Can. J. Math. Vol. 44 (1), 1992 pp. 104–118

ON CERTAIN STABLE WEDGE SUMMANDS OF $B(\mathbf{Z}/p)^n_+$

JOHN C. HARRIS

ABSTRACT. Campbell and Selick have given a natural decomposition of the cohomology of an elementary abelian p-group over the Steenrod algebra. We study the corresponding stable wedge summands of the classifying space $B(\mathbf{Z}/p)^n_+$ using representation theory and explicit idempotents in the group ring $\mathbf{F}_p[\mathrm{GL}_n(\mathbf{Z}/p)]$.

Introduction. Let $B(\mathbf{Z}/p)^n_+$ be the classifying space of the elementary abelian p-group $(\mathbf{Z}/p)^n$, together with a disjoint basepoint. Let $H = H^*(B(\mathbf{Z}/p);\mathbf{F}_p)$ regarded as a module over the mod-p Steenrod algebra, $\mathcal{A}$. And write $H^{\otimes n} \cong \mathbf{F}_p[t_0,\ldots,t_{n-1}]\otimes \mathrm{Ext}[u_0,\ldots,u_{n-1}]$, with $\beta(u_k) = t_k$ and $\mathcal{P}^1(t_k) = t_k^p$ (if $p = 2$, take $\mathcal{P}^1 = Sq^1$ and $\mathrm{Ext} = 0$). Note that $H^{\otimes n}$ is the reduced cohomology of $B(\mathbf{Z}/p)^n_+$.

Consider the following three problems:

1) Find a decomposition $B(\mathbf{Z}/p)^n_+ \simeq X_1 \vee \cdots \vee X_N$ into indecomposable *stable* wedge summands.
2) Find a decomposition $H^{\otimes n} \cong I_1 \oplus \cdots \oplus I_N$ into indecomposable modules over the Steenrod algebra.
3) Find a decomposition $1 = e_1+\cdots+e_N$ in $\mathbf{F}_p[\mathrm{M}_{n,n}(\mathbf{Z}/p)]$ into primitive orthogonal idempotents, where $\mathrm{M}_{n,n}(\mathbf{Z}/p)$ is the multiplicative semigroup of $n\times n$ matrices.

The first and third problems are shown to be equivalent in [HK], where a solution to the third is given in terms of the modular representation theory of $\mathrm{M}_{n,n}(\mathbf{Z}/p)$. The second and third are equivalent by a result of Adams, Gunawardena, and Miller ([AGM], [LZ2], [Wo]). The correspondence from 1) to 2) is given by taking reduced mod-p cohomology.

The importance of the modules I_k comes from results of Carlsson, Miller, Lannes, Zarati, and Schwartz ([Ca], [Mi], [LZ1], [LS]). In his proof of the Segal conjecture, Carlsson used a certain splitting which Miller later observed, in his proof of the Sullivan conjecture, was equivalent to the fact that the module H is injective in the category of unstable $\mathcal{A}$-modules, $\mathcal{U}$. Using this, Lannes and Zarati showed that $H^{\otimes n}$ (hence any direct summand of $H^{\otimes n}$) is also injective in $\mathcal{U}$. Then Lannes and Schwartz classified all of the injectives in $\mathcal{U}$, showing in particular that the modules I_k are exactly the indecomposable *reduced* injectives.

In [CS], Campbell and Selick give a very natural decomposition of $H^{\otimes n}$ into a direct sum of (p^n-1) $\mathcal{A}$-modules, called the weight summands, $M_n(j)$, for $j \in \mathbf{Z}/(p^n-1)$. These summands are particularly easy to work with because they have bases consisting of monomials in a certain finitely generated algebra. By the above correspondence of 1)

Received by the editors November 1, 1989.
AMS subject classification: 55P42, 55S10.

and 2), the Campbell and Selick weight summands give a decomposition of $B(\mathbf{Z}/p)^n_+$ into $(p^n - 1)$ stable wedge summands, which we call $Y_n(j)$, for $j \in \mathbf{Z}/(p^n - 1)$.

The purpose of this paper is to describe the $Y_n(j)$. To do this, we produce a set of orthogonal idempotents in $\mathbf{F}_p[\mathrm{M}_{n,n}(\mathbf{Z}/p)]$ inducing Campbell and Selick's decomposition of $H^{\otimes n}$, hence inducing the $Y_n(j)$'s. Then we relate these idempotents to the irreducible $\mathrm{M}_{n,n}(\mathbf{Z}/p)$ representations to give the complete decompositions of the $Y_n(j)$'s.

Our construction of the idempotents was inspired by the work of Witten ([W]). She produces $(p^n - 1)$ orthogonal idempotents in a certain group ring $\mathbf{F}_p[G]$ (with $G \cong (\mathbf{F}_{p^n})^* \rtimes \mathrm{Gal}(\mathbf{F}_{p^n} : \mathbf{F}_p) \subseteq \mathrm{M}_{n,n}(\mathbf{Z}/p)$) inducing a decomposition of $B(\mathbf{Z}/p)^n$ into wedge summands, each of which has rank 1 mod p K-theory. Her idempotents are not uniquely specified, and it turns out that her summands are only well defined up to K-theoretically trivial pieces.

THEOREM A. *An appropriate choice of Witten's idempotents induces the Campbell and Selick decomposition of* $B(\mathbf{Z}/p)^n_+$.

It follows that $Y_n(0)$ has rank 2 mod p K-theory, and, for $j \neq 0$, $Y_n(j)$ has rank 1 mod p K-theory. (Note that $B(\mathbf{Z}/p)^n_+ \simeq B(\mathbf{Z}/p)^n \vee S^0$, S^0 has rank 1 mod p K-theory, and if we write $Y_n(0) \simeq \overline{Y}_n(0) \vee S^0$, then $\overline{Y}_n(0)$ is the Witten summand with rank 1 mod p K-theory.)

Results of Kuhn and Carlisle ([K], [CK]) can be used to determine which indecomposable summands have rank 1 mod p K-theory. In Section 4, we show how these summands distribute themselves among the Campbell and Selick summands when $p = 2$.

From the Campbell and Selick description it is easy to see that $Y_n(j) \simeq Y_n(jp)$; we let $\widehat{Y}_n(i) \simeq Y_n(i) \vee \cdots \vee Y_n(ip^{z_i-1})$, where z_i is the smallest positive exponent k with $ip^k \equiv i \pmod{p^n - 1}$.

THEOREM B. *There are (unique) orthogonal idempotents in* $\mathbf{F}_p[C]$, *where* $C \cong (\mathbf{F}_{p^n})^* \subseteq G$ *is a cyclic subgroup of order* $(p^n - 1)$, *inducing the wedge summands* $\widehat{Y}_n(i)$.

