabilizer in one of the Aschbacher classes $\mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4, \mathscr{C}_5, \mathscr{C}_6, \mathscr{C}_7$. For this purpose, we use [14, Tables 3.5.A. 3.5.B, 3.5.C, 3.5.D, 3.5.E and 3.5.F] (and the similar tables in [4] if the dimension of G is up to 12). We need to be careful because a subgroup H in one of the given Aschbacher classes of G may not actually be maximal in G. As it is explained in [14, Section 3.4], to avoid this possibility, we need to select H in such a way that when we look to the corresponding row in the table, we do not find restrictive conditions in column VI and the homomorphism π described in column V is the identity. A subgroup with these properties can be found, except in the following three cases:

(1) $S = \Omega_{2p}^{+}(2)$ and p is an odd prime (and we may assume $p \geq 5$, since $\Omega_6^{+}(2) \cong \operatorname{Alt}(8)$). In this case, $|G:S| \leq 2$. Let V be the natural module for G, and let Ω be the set of nondegenerate plus-type subspaces of dimension $p+1$. Then G acts primitively on this set, and by the proof of [6, Theorem 6.13], it contains three maximal subgroups M_1, M_2, M_3 such that $M_1 \cap M_2 \cap M_3 \cap S = 1$, so $t \leq 4$.
(2) $S = \mathrm{P}\Omega_{2p}^{+}(5)$ and p is an odd prime. Again, let V be the natural module for S, and let Ω be the set of nondegenerate plus-type subspaces of dimension $p+1$. Then G acts primitively on this set. Arguing as in the proof of [6, Theorem 6.13], three subspaces in Ω can be exhibited with the property that if $g \in \mathrm{O}_{2p}^{+}(5)$ stabilizes each of them, then, with respect to a suitable basis, g is represented either by a scalar matrix or by the matrix

$$\pm\begin{pmatrix} I_{2p-2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let M_1, M_2, M_3 be the stabilizers in G of these subspaces. We have $|M_1 \cap M_2 \cap M_3 \cap \mathrm{PO}^+_{2p}(5)| \le 2$, so $|M_1 \cap M_2 \cap M_3 \cap G| \le 4$ and consequently there exist M_4 and M_5 such that $M_1 \cap M_2 \cap M_3 \cap M_4 \cap M_5 = 1$.

(3) $S = \Omega_p(q)$ with $p \ge 7$ a prime, $q = q_0^t$ with q_0 an odd prime, and t a power of 2. In this case, let V be the natural module for S and Ω the set of the $2m$-dimensional nondegenerate subspaces of V of plus-type if $p = 4m + 1$, or the set of the $(2m + 1)$-dimensional nondegenerate subspaces X of V with the property that $X^\perp$ has plus type if $p = 4m + 3$. Then, G acts primitively on Ω, and by [**6**, Theorem 6.11], the restriction of this action to S has a base of size 2. By [**11**, Theorem 1.2], each element of G has a regular cycle. Since G/S is metacyclic, it follows that the action of G on Ω has a base of size at most 4. As a consequence, we can find four point stabilizers with trivial intersection. □

LEMMA 16. *If G is a finite monolithic primitive group with non-abelian socle, then there is no finite nilpotent group X with $\mathcal{M}(G) \cong \mathcal{M}(X)$.*

Proof. Assume, by contradiction, that there exists a finite nilpotent group X with $\mathcal{M}(X) \cong \mathcal{M}(G)$. Since $\mathcal{M}(X) \cong \mathcal{M}(X/\mathrm{Frat}(X))$, we may assume $\mathrm{Frat}(X) = 1$. There exists a finite nonabelian simple group S such that $N = \mathrm{soc}(G) = S_1 \times \ldots \times S_n$, with $S_i \cong S$ for $1 \le i \le n$.