In fact, the $\widehat{Y}_n(i)$ correspond to the distinct irreducible representations of $\mathbf{F}_p[C]$. (Note that these representations are not necessarily one dimensional since $\mathbf{F}_p$ is not algebraically closed.) By comparing these to the irreducible $\mathbf{F}_p[\mathrm{GL}_n(\mathbf{Z}/p)]$ representations, we describe complete decompositions of the $\widehat{Y}_n(i)$. Of course, complete decompositions of the $Y_n(j)$ follow.

The paper is organized as follows. In Section 1, we recall the methods from [HK] giving the complete decomposition of $B(\mathbf{Z}/p)^n_+$ using $\mathbf{F}_p[\mathrm{M}_{n,n}(\mathbf{Z}/p)]$ and giving a partial decomposition using $\mathbf{F}_p[\mathrm{GL}_n(\mathbf{Z}/p)]$. In Section 2, the Campbell and Selick decomposition of $H^{\otimes n}$ is given. In Section 3, we first define the subgroups C and G of $\mathrm{GL}_n(\mathbf{Z}/p)$. Then we describe their irreducible representations over $\mathbf{F}_p$ and construct our idempotents. The relationship between the $\mathbf{F}_p[C]$ irreducibles and the $\mathbf{F}_p[\mathrm{GL}_n(\mathbf{Z}/p)]$ irreducibles is given in (3.8). Section 4 contains the main results. Theorem B is given as 4.4 and Theorem A as 4.5. Theorem 4.6 gives the complete decompositions of the $\widehat{Y}_n(i)$. In Section 5, we give expressions for the Poincaré series of the $Y_n(j)$ using Molien's theorem. Finally, in Section 6, we give some examples of our results for small cases.

For F a field, we let $F\langle v_1, \dots, v_n \rangle$ denote the F-vector space with basis $\{ v_1, \dots, v_n \}$. We let $F[v_1, \dots, v_n] \otimes \mathrm{E}[v_1, \dots, v_n]$ denote the tensor product of the polynomial ring and the exterior algebra over F. All cohomology groups will have coefficients in $\mathbf{F}_p$ unless otherwise stated. All spectra are assumed to be completed at p. And *K-theory* will mean mod$p$ K-theory.

The author would like to thank Paul Selick for a preliminary version of and many conversations about his paper with Campbell.

1. **Preliminaries on stable splittings.** A reference for this section is [HK]. Let G be a finite group, BG_+ its classifying space with a disjoint basepoint. By a standard telescope construction, idempotents in $\{ BG_+, BG_+ \}$, the ring of stable self-maps, correspond to stable wedge summands: $BG_+ \simeq eBG_+ \vee (1-e)BG_+$ [Co].

When G is a p-group, the summands can be found from idempotents in $\{ BG_+, BG_+ \} \otimes \mathbf{F}_p$. There is a generalized Burnside ring, denoted $A(G,G)$, with a natural map to $\{ BG_+, BG_+ \}$. The following theorem was proven by Lewis, May, and McClure.

THEOREM 1.1 ([M], 15). *If G is a p-group, then the map $A(G,G) \otimes \mathbf{F}_p \longrightarrow \{ BG_+, BG_+ \} \otimes \mathbf{F}_p$ is an isomorphism.*

Now let $G = (\mathbf{Z}/p)^n$. From the description of $A(G,G)$ in [M], it is easy to see that the semigroup ring $\mathbf{F}_p[\mathrm{M}_{n,n}(\mathbf{Z}/p)]$ is contained in $A(G,G) \otimes \mathbf{F}_p$. The following theorem was proven independently by the author and Nishida.

THEOREM 1.2 ([HK], 2.6). *If $e \in \mathbf{F}_p[\mathrm{M}_{n,n}(\mathbf{Z}/p)]$ is a primitive idempotent, then its image in $A(G,G) \otimes \mathbf{F}_p$ is also primitive, so $eB(\mathbf{Z}/p)^n_+$ is indecomposable.*

It follows that a formula $1 = \sum e_k$ in $\mathbf{F}_p[\mathrm{M}_{n,n}(\mathbf{Z}/p)]$, writing the identity as a sum of primitive orthogonal idempotents, gives a complete decomposition $B(\mathbf{Z}/p)^n_+ \simeq \bigvee e_k B(\mathbf{Z}/p)^n_+$.

From general representation theory (e.g. [CR1]), a primitive idempotent e in a finite dimensional algebra R over $\mathbf{F}_p$ corresponds to a projective indecomposable left ideal Re, which in turn corresponds to the irreducible $\mathbf{F}_p$ representation Re/Je, where J is the radical of R. There is a one-to-one correspondence between isomorphism types of projective indecomposables and isomorphism types of irreducible representations, and the number of times a given projective occurs in a complete decomposition of R equals the dimension of its associated irreducible over its endomorphism ring.

THEOREM 1.3 ([HK], A). *In a complete stable decomposition of $B(\mathbf{Z}/p)^n_+$, there are wedge summands of p^n distinct homotopy types. These correspond to the p^n irreducible left $\mathbf{F}_p[\mathrm{M}_{n,n}(\mathbf{Z}/p)]$-modules, and a given homotopy type appears with multiplicity equal to the dimension of the corresponding module.*

By the following theorem, there is a similar result for the $\mathcal{A}$-module summands of $H^{\otimes n}$.

THEOREM 1.4 ([AGM], P. 438). $\mathbf{F}_p[\mathrm{M}_{n,n}(\mathbf{Z}/p)] \cong \mathrm{Hom}_{\mathcal{A}}(H^{\otimes n}, H^{\otimes n})$.

Of course, orthogonal idempotents in any subring of $\mathbf{F}_p[\mathrm{M}_{n,n}(\mathbf{Z}/p)]$ will induce stable decompositions of $B(\mathbf{Z}/p)^n_+$ into (possibly decomposable) wedge summands. The most important subring for our purposes is $\mathbf{F}_p[\mathrm{GL}_n(\mathbf{Z}/p)]$.

The irreducible representations of $\mathbf{F}_p[\mathrm{M}_{n,n}(\mathbf{Z}/p)]$ and $\mathbf{F}_p[\mathrm{GL}_n(\mathbf{Z}/p)]$ can be described using Young diagrams. We adopt the following notations (see [HK], Section 6).

$$\begin{aligned} (1.5) \qquad &\mathrm{Irr}(\mathbf{F}_p[\mathrm{M}_{n,n}(\mathbf{Z}/p)]) = \{ S_{\lambda_1,\dots,\lambda_n} \mid 0 \le \lambda_k \le p-1 \} \\ &\mathrm{Irr}(\mathbf{F}_p[\mathrm{GL}_n(\mathbf{Z}/p)]) = \{ S'_{\lambda_1,\dots,\lambda_n} \mid 0 \le \lambda_k \le p-1, \text{ and } \lambda_n \le p-2 \} \end{aligned}$$

Denote the stable summand corresponding to $S_{(\lambda)}$ (resp. $S'_{(\lambda)}$) by $X_{(\lambda)}$ (resp. $X'_{(\lambda)}$). These notations give the following decompositions where the first is complete.

$$(1.6) \qquad B(\mathbf{Z}/p)^n_+ \simeq \bigvee_{(\lambda)} \dim(S_{(\lambda)}) X_{(\lambda)} \qquad B(\mathbf{Z}/p)^n_+ \simeq \bigvee_{(\lambda)} \dim(S'_{(\lambda)}) X'_{(\lambda)}$$

(The indexing sets are those given in 1.5.)

PROPOSITION 1.7 ([HK], 6.2). *With the above notations, we have*

(i) $X_{\lambda_1,\dots,\lambda_{n-1},0} \simeq X_{\lambda_1,\dots,\lambda_{n-1}}$,

(ii) $X'_{\lambda_1,\dots,\lambda_n} \simeq X_{\lambda_1,\dots,\lambda_n}$, *if* $\lambda_n \ne 0$ or $p-1$, *and*

(iii) $X'_{\lambda_1,\dots,\lambda_{n-1},0} \simeq X_{\lambda_1,\dots,\lambda_{n-1},0} \vee X_{\lambda_1,\dots,\lambda_{n-1},p-1}$.

2. **The Campbell and Selick Summands.** Let H be the mod-p cohomology of the classifying space $B(\mathbf{Z}/p)$. One of the results of the paper of Campbell and Selick is to give a decomposition of $H^{\otimes n}$ into a direct sum of (p^n-1) modules over the Steenrod algebra. This section gives a sketch of their argument.

In $\mathbf{F}_{p^n}$, choose an element ω so that ω generates the cyclic group of units in $\mathbf{F}_{p^n}$ and $\{\omega, \phi(\omega), \dots, \phi^{n-1}(\omega)\}$ forms a basis for $\mathbf{F}_{p^n}$ over $\mathbf{F}_p$ ([D]), where $\phi(a) = a^p$ is the Frobenius. Let $p(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n$ be the minimal polynomial for ω. Let

$$(2.1) \qquad T = \begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{pmatrix}$$

be the $n \times n$ matrix over $\mathbf{F}_p$ representing multiplication by ω in the basis $\{1, \omega, \dots, \omega^{n-1}\}$. Regard T as a linear transformation on the vector space $\mathbf{F}_{p^n}\langle t_0, \dots, t_{n-1}\rangle$. The eigenvalues of T are $\omega, \omega^p, \dots, \omega^{p^{n-1}}$ defined over $\mathbf{F}_{p^n}$. A basis of nonzero eigenvectors of T, $\{x_0, \dots, x_{n-1}\}$, can be chosen with $T(x_k) = \omega^{p^k} x_k$ and $x_k = \phi(x_{k-1})$ (here the Frobenius acts trivially on the t's). Let B be the matrix in $\mathrm{GL}_n(\mathbf{F}_{p^n})$ giving the x's in terms of the t's, $B\colon \mathbf{F}_{p^n}\langle t_0, \dots, t_{n-1}\rangle \longrightarrow \mathbf{F}_{p^n}\langle x_0, \dots, x_{n-1}\rangle$, and note that

BTB^{-1} is the diagonal matrix $\mathrm{diag}(\omega, \omega^p, \ldots, \omega^{p^{n-1}})$ in $GL_n(\mathbf{F}_{p^n})$. Extend B multiplicatively to polynomial algebras to give

(2.2) $$B\colon \mathbf{F}_{p^n}[t_0, \ldots, t_{n-1}] \cong \mathbf{F}_{p^n}[x_0, \ldots, x_{n-1}].$$

Give $\mathbf{F}_p[t_0, \ldots, t_{n-1}]$ the usual $\mathcal{A}$-algebra structure (thought of as the polynomial part of the cohomology of $B(\mathbf{Z}/p)^n$) $\mathcal{P}^1(t_i) = t_i^p$, and extend to $\mathbf{F}_{p^n}[t_0, \ldots, t_{n-1}]$ so that the action of $\mathcal{A}$ is $\mathbf{F}_{p^n}$-linear. The induced $\mathcal{A}$-module action on the x's is specified by $\mathcal{P}^1(x_i) = x_{i-1}^p$, where the subscripts are taken modulo n. The Cartan formula applies, so $\mathbf{F}_p[x_0, \ldots, x_{n-1}]$ is an $\mathcal{A}$-submodule of $\mathbf{F}_{p^n}[t_0, \ldots, t_{n-1}]$. (If p is odd, the Bockstein acts trivially, and if $p = 2$, take $\mathcal{P}^1 = Sq^1$.)

THEOREM 2.3 ([CS], 1). *$\mathbf{F}_p[x_0, \ldots, x_{n-1}] \cong \mathbf{F}_p[t_0, \ldots, t_{n-1}]$ as $\mathcal{A}$-modules.*

The proof uses the composition

(2.4)
$$\Psi\colon \mathbf{F}_p[x_0, \ldots, x_{n-1}] \hookrightarrow \mathbf{F}_{p^n}[x_0, \ldots, x_{n-1}] \xrightarrow{B^{-1}} \mathbf{F}_{p^n}[t_0, \ldots, t_{n-1}] \xrightarrow{\lambda} \mathbf{F}_p[t_0, \ldots, t_{n-1}],$$

where $\lambda : \mathbf{F}_{p^n}[t_0, \ldots, t_{n-1}] \longrightarrow \mathbf{F}_{p^n}[t_0, \ldots, t_{n-1}]$ is given by $\lambda(y) = \omega y + \phi(\omega y) + \cdots + \phi^{n-1}(\omega y)$ and ϕ is the Frobenius (acting trivially on the t's). Note that λ is $\mathcal{A}$-linear but *not* multiplicative.