Suppose first that $n \ge 2$. Let ψ be the map from $N_G(S_1)$ to $\mathrm{Aut}(S)$ induced by the conjugacy action on S_1. Set $H = \psi(N_G(S_1))$, and note that H is an almost simple group with socle $S = \mathrm{Inn}(S) = \psi(S_1)$. Let $T := \{t_1, \ldots, t_n\}$ be a right transversal of $N_G(S_1)$ in G; the map

$$\phi_T : G \to H \wr \mathrm{Sym}(n)$$

given by

$$g \mapsto (\psi(t_1 g t_{1\pi_g}^{-1}), \ldots, \psi(t_n g t_{n\pi_g}^{-1}))\pi_g,$$

where $\pi_g \in \mathrm{Sym}(n)$ satisfies $t_i g t_{i\pi_g}^{-1} \in N_G(S_1)$ for all $1 \le i \le n$, is an injective homomorphism. So we may identify G with its image in $H \wr \mathrm{Sym}(n)$; in this identification, N is contained in the base subgroup H^n and S_i is a subgroup of the ith component of H^n. By Lemma 13, $\mathrm{Frat}(G/N) = 1$ and so there exist u maximal subgroups $M_1, \ldots, M_u$ of G such that

$$N = M_1 \cap \cdots \cap M_u < M_1 \cap \cdots \cap M_{u-1} < \cdots < M_1 \cap M_2 < M_1 < G.$$

Let R be a maximal subgroup of H with $H = RS$ and set $K = R \cap S$. We must have $K \ne 1$ (see, for example, the last paragraph of the proof of the main theorem in [**16**]). Notice that $L := G \cap (R \wr \mathrm{Sym}(n))$ is a maximal subgroup of G ([**2**] Proposition 1.1.44). We have $D := L \cap M_1 \cap \cdots \cap M_u = L \cap N = K^n$. Choose a subset $\{s_1, \ldots, s_m\}$ of S with minimal cardinality with respect to the property $K \cap K^{s_1} \cap \cdots \cap K^{s_m} = 1$. Set

$$\alpha_1 = (s_1, \ldots, s_1), \alpha_2 = (s_2, \ldots, s_2), \ldots, \alpha_m = (s_m, \ldots, s_m),$$
$$\beta_1 = (s_1, 1, \ldots, 1), \beta_2 = (s_2, 1, \ldots, 1), \ldots, \beta_m = (s_m, 1, \ldots, 1),$$
$$\gamma_1 = (1, s_1, \ldots, s_1), \gamma_2 = (1, s_2, \ldots, s_2), \ldots, \gamma_m = (1, s_m, \ldots, s_m).$$

For $1 \le i \le m$, set

$$A_i := L^{\alpha_i} \cap \cdots \cap L^{\alpha_m} \cap D,$$
$$B_i := L^{\beta_i} \cap \cdots \cap L^{\beta_m} \cap L^{\gamma_1} \cap \cdots \cap L^{\gamma_m} \cap D,$$
$$C_i := L^{\gamma_i} \cap \cdots \cap L^{\gamma_m} \cap D.$$

We have

$$1 = A_1 < \cdots < A_m < D, \quad 1 = B_1 < \cdots < B_m < C_1 < \cdots < C_m < D.$$

In particular,

$$\{M_1, \ldots, M_t, L, L^{\alpha_1}, \ldots, L^{\alpha_m}\}, \quad \{M_1, \ldots, M_t, L, L^{\beta_1}, \ldots, L^{\beta_m}, L^{\gamma_1}, \ldots, L^{\gamma_m}\}$$

are two families of maximal subgroups of G that are minimal with respect to the property that their intersection is the trivial subgroup. However, the assumption $\mathcal{M}(G) \cong \mathcal{M}(X)$ implies that all the families of maximal subgroups of G with this property must have the same size.

We may therefore assume that G is a finite almost simple group. Since $\mathrm{Frat}(X) = 1$, by Corollary 12, $0 \neq \mu_X(1) = \mu_G(1)$. By Lemma 15, G contains $t \le 5$ maximal subgroups with trivial intersection. But then X satisfies the same properties, and consequently, by Lemma 10, $|X|$ is the product of at most $t \le 5$ primes. It follows from Corollary 12 that $\mu_X(1) = \mu_G(1)$ is divisible by at most two different primes. By [**13**, Theorem 4.5], $|G|$ divides $m \cdot \mu_G(1)$, where m is the square-free part of $|G/G'|$. So, if $S = \mathrm{soc}(G)$, then, since $S \le G'$, m divides $|G/S|$ and consequently $|S|$ divides $\mu_G(1) = \mu_X(1)$. But then $|S|$ is divisible by at most two different primes, so it is soluble by Burnside's $p^a q^b$-theorem, a contradiction. □

Proof of Theorem 4. We prove our statement by induction on the order of G. If $\mathrm{Frat}(G) \neq 1$, then $\mathcal{M}(G/\mathrm{Frat}(G)) \cong \mathcal{M}(X/\mathrm{Frat}(X))$, so $G/\mathrm{Frat}(G)$ is supersoluble by induction. But this implies that G itself is supersoluble. So we may assume $\mathrm{Frat}(G) = 1$. Assume, by contradiction, that G is not soluble. Then, there exists a non-abelian chief factor R/S of G. Let $L = G/C_G(R/S)$. Notice that L is a primitive monolithic group whose socle is isomorphic to R/S. By Lemma 13, $C_G(R/S)$ is a maximal intersection in G. But then $\mathcal{M}(L) \cong \mathcal{M}(X/Y)$ for a suitable normal subgroup Y of X, in contradiction with Lemma 16. So we may assume that G is soluble. Assume by contradiction that G is not supersoluble. Let $1 = N_0 < N_1 < \cdots < N_u = G$ be a chief series of G, and let j be the largest positive integer with the property that the chief factor N_j/N_{j-1} is not cyclic. Let $V = N_j/N_{j-1}$ and $H = G/C_G(V)$. By Lemma 13 and Proposition 11, N_j/N_{j-1} is a complemented chief factor of G. Let K/N_{j-1} be a complement of N_j/N_{j-1} in G/N_{j-1} and set $M = N_{j-1}C_K(V)$. It turns out that $G/M \cong V \rtimes H$. Again by Lemma 13, M is a maximal intersection in G, so there exists $Y \le X$ such that $\mathcal{M}(G/M) \cong \mathcal{M}(X/Y)$. By our choice of the index j, the factor group G/N_j is supersoluble. Since $N_j \le C_G(V)$, also H is supersoluble. But then it follows from Lemma 14 that V is cyclic of prime order, in contradiction with our assumption. □

https://doi.org/10.1017/S0017089520000415 Published online by Cambridge University Press

4. Frattini-free groups in $\mathfrak{D}$ and $\mathfrak{M}$.

Proof of Proposition 6. Assume that X is a finite nilpotent group with $\Delta(X) \cong \Delta(G)$. Since $\mathrm{Frat}(G) = 1$, the unique isolated vertex in $\Delta(G)$ is the one corresponding to the identity subgroup. The same must be true in $\Delta(X)$ and therefore $\mathrm{Frat}(X) = 1$. Hence, X is a direct product of elementary abelian groups. In particular, every subgroup of X is a maximal intersection in X, so the lattice $\mathcal{M}(X)$ coincides with the entire subgroup lattice $\mathcal{L}(X)$ of X. This is equivalent to say that if Y_1 and Y_2 are different subgroups of G, then $\mathcal{N}_G(Y_1) \neq \mathcal{N}_G(Y_2)$. Again, the same property holds for $\Delta(G)$ and consequently $\mathcal{M}(G) \cong \mathcal{L}(G)$. So by Proposition 2, $\mathcal{L}(G) \cong \mathcal{L}(X)$, and the conclusion follows from [17, Theorem 2.5.10]. □

Lemma 17. *Suppose that X_1 and X_2 are finite groups. If no simple group is a homomorphic image of both X_1 and X_2 then $\mathcal{M}(X_1 \times X_2) \cong \mathcal{M}(X_1) \times \mathcal{M}(X_2)$.*