Let $M_n = \mathbf{F}_p[x_0, \ldots, x_{n-1}]$ and define weights $w(m)$ in $\mathbf{Z}/(p^n - 1)$ for monomials m in M_n by $w(1) = 0$, $w(x_k) = p^k$, and $w(yz) = w(y) + w(z)$. Let $M_n(j)$ be the subspace of M_n having the monomials of weight j as basis. Since $\mathcal{P}^1$ preserves weights (and β acts trivially if $p > 2$), there is a decomposition

(2.5) $$M_n = \bigoplus_{j \in \mathbf{Z}/(p^n-1)} M_n(j)$$

as $\mathcal{A}$-modules. Note that $M_n(0)$ is a ring, and each $M_n(j)$ is an $M_n(0)$-module.

The self mapping $x_l \longrightarrow x_{l+1}$ of M_n shows that $M_n(j)$ is isomorphic to $M_n(jp)$. Let $\widehat{M}_n(i) = M_n(i) \oplus \cdots \oplus M_n(ip^{z_i - 1})$, where z_i is the smallest positive exponent k with $ip^k \equiv i \pmod{p^n - 1}$.

If we let $\mathbf{Z}/n = \langle\, \phi \,\rangle$ act on $\mathbf{Z}/(p^n - 1)$ by $\phi(i) = ip$, then the $\widehat{M}_n(i)$ can be described as follows. Let J_i be the orbit containing i, and let I be a set consisting of one element from each orbit. Then $\widehat{M}_n(i) = \oplus_{j \in J_i} M_n(j)$, z_i is the cardinality of J_i, and $M_n = \oplus_{i \in I} \widehat{M}_n(i)$. We will see in the next section that this last decomposition of M_n has a particularly nice description in terms of idempotents.

If $p > 2$, let $\{u_0, \ldots, u_{n-1}\}$ denote generators for an exterior algebra with $\beta(u_k) = t_k$. Then $\mathbf{F}_p[t_0, \ldots, t_{n-1}] \otimes \mathrm{E}[u_0, \ldots, u_{n-1}]$ gives the cohomology of $B(\mathbf{Z}/p)^n$. Define a new basis $\{y_0, \ldots, y_{n-1}\}$ from the $\{u_0, \ldots, u_{n-1}\}$ as the $\{x_0, \ldots, x_{n-1}\}$ were defined from the $\{t_0, \ldots, t_{n-1}\}$. With $\beta(y_k) = x_k$ Theorem 2.3 extends to give $\mathbf{F}_p[x_0, \ldots, x_{n-1}] \otimes \mathrm{E}[y_0, \ldots, y_{n-1}] \cong \mathbf{F}_p[t_0, \ldots, t_{n-1}] \otimes \mathrm{E}[u_0, \ldots, u_{n-1}]$ as $\mathcal{A}$-modules. With $w(y_j) = p^j$ the weight decomposition also extends.

We will use the notations ME_n for $\mathbf{F}_p[x_0, \ldots, x_{n-1}] \otimes \mathrm{E}[y_0, \ldots, y_{n-1}]$, $ME_n(j)$ for the weight j summand, and $\widehat{ME}_n(i)$ for the summand $\oplus_{j \in J_i} ME_n(j)$.

The modules $M_n(j)$ and $ME_n(j)$ are easy to work with because they have $\mathbf{F}_p$-bases consisting of monomials. For example, here we find the monomial of least degree in $M_n(j)$ or $ME_n(j)$. Let $j = (j_{n-1}j_{n-2}\ldots j_0)$ be the base-p representation of j, let $\sigma(j) = j_0 + \cdots + j_{n-1}$, and let $\alpha(j)$ be the cardinality of $\{\, k \mid j_k \neq 0 \,\}$. (Here we use $\{\, 0, \ldots, p^n - 2 \,\}$ to represent $\mathbf{Z}/(p^n - 1)$.) Note that $\sigma(j) = \alpha(j)$ when $p = 2$.

PROPOSITION 2.6. *The monomial of least degree in $M_n(j)$ is $x_0^{j_0}x_1^{j_1}\ldots x_{n-1}^{j_{n-1}}$; it has degree $\sigma(j)$.*

For p odd, let the x_k's and y_k's in ME_n have degrees 2 and 1, respectively.

PROPOSITION 2.7. *The monomial of least degree in $ME_n(j)$ is obtained by replacing $x_k^{j_k}$ by $x_k^{j_k-1}y_k$ (when $j_k \neq 0$) in $x_0^{j_0}x_1^{j_1}\ldots x_{n-1}^{j_{n-1}}$; it has degree $2\sigma(j) - \alpha(j)$.*

It is often convenient to eliminate the degree zero elements (spanned by the identity) from the modules $M_n(0)$ and $ME_n(0)$. We let $\overline{M}_n(0) = M_n(0)/\mathbf{F}_p \cdot 1$ and $\overline{ME}_n(0) = ME_n(0)/\mathbf{F}_p \cdot 1$. Note that $M_n(0) \cong \mathbf{F}_p \oplus \overline{M}_n(0)$ and $ME_n(0) \cong \mathbf{F}_p \oplus \overline{ME}_n(0)$ as A-modules.

PROPOSITION 2.8. *(i) The monomial of least degree in $\overline{M}_n(0)$ is $x_0^{p-1}\cdots x_{n-1}^{p-1}$; it has degree $np - n$. (ii) The monomial of least degree in $\overline{ME}_n(0)$ is $x_0^{p-2}y_0\cdots x_{n-1}^{p-2}y_{n-1}$; it has degree $2np - 3n$.*

3. **Some Representation Theory.** In this section we construct the idempotents in $\mathbf{F}_p[\mathrm{GL}_n(\mathbf{Z}/p)]$ that we will need. First we define subgroups C and G of $\mathrm{GL}_n(\mathbf{Z}/p)$. The idempotents for Theorem B are the (unique) primitive idempotents in $\mathbf{F}_p[C]$ and are given in (3.4). The idempotents for Theorem A are less canonical and lie in $\mathbf{F}_p[G]$ (see 3.18).

Let $G = \langle c, d \mid c^{p^n-1} = d^n = 1, d^{-1}cd = c^p \rangle$ and let $C \subseteq G$ be the subgroup generated by c. To fix an inclusion of G in $\mathrm{GL}_n(\mathbf{Z}/p)$, consider the $\mathbf{F}_p$ vector space $\mathbf{F}_{p^n}$ with basis $\{\, 1, \omega, \ldots, \omega^{n-1} \}$ and identify c with multiplication by ω and d with the Frobenius, ϕ. (Thus G is isomorphic to the semidirect product $(\mathbf{F}_{p^n})^* \rtimes \mathrm{Gal}(\mathbf{F}_{p^n} : \mathbf{F}_p)$.)

We now give some elementary facts about the action of $(\mathbf{F}_{p^z})^*$ on $\mathbf{F}_{p^z}$. Let ζ be a primitive $(p^z - 1)$-st root of unity in $\mathbf{F}_{p^z}$ and take $\{\, 1, \zeta, \ldots, \zeta^{z-1} \}$ as a basis for the vector space $\mathbf{F}_{p^z}$ over $\mathbf{F}_p$. Consider the z-dimensional $\mathbf{F}_p$-representation of the group $(\mathbf{F}_{p^z})^*$ on the vector space $\mathbf{F}_{p^z}$ (given by left multiplication). We call this representation B_z.

LEMMA 3.1. *Let μ be any element in $\mathbf{F}_{p^z}$ with $\mathbf{F}_{p^z} = \mathbf{F}_p(\mu)$. Then the representation B_z restricted to the cyclic group $\langle \mu \rangle$ is irreducible.*

PROOF. If v_1 and v_2 are non-zero vectors in $\mathbf{F}_{p^z}$, then $\zeta^j \cdot v_1 = v_2$ for some j. Since the set $\{\, 1, \mu, \ldots, \mu^{z-1} \}$ is a basis for $\mathbf{F}_{p^z}$ over $\mathbf{F}_p$, there exist a_k in $\mathbf{F}_p$ such that $\zeta^j = \sum_{k=0}^{z-1} a_k\mu^k$. The element $\sum a_k\mu^k$ in the group ring $\mathbf{F}_p[\langle \mu \rangle]$ takes v_1 to v_2. Since the group ring acts transitively on the non-zero vectors in $\mathbf{F}_{p^z}$, there are no non-trivial invariant subspaces. ∎

LEMMA 3.2. *The eigenvalues of the endomorphism $B_z(\zeta^j)$ acting on the vector space $\mathbf{F}_{p^z}$ are $\{\zeta^j, \zeta^{jp}, \ldots, \zeta^{jp^{z-1}}\}$.*

PROOF. Let $p(x) = a_0 + a_1 x + \cdots + a_{z-1}x^{z-1} + x^z$ be the minimum polynomial for ζ (so its roots are $\{\zeta, \zeta^p, \ldots, \zeta^{p^{z-1}}\}$). In the basis $\{1, \zeta, \ldots, \zeta^{z-1}\}$ the endomorphism given by left multiplication by ζ has the matrix

$$\begin{pmatrix} 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & \ldots & 0 & -a_1 \\ 0 & 1 & \ldots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -a_{z-1} \end{pmatrix} \tag{3.3}$$

whose characteristic polynomial is $p(x)$. ∎

We now describe the mod-p representation theory of C and give the (unique) primitive orthogonal idempotents in $\mathbf{F}_p[C]$.

Since C is abelian and p does not divide the order of C, there are $p^n - 1$ distinct one dimensional representations of C defined over $\mathbf{F}_{p^n}$. Label them by R_j, for $j \in \mathbf{Z}/(p^n - 1)$, with $R_j(c) = \omega^j$. Explicit idempotents in $\mathbf{F}_{p^n}[C]$ associated to these are $e_j = \frac{1}{p^n-1}\sum_{k=0}^{p^n-2} R_j(c^{-k})c^k = -1\sum_{k=0}^{p^n-2}\omega^{-kj}c^k$ ([CR1], 33.8). Again consider the sets J_i and I from Section 2. The action of $\mathbf{Z}/n = \langle \phi \rangle$ on $\mathbf{F}_{p^n}$ sends R_j to R_{jp} and e_j to e_{jp}.

DEFINITION 3.4. *For $i \in I$, let $f_i = \sum_{j \in J_i} e_j$.*

PROPOSITION 3.5. *(i) $f_i \in \mathbf{F}_p[C]$, (ii) $\mathbf{F}_p[C]f_i$ is an irreducible $\mathbf{F}_p[C]$-module, and (iii) the idempotents f_i are primitive in $\mathbf{F}_p[C]$.*

PROOF. (i) The f_i are invariant under ϕ.

(ii) The representation $R_i\colon C \longrightarrow (\mathbf{F}_{p^n})^*$ takes c to ω^i. Since $z_i = \min\{ k > 0 \mid \omega^i = (\omega^i)^{p^k} = \phi^k(\omega^i)\}$, we have $\mathbf{F}_p(\omega^i) = \mathbf{F}_{p^{z_i}}$. Consider the $\mathbf{F}_p[C]$-representation Γ_i given by

$$\Gamma_i\colon C \xrightarrow{R_i} (\mathbf{F}_{p^{z_i}})^* \xrightarrow{B_{z_i}} \mathrm{Hom}_{\mathbf{F}_p}(\mathbf{F}_{p^{z_i}}, \mathbf{F}_{p^{z_i}}).$$

For each j, the eigenvalues of the endomorphism $\Gamma_i(c^j)$ are $\{\omega^{ij}, \omega^{ijp}, \ldots, \omega^{ijp^{z_i-1}}\}$ by Lemma 3.2. These are the same as the eigenvalues of c^j acting on $\mathbf{F}_p[C]f_i$. Hence these two representations have the same composition factors ([CR1], 30.16).

Since the image of R_i is the group $\langle \omega^i \rangle$, Lemma 3.1 implies that Γ_i is irreducible, so $\mathbf{F}_p[C]f_i$ is also.

(iii) Follows from (ii). ∎

REMARKS 3.7. (i) Since $\mathbf{F}_p[C]$ is semisimple and commutative, it must be equal to a direct sum of fields. $\mathbf{F}_p[C] \cong \oplus \mathbf{F}_p[C]f_i$ realizes this decomposition.
(ii) The above ideas can be used to describe the $\mathbf{F}_p$ representations of any cyclic group whose order is prime to p.

Now we relate the irreducible $\mathbf{F}_p[C]$-representations to the irreducible $\mathbf{F}_p[\mathrm{GL}_n(\mathbf{Z}/p)]$-representations. This relationship will be used in the next section to give complete decompositions for the weight summands. Let $S'_{(\lambda)}$ be the irreducibles for $\mathbf{F}_p[\mathrm{GL}_n(\mathbf{Z}/p)]$

as in Section 1, and let $P'_{(\lambda)}$ be their projective covers. To simplify notation, let $R = \mathbf{F}_p[\mathrm{GL}_n(\mathbf{Z}/p)]$ and let $S = \mathbf{F}_p[C]$.

THEOREM 3.8. *$Rf_i \cong \oplus_{(\lambda)} z_i a'_{(\lambda)} P'_{(\lambda)}$, where $a'_{(\lambda)}$ is the number of times the irreducible Sf_i occurs in a composition series for $\mathrm{Res}^R_S(S'_{(\lambda)})$, the restriction of $S'_{(\lambda)}$ from R to S.*

This follows from the following four lemmas.