Proof. A maximal subgroup M of a direct product $X_1 \times X_2$ is of standard type if either $M = Y_1 \times X_2$ with Y_1 a maximal subgroup of X_1 or $M = X_1 \times Y_2$ with Y_2 a maximal subgroup of X_2. A maximal subgroup M of $X_1 \times X_2$ is of diagonal type if there exist a maximal normal subgroup N_1 of X_1, a maximal normal subgroup N_2 of X_2, and an isomorphism $\phi : X_1/N_1 \to X_2/N_2$ such that $M = \{(x_1, x_2) \in H_1 \times H_2 \mid \phi(x_1N_1) = x_2N_2\}$. By [18, Chapter 2, (4.19)], a maximal subgroup of $X_1 \times X_2$ is either of standard type or of diagonal type. If no simple group is a homomorphic image of both X_1 and X_2, then all the maximal subgroups of $X_1 \times X_2$ are of standard type. In particular, $K \in \mathcal{M}(X_1 \times X_2)$ if and only if $K = K_1 \times K_2$, with $K_1 \in \mathcal{M}(X_1)$ and $K_2 \in \mathcal{M}(X_2)$. □

Lemma 18. *The following hold:*

(1) If $G = H_1 \times \cdots \times H_{t-1} \in \Lambda(p_1, \ldots, p_t)$, with $H_i \cong C_{p_i}^{n_i} \rtimes C_{p_i+1}$, then $\mathcal{M}(G) \cong \mathcal{M}(C_{p_1}^{n_1+1} \times \cdots \times C_{p_{t-1}}^{n_{t-1}+1})$.
(2) If $G = H_1 \times \cdots \times H_{t-1} \times C_{p_1} \in \Lambda^*(p_1, \ldots, p_t)$ with $H_i \cong C_{p_i}^{n_i} \rtimes C_{p_i+1}$, then $\mathcal{M}(G) \cong \mathcal{M}(C_{p_1}^{n_1+1} \times \cdots \times C_{p_{t-1}}^{n_{t-1}+1} \times C_{p_t})$.

Proof. Let $H \cong C_p^n \rtimes C_q$ be a P-group. By [17, Theorem 2.2.3], the subgroup lattices of H and C_p^{n+1} are isomorphic, and consequently, $\mathcal{M}(H) \cong \mathcal{M}(C_p^{n+1})$. Now assume $G = H_1 \times \cdots \times H_{t-1} \in \Lambda(p_1, \ldots, p_t)$, with $H_i \cong C_{p_i}^{n_i} \rtimes C_{p_i+1}$. By Lemma 17,

$$\begin{aligned}\mathcal{M}(G) &\cong \mathcal{M}(H_1 \times \cdots \times H_{t-1}) \cong \mathcal{M}(H_1) \times \cdots \times \mathcal{M}(H_{t-1})\\ &\cong \mathcal{M}(C_{p_1}^{n_1+1}) \times \cdots \times \mathcal{M}(C_{p_{t-1}}^{n_{t-1}+1}) \cong \mathcal{M}(C_{p_1}^{n_1+1} \times \cdots \times C_{p_{t-1}}^{n_{t-1}+1}).\end{aligned}$$

This proves (1). If $G = H_1 \times \cdots \times H_{t-1} \times C_{p_1} \in \Lambda^*(p_1, \ldots, p_t)$ with $H_i \cong C_{p_i}^{n_i} \rtimes C_{p_i+1}$, then, again by Lemma 17,

$$\begin{aligned}\mathcal{M}(G) &\cong \mathcal{M}(H_1 \times \cdots \times H_{t-1} \times C_{p_1})\\ &\cong \mathcal{M}(H_1) \times \cdots \times \mathcal{M}(H_{t-1}) \times \mathcal{M}(C_{p_1})\\ &\cong \mathcal{M}(C_{p_1}^{n_1+1}) \times \cdots \times \mathcal{M}(C_{p_{t-1}}^{n_{t-1}+1}) \times \mathcal{M}(C_{p_1})\\ &\cong \mathcal{M}(C_{p_1}^{n_1+1}) \times \cdots \times \mathcal{M}(C_{p_{t-1}}^{n_{t-1}+1}) \times \mathcal{M}(C_{p_t})\\ &\cong \mathcal{M}(C_{p_1}^{n_1+1} \times \cdots \times C_{p_{t-1}}^{n_{t-1}+1} \times C_{p_t}).\end{aligned}$$

So (2) is also proved. □

https://doi.org/10.1017/S0017089520000415 Published online by Cambridge University Press

Proof of Proposition 7. First, we prove by induction on the order of G that if $G \in \mathfrak{M}$, then G is as described in the statement. Let M be a normal subgroup of G. By Lemma 13, $\mathrm{Frat}(G/M) = 1$ and $G/M \in \mathfrak{M}$. Hence, G/M satisfies the same assumptions as G. During the proof, we will use several times, without an explicit mention, this remark.