LEMMA 3.9. *The number of times $P'_{(\lambda)}$ occurs as a direct summand in Rf_i equals the $\mathbf{F}_p$-dimension of $\mathrm{Hom}_R(Rf_i, S'_{(\lambda)})$.*

PROOF. Write f_i as an orthogonal sum of primitive idempotents $\{\epsilon_j\}$ in R. Then $R\epsilon_j$ has a unique maximal submodule and maps to $S'_{(\lambda)}$ if and only if $R\epsilon_j \cong P'_{(\lambda)}$ ([CR1], 54.11, 54.14). Also $\dim_{\mathbf{F}_p} \mathrm{Hom}_R(S'_{(\lambda)}, S'_{(\lambda)}) = 1$, since $\mathbf{F}_p$ is a splitting field for $\mathrm{GL}_n(\mathbf{Z}/p)$. ∎

LEMMA 3.10. $\mathrm{Hom}_R(Rf_i, S'_{(\lambda)}) \cong \mathrm{Hom}_S(Sf_i, \mathrm{Res}^R_S(S'_{(\lambda)}))$

PROOF. Since $Rf_i \cong R \otimes_S Sf_i$, this is standard ([CR2], 2.19, 2.6). ∎

LEMMA 3.11. *The $\mathbf{F}_p$-dimension of $\mathrm{Hom}_S(Sf_i, \mathrm{Res}^R_S(S'_{(\lambda)}))$ equals the multiplicity of Sf_i as a composition factor in $\mathrm{Res}^R_S(S'_{(\lambda)})$ times the $\mathbf{F}_p$-dimension of $\mathrm{Hom}_S(Sf_i, Sf_i)$.*

PROOF. Since the radical of S is zero and Sf_i is irreducible (3.5), this follows from ([CR1], 54.15, 54.19). ∎

LEMMA 3.12. *$\mathrm{Hom}_S(Sf_i, Sf_i) \cong Sf_i$, so has $\mathbf{F}_p$-dimension z_i.*

PROOF. $\mathrm{Hom}_S(Sf_i, Sf_i) \cong \mathrm{Hom}_{Sf_i}(Sf_i, Sf_i) \cong Sf_i$ since f_i is a primitive central idempotent in S. ∎

We now describe the $\mathbf{F}_p$-representation theory of G and construct the idempotents for Theorem A. The argument goes as follows. First the absolutely irreducible representations over a field of characteristic zero are described. These are then used to define the irreducible representations in characteristic p. (This step is non-trivial only if p divides n.) We then observe that these representations are in fact defined over $\mathbf{F}_p$. Finally, we give a decomposition of f_i into z_i orthogonal idempotents in $\mathbf{F}_p[G]$.

Let $\tilde{K}$ be an algebraic number field which is a splitting field for G (*ie.* every irreducible $\tilde{K}[G]$-representation remains irreducible over any field extension); $\mathbf{Q}(\sqrt[|G|]{1})$ is such a field. Let R be the algebraic integers in $\tilde{K}$, and let P be a prime ideal in R with p the unique rational prime in P. The residue field $K = R/P$ is a finite field which is also a splitting field for G. Let $\tilde{\omega}$ be a primitive $(p^n - 1)$-st root of unity in $\tilde{K}$ chosen so that the reduction $R \longrightarrow R/P$ takes $\tilde{\omega}$ to ω. Also let $\tilde{\theta}$ be a primitive n-th root of unity in $\tilde{K}$. Define $\tilde{f}_i$ by the formula in Definition 3.4 with ω replaced by $\tilde{\omega}$. Define $\tilde{K}[G]$ representations $\tilde{\Gamma}_{ik}$, for $i \in I$, and $k = 1, \ldots, r_i = \frac{n}{z_i}$, by the matrices

$$(3.13) \qquad c \longmapsto \begin{pmatrix} \tilde{\omega}^i & 0 & \ldots & 0 \\ 0 & \tilde{\omega}^{ip} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \tilde{\omega}^{ip^{z_i-1}} \end{pmatrix} \qquad d \longmapsto \begin{pmatrix} 0 & 0 & \ldots & 0 & \tilde{\theta}^{kz_i} \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

LEMMA 3.14. *The $\tilde{\Gamma}_{ik}$ are irreducible, distinct, and give a full set of irreducible representations of G over $\tilde{K}$.*

PROOF. Since G is a semidirect product of cyclic groups, its irreducible characters are all induced from one dimensional characters of normal subgroups containing C ([CR2], 11.8, [T]). The matrix representations can be found using the methods in ([CR1], Section 47). ■

LEMMA 3.15. *The projective representation $\tilde{K}[G]\tilde{f}_i$ is isomorphic to $\oplus_{k=1}^{r_i} z_i \tilde{\Gamma}_{ik}$.*

PROOF. $\tilde{f}_i$ is the central idempotent to which all of the $\tilde{\Gamma}_{ik}$ belong. ■

Now let Γ_{ik} be the K-representation of G given by applying the map $R \longrightarrow R/P = K$ to the matrices in (3.13) defining $\tilde{\Gamma}_{ik}$. Note that θ, the reduction of $\tilde{\theta}$ will be a primitive s-th root of unity, where $n = sp^l$ with $(s,p) = 1$. Let $r_i = \frac{n}{z_i} = s_i p^l$, with $(s_i, p) = 1$.

THEOREM 3.16. *(i) The Γ_{ik} are irreducible, and (ii) $\{ \Gamma_{ik} \mid i \in I$ and $k = 1, \ldots, s_i \}$ is a complete set of distinct irreducibles for G over K.*

PROOF. (i) Let $\{ v_j \}_{j \in J_i}$ be a basis for Γ_{ik} having the given matrix representation. Suppose $w = \sum_{j \in J_i} a_j v_j$ is a non-zero vector in an invariant subspace W. If Γ_{ik} is restricted to C, then v_j is an eigenvector with eigenvalue ω^j, so $e_j \cdot w = a_j v_j$ (Here we assume $\mathbf{F}_{p^n} \subseteq K$, so $e_j \in K[C]$). If $a_{j_0} \neq 0$, then $(a_{j_0})^{-1} e_{j_0} \cdot w = v_{j_0}$, so v_{j_0} is in W. The action of d permutes the v_j's (with multiplication by θ^{kz_i} in one case), so all of the v_j's are in W.