Let N be a minimal normal subgroup of G. By Theorem 4, there exists a prime p such that $N \cong C_p$. Moreover, since $\mathrm{Frat}(G) = 1$, N has a complement, say K in G. Since $K \cong G/N$, by induction $K = H_1 \times \cdots \times H_u$, where $H_1, \ldots, H_u$ have coprime orders and are as described in the statement.

First assume that N is central in G. If p does not divide the order of K, then $G = H_1 \times \cdots \times H_u \times N$ is a factorization with the required properties. Otherwise, there exists a unique i such that p divides $|H_i|$. It is not restrictive to assume $i = u$. If H_u is either elementary abelian or $H_u \in \Lambda(p_1, \ldots, p_t)$ with $p_1 = p$, then we set $\tilde{H}_u = H_u \times N \cong H_u \times C_p$ and the factorization $G = H_1 \times \cdots \times H_{u-1} \times \tilde{H}_u$ satisfies the required properties. In the other cases, there exist a prime $q \neq p$ and a normal subgroup L of H_u such that $J = H_u/L$ is isomorphic either to $C_q \rtimes C_p$ or to $(C_p \rtimes C_q) \times C_p$. Since $T = N \times J \cong G/(H_1 \times \cdots \times H_{u-1} \times L) \in \mathfrak{M}$, there exists a Frattini-free nilpotent group X with $\mathcal{M}(X) \cong \mathcal{M}(T)$. Notice that since $\mathrm{Frat}(X) = 1$, X is a direct product of elementary abelian groups, so we may apply Corollary 12 when it is needed. If $J \cong C_q \rtimes C_p$, then $\mu_X(1) = \mu_T(1) = -p \cdot q$ and $|X|$ is the product of three primes, but this possibility is excluded by Corollary 12. If $J \cong (C_p \rtimes C_q) \times C_p$, then $\mu_X(1) = \mu_T(1) = p^2$, again in contradiction with Corollary 12.

Now assume that N is not central. Notice that $G/C_G(N)$, being isomorphic to a subgroup of $\mathrm{Aut}(N)$, is cyclic. Since $\mathrm{Frat}(G/C_G(N)) = 1$, we deduce $G/C_G(N) \cong C_q$, where q is a square-free positive integer. Moreover, there exists a Frattini-free nilpotent group X such that $\mathcal{M}(X) \cong \mathcal{M}(G/C_K(N))$. Since $G/C_K(N) \cong C_p \rtimes C_q$, the identity subgroup of $G/C_K(N)$ can be obtained as the intersection of two conjugated subgroups of order q. By Lemma 10, $|X|$ is the product of two primes, and consequently, $\mathcal{M}(G/C_K(N)) \cong \mathcal{M}(X)$ cannot contain chains of length > 2. But then q is a prime. In particular, there exists a unique i such that q divides $|H_i|$. It is not restrictive to assume $i = u$. Notice that $C_q \cong H_u/C_{H_u}(N)$, so q divides $|H_u/H_u'|$. We distinguish the different possibilities for H_u and determine the structure of NH_u in each case.

First assume $H_u = C_q^t$, for some $t \in \mathbb{N}$. Then, $G/(H_1 \times \cdots \times H_{u-1}) \cong NH_u \cong (C_p \rtimes C_q) \times C_q^{t-1}$. If $t \geq 2$, then $Y_1 = (C_p \rtimes C_q) \times C_q$ would be an epimorphic image of G. Consequently, by Lemma 10, there would exist a nilpotent group X whose order is the product of three primes such that $\mu_X(1) = \mu_{Y_1}(1) = -p \cdot q$, in contradiction with Corollary 12. Thus, $t = 1$, and consequently, $NH_u \in \Lambda(p, q)$.

Assume $H_u = T_1 \times \cdots \times T_{t-1} \in \Lambda(p_1, \ldots, p_t)$, with $T_j \cong C_{p_j}^{n_j} \rtimes C_{p_{j+1}}$. Since H_u is a direct product of non-abelian P-groups, $|H_u/H_u'|$ is not divisible by p_1. On the other hand, q divides $|H_u/H_u'|$, hence $q \neq p_1$ and there exists $1 \leq i \leq t-1$ such that $q = p_{i+1}$. Moreover, since $H_u/C_{H_u}(N) \cong C_q$, it follows that $C_{H_u}(N) = \left(\prod_{j \neq i} T_j\right) \times C_{p_i}^{n_i}$. Let $r = p_i$ and R a (non-central) normal subgroup of T_i with order r. A Sylow q-subgroup Q of T_i centralizes neither N nor R. The semidirect product $Y_2 = (N \times R) \rtimes Q \cong (C_p \times C_r) \rtimes C_q$ is an epimorphic image of G, and consequently, there exists a nilpotent group X whose order is the product of three primes (by Lemma 10) such that $\mu_X(1) = \mu_{Y_2}(1)$ is divisible by $p \cdot r$. By Corollary 12 and Proposition 11, this is possible only if $p = r$, $X \cong C_p^3$, $\mu_X(1) = -p^3$ and N and R are Q-isomorphic (and consequently G-isomorphic). But then $NT_i \cong C_p^{1+n_i} \rtimes C_q$ is a P-group and $NH_u = T_1 \times \cdots \times T_{i-1} \times NT_i \times T_{i+1} \times \cdots \times T_{t-1} \in \Lambda(p_1, \ldots, p_t)$.

Assume $H_u = T_1 \times \cdots \times T_{t-1} \times L \in \Lambda^*(p_1, \ldots, p_t)$, with $T_j \cong C_{p_j}^{n_j} \rtimes C_{p_{j+1}}$ and L a group of order p_1. If $q \neq p_1$, then $q = p_{i+1}$ for some $1 \leq i \leq t$, and we may repeat the previous argument to deduce that NT_i is a P-group and $NH_u = T_1 \times \cdots \times T_{i-1} \times NT_i \times T_{i+1} \times \cdots \times T_{t-1} \times L \in \Lambda^*(p_1, \ldots, p_t)$. If $q = p_1$, then NL is a P-group of order $p \cdot p_1$ and $NH_u = NL \times T_1 \times \cdots \times T_{t-1} \in \Lambda(p, p_1, \ldots, p_t)$.

We conclude that in any case one of the following occurs:

(1) $NH_u \in \Lambda(p, p_1, \ldots, p_t)$,
(2) $NH_u \in \Lambda(p_1, \ldots, p_t)$,
(3) $NH_u \in \Lambda^*(p_1, \ldots, p_t)$.

If p does not divide $|H_1| \cdots |H_{u-1}|$, then the factorization $H_1 \times \ldots H_{u-1} \times NH_u$ satisfies the requirements of the statement. Otherwise, we may assume that p divides $|H_1|$. Notice that in this case p does not divide H_u, so $NH_u \in \Lambda(p, p_1, \ldots, p_t)$. If H_1 admits a non-central chief factor of order p, then there exists a prime r such that $Y_3 = (C_p \rtimes C_q) \times (C_p \rtimes C_r)$ is an epimorphic image of G. There would exist a nilpotent group X with $\mu_X(1) = \mu_{Y_3}(1)$. However by Proposition 11, $\mu_{Y_3}(1) = p^2 \cdot q^\eta$, with $\eta = 1$ if $q = r$, $\eta = 0$ otherwise, while by Corollary 12, p cannot divide $\mu_X(1)$ with multiplicity equal to 2. The only possibility that remains is $H_1 \cong C_p^t$. If $t \geq 2$, then $Y_4 = (C_p \rtimes C_q) \times C_p^2$ is an epimorphic image of G, and there would exist a nilpotent group X with $\mu_X(1) = \mu_{Y_4}(1) = p^2$, again in contradiction with Corollary 12. So $t = 1$ and $H_1 \times NH_u \in \Lambda^*(p, p_1, p_2, \ldots, p_t)$. Setting $\tilde{H}_1 = H_1 \times NH_u$, we conclude that $\tilde{H}_1 \times H_2 \times \cdots \times H_{u-1}$ is the factorization we are looking for.