(ii) Two irreducible matrix representations over K are isomorphic if and only if they have the same characteristic roots ([CR1], 30.16). The result then follows from Lemma 3.14 and the fact that $s_i = \min\{ l > 0 \mid \theta^{lz_i} = 1 \}$. ■

COROLLARY 3.17. *The projective representation $K[G]f_i$ has a composition series with n quotients: each Γ_{ik}, for $k = 1, \ldots, s_i$, occurs $\frac{n}{s_i}$ times.*

PROOF. This follows from Lemma 3.15. (Since these representations are modular, they may not be completely reducible, so we cannot conclude as in 3.15 that this is a direct sum decomposition.) ■

The representations Γ_{ik} have characters in $\mathbf{F}_p$, so they are defined over $\mathbf{F}_p$ ([HB], 1.17). Hence $\mathbf{F}_p$ is a splitting field for G. It follows from (3.16) and the fact that $\dim_{\mathbf{F}_p}(\Gamma_{ik}) = |J_i|$, that there are primitive orthogonal idempotents $\{ \epsilon_{ijk} \mid i \in I,\ j \in J_i,$ and $k = 1, \ldots, s_i \}$ in $\mathbf{F}_p[G]$, with $\mathbf{F}_p[G]\epsilon_{ijk}$ a projective indecomposable associated to Γ_{ik} for each $j \in J_i$, and with $f_i = \sum_{j,k} \epsilon_{ijk}$ for each $i \in I$. (Note that when $s_i < r_i$, $\tilde{f}_i$ has a finer decomposition than f_i.

DEFINITION 3.18. For $j \in J_i$, let $d_j = \sum_k \epsilon_{ijk}$.

PROPOSITION 3.19 ([W], THEOREM 4.1). *The summand $d_j B(\mathbf{Z}/p)^n_+$ has rank 1 K-theory for $i \neq 0$ and rank 2 K-theory for $i = 0$. (Note that $d_0 B(\mathbf{Z}/p)^n_+ \simeq d_0 B(\mathbf{Z}/p)^n \vee S^0$; each of these summands has rank 1 K-theory.)*

REMARK 3.20. Witten doesn't construct the idempotents d_j as above. Instead she uses the $K[G]$-representation Γ_{i0} above and standard facts about lifting idempotents to show that the f_i can be written as a sum of z_i orthogonal idempotents projecting to the primitive idempotents in $\mathbf{F}_p[G]f_i/\operatorname{Ann}(\Gamma_{i0})$. She then shows that any such idempotent decomposition of f_i gives the K-theory result.

The specific idempotents $\{d_j\}$ allow us to prove the following.

COROLLARY 3.21. *The modules $\mathbf{F}_p[G]d_j$, for $j \in J_i$ are isomorphic.*

PROOF. For fixed i and k, the idempotents ϵ_{ij_1k} and ϵ_{ij_2k} are conjugate in $\mathbf{F}_p[G]$. ∎

REMARK 3.22. It is easy to see that the idempotents f_i in Definition 3.4 are in the center of $\mathbf{F}_p[G]$, thus $\mathbf{F}_p[G]$ is isomorphic to $\oplus_{i\in I}\mathbf{F}_p[G]f_i$ as *rings* (compare [W], p. 42). In general, some of the f_i are not centrally primitive and can be further decomposed into the *block idempotents*. These can be determined from the complex character table (see [CR1], Section 85) and could be used to give a finer decomposition than the f_i give. We do not pursue this here.

4. **Main Results.** To begin this section, we recall the $\mathrm{GL}_n(\mathbf{Z}/p)$ actions on the polynomial rings in the Campbell and Selick composition (2.4):

$$(4.1)\qquad \begin{aligned}\Psi\colon \mathbf{F}_p[x_0,\ldots,x_{n-1}] &\hookrightarrow \mathbf{F}_{p^n}[x_0,\ldots,x_{n-1}]\\ &\xrightarrow{B^{-1}} \mathbf{F}_{p^n}[t_0,\ldots,t_{n-1}] \xrightarrow{\lambda} \mathbf{F}_p[t_0,\ldots,t_{n-1}].\end{aligned}$$

$\mathrm{GL}_n(\mathbf{Z}/p)$ acts in the usual way on the vector space $\mathbf{F}_p\langle t_0,\ldots,t_{n-1}\rangle$. Extending multiplicatively to polynomial rings gives actions of $\mathbf{F}_p[\mathrm{GL}_n(\mathbf{Z}/p)]$ on $\mathbf{F}_p[t_0,\ldots,t_{n-1}]$ and of $\mathbf{F}_{p^n}[\mathrm{GL}_n(\mathbf{Z}/p)]$ on $\mathbf{F}_{p^n}[t_0,\ldots,t_{n-1}]$. Let $\mathrm{GL}_n(\mathbf{F}_{p^n})$ act in the usual way on the vector space $\mathbf{F}_{p^n}\langle x_0,\ldots,x_{n-1}\rangle$. Include $\mathrm{GL}_n(\mathbf{Z}/p)$ in $\mathrm{GL}_n(\mathbf{F}_{p^n})$ by $(a_{ij})\mapsto B(a_{ij})B^{-1}$, where B is the matrix in (2.2). Then $\mathbf{F}_{p^n}[\mathrm{GL}_n(\mathbf{Z}/p)]$ acts on $\mathbf{F}_{p^n}[x_0,\ldots,x_{n-1}]$. Note that this action does not restrict to an action of $\mathrm{GL}_n(\mathbf{Z}/p)$ on $\mathbf{F}_p[x_0,\ldots,x_{n-1}]$ (e.g. $T(x_0)=\omega x_0$).

LEMMA 4.2. *The map B^{-1} is $\mathbf{F}_{p^n}[\mathrm{GL}_n(\mathbf{Z}/p)]$-linear, and the map λ is $\mathbf{F}_p[\mathrm{GL}_n(\mathbf{Z}/p)]$ linear.*

PROOF. The linearity of B^{-1} follows from the definitions, and the linearity of λ is easy to check. ∎

Recall the definition of $\widehat{M}_n(i)$ given after (2.5).

THEOREM 4.3. *$\widehat{M}_n(i) \cong f_i\mathbf{F}_p[t_0,\ldots,t_{n-1}]$ as $\mathcal{A}$-modules.*

PROOF. The first two rings in the composition for Ψ decompose into (p^n-1) weight summands. As an $\mathbf{F}_{p^n}[C]$-module, the weight j summand in $\mathbf{F}_{p^n}[x_0,\ldots,x_{n-1}]$ is a direct sum of infinitely many copies of the representation R_j. Hence the idempotents e_j decompose the ring $\mathbf{F}_{p^n}[x_0,\ldots,x_{n-1}]$ into its weight summands. Unfortunately, the e_j's do not act on the ring $\mathbf{F}_p[t_0,\ldots,t_{n-1}]$, so we cannot use them to decompose it. However, the idempotents f_i *do* act, the f_i are in $\mathbf{F}_p[\mathrm{GL}_n(\mathbf{Z}/p)]$, and B^{-1} and λ are $\mathbf{F}_p[\mathrm{GL}_n(\mathbf{Z}/p)]$-module maps. The result follows. ∎

Recall that $ME_n = \mathbf{F}_p[x_0,\ldots,x_{n-1}]\otimes \mathrm{E}[y_0,\ldots,y_{n-1}]$ when p is odd. The above theorem extends in the obvious way to $\widehat{ME}_n(i)$. In terms of summands of $B(\mathbf{Z}/p)^n_+$ we have the following.