Conversely, assume that $G = H_1 \times \cdots \times H_u$ is a factorization with the properties described by the statement. By Lemma 18, for every $1 \leq i \leq u$, there exists a nilpotent group X_i such that $\mathcal{M}(H_i) = \mathcal{M}(X_i)$ and $|X_i|$ and $|H_i|$ have the same prime divisors. But then, by Lemma 17, $\mathcal{M}(G) \cong \mathcal{M}(H_1) \times \cdots \times \mathcal{M}(H_u) \cong \mathcal{M}(X_1) \times \cdots \times \mathcal{M}(X_u) \cong \mathcal{M}(X_1 \times \cdots \times X_u)$. □

REFERENCES

1. H. Ahmadi and B. Taeri, A graph related to the join of subgroups of a finite group, *Rend. Semin. Mat. Univ. Padova* **131** (2014), 281–292.

2. A. Ballester-Bolinches and L. M. Ezquerro, Classes of finite groups, in *Mathematics and Its Applications*, vol. 584 (Springer, Dordrecht, 2006).

3. R. Baer, The significance of the system of subgroups for the structure of the group, *Amer. J. Math.* **61**(1) (1939), 1–44.

4. J. Bray, D. Holt and C. Roney-Dougal, *The maximal subgroups of the low-dimensional finite classical groups*. With a foreword by Martin Liebeck. London Mathematical Society Lecture Note Series, vol. 407 (Cambridge University Press, Cambridge, 2013).

5. T. Burness, On base sizes for almost simple primitive groups, *J. Algebra* **516** (2018), 38–74.

6. T. Burness, M. Garonzi and A. Lucchini, On the minimal dimension of a finite simple group. With an appendex by T. C. Burnes and R. M. Guralnick, *J. Combin. Theory Ser. A* **171** (2020), 105175.

7. T. Burness, R. Guralnick and J. Saxl, On base sizes for symmetric groups, *Bull. Lond. Math. Soc.* **43**(2) (2011), 386–391.

8. T. Burness, M. Liebeck and A. Shalev, Base sizes for simple groups and a conjecture of Cameron, *Proc. Lond. Math. Soc. (3)* **98**(1) (2009), 116–162.

9. M. Garonzi and A. Lucchini, Maximal irredundant families of minimal size in the alternating group, *Arch. Math. (Basel)* **113**(2) (2019), 119–126.

10. W. Gaschütz, Die Eulersche Funktion endlicher auflösbarer Gruppen, *Illinois J. Math.* **3** (1959), 469–476.

11. S. Guest and P. Spiga, Finite primitive groups and regular orbits of group elements, *Trans. Amer. Math. Soc.* **369**(2) (2017), 997–1024.

12. P. Hall, The eulerian functions of a group, *Quart. J. Math.* **7** (1936), 134–151.

13. T. Hawkes, I. M. Isaacs and M. Özaydin, On the Möbius function of a finite group, *Rocky Mountain J. Math.* **19**(4) (1989), 1003–1034.

14. P. Kleidman and M. Liebeck, *The subgroup structure of the finite classical groups*. London Mathematical Society Lecture Note Series, 129. (Cambridge University Press, Cambridge, 1990).

15. C. Kratzer and J. Thvenaz, Fonction de Möbius d'un groupe fini et anneau de Burnside, *Comment. Math. Helv.* **59**(3) (1984), 425–438.

16. M. Liebeck, C. Praeger and J. Saxl, On the O'Nan-Scott theorem for finite primitive permutation groups, *J. Austral. Math. Soc. Ser. A* **44**(3) (1988), 389–396.

17. R. Schmidt, *Subgroup lattices of groups*, De Gruyter Expositions in Mathematics, vol. 14 (Walter de Gruyter, Berlin, 1994).

18. M. Suzuki, *Group theory I*, Grundlehren der Mathematischen Wissenschaften vol. 247 (Springer-Verlag, Berlin, 1982).

19. T. R. Wolf, Large orbits of supersolvable linear groups, *J. Algebra* **215**(1) (1999), 235–247.