THEOREM 4.4.

$$\tilde{H}^*(f_iB(\mathbf{Z}/p)^n_+) \cong \begin{cases} \widehat{ME}_n(i), & \text{if } p \text{ is odd;} \\ \widehat{M}_n(i), & \text{if } p = 2. \end{cases}$$

COROLLARY 4.5.

$$\tilde{H}^*(d_jB(\mathbf{Z}/p)^n_+) \cong \begin{cases} ME_n(j), & \text{if } p \text{ is odd;} \\ M_n(j), & \text{if } p = 2. \end{cases}$$

We let $Y_n(j) = d_jB(\mathbf{Z}/p)^n_+$ and $\widehat{Y}_n(i) = f_iB(\mathbf{Z}/p)^n_+ \simeq \bigvee_{j\in J_i} Y_n(j)$.

Now we give some applications of these results. Since the f_i are in the group ring (as opposed to the semigroup ring), the complete decompositions of the $\widehat{Y}_n(i) \simeq f_iB(\mathbf{Z}/p)^n_+$ can be given in terms of the $X'_{(\lambda)}$ described in Section 1. The next result follows from Theorems 3.8 and 4.4.

THEOREM 4.6. *$\widehat{Y}_n(i) \simeq \bigvee_{(\lambda)} z_i a'_{(\lambda)} X'_{(\lambda)}$, where $a'_{(\lambda)}$ is the number of times the representation $\mathbf{F}_p[C]f_i$ occurs in a composition series for $\mathrm{Res}_C^{\mathrm{GL}_n(\mathbf{Z}/p)}(S'_{(\lambda)})$.*

To apply this theorem, one calculates the eigenvalues of the action of the element c on the representation space $S'_{(\lambda)}$, then compares to the eigenvalues of c on $\mathbf{F}_p[C]f_i$, which are $\{\omega^i, \dots, \omega^{ip^{z_i-1}}\}$. The case $i = 0$ is particularly simple:

COROLLARY 4.7. *$Y_n(0) \simeq \bigvee_{(\lambda)} a'_{(\lambda)} X'_{(\lambda)}$, where $a'_{(\lambda)} = \dim(S'_{(\lambda)})^C$.*

This corollary is a special case of ([HK], 5.1). We mention that Campbell and Selick show that $\tilde{H}^*(Y_n(0)) \cong (\tilde{H}^*(B(\mathbf{Z}/p)^n_+))^C$, so $Y_n(0)$ is equivalent to $B((\mathbf{Z}/p)^n \rtimes C)_+$ and to $B(\mathrm{GL}_2(\mathbf{F}_{p^n}))_+$ ([A]). For p odd (resp. $p = 2$) this cohomology is isomorphic to $ME_n(0)$ (resp. $M_n(0)$) as A-modules, but *not* as rings. However, if we tensor with $\mathbf{F}_{p^n}$ we do get that the *rings* $ME_n(0)\otimes\mathbf{F}_{p^n}$ (resp. $M_n(0)\otimes\mathbf{F}_{2^n}$) and $\tilde{H}^*(Y_n(0); \mathbf{F}_{p^n})$ are isomorphic. (Compare with Aguadé [A]).

From Propositions 2.6, 2.7, 2.8, and Theorem 4.4, we have

THEOREM 4.8. *For $0 \leq j \leq (p^n - 2)$, the bottom cell of $Y_n(j)$ is in dimension $2\sigma(j) - \alpha(j)$. The second cell in $Y_n(0)$ is in dimension $2pn - 3n$.*

From Proposition 3.19, we have

THEOREM 4.9. *$Y_n(j)$ has rank 1 K-theory if $j \neq 0$, and rank 2 K-theory if $j = 0$.*

The K-theory of the indecomposable summands of $B(\mathbf{Z}/p)^n_+$ are given by Kuhn and Carlisle.

PROPOSITION 4.10 ([K], 1.5; [CK], 6.1). *The indecomposable summands $X_{i,0,\dots,0}$, for $0 \leq i \leq (p-1)$, and $X_{0,\dots,0,j,k,0,\dots,0}$, for $j + k = (p-1)$, each have rank 1 K-theory. All other indecomposables have zero K-theory.*

We now restrict to $p = 2$. For $1 \leq k \leq n$, let $S(k)$ denote the irreducible $\mathbf{F}_2[\mathrm{M}_{n,n}(\mathbf{Z}_2)]$-representation $S_{0,\dots,0,1,0,\dots,0}$, where the 1 is in the k-th position. Let $S(0) = S_{0,\dots,0}$. For $0 \leq k \leq n$, let $X(k)$ denote the indecomposable wedge summand of $B(Z/2)^n_+$ corresponding to $S(k)$.

THEOREM 4.11. *Let $p = 2$. For $0 \leq j \leq (2^n - 2)$, $Y_n(j)$ contains exactly one copy of the summand $X(k)$ if and only if $k = \alpha(j)$. Also, $Y_n(0)$ contains the copy of $X(n)$.*

PROOF. The irreducible $S(k)$ has dimension $\binom{n}{k}$ ([JK], 8.3.9), so $X(k)$ has multiplicity $\binom{n}{k}$ in $B(Z/2)^n_+$. The number of $Y_n(j)$'s with $\alpha(j) = k$ is also $\binom{n}{k}$.

The bottom cell of $X(k)$ is in dimension k ([CK], 1.1). and the bottom cell of $Y_n(j)$ is in dimension $\alpha(j)$. Therefore, the $X(k)$ must be distributed among the $Y_n(j)$ as stated to avoid contradicting Theorem 4.9. ■

5. **Poincaré Series.** It is easy to determine the beginning coefficients in the Poincaré series for the $M_n(j)$ or the $ME_n(j)$ since these modules are generated by monomials. One just writes down all of the monomials and calculates their weights. Here we obtain a closed form for these series using invariant theory.

Let K be any field, and let W be an irreducible $K[Q]$-module, where Q is a finite group. For N a graded $K[Q]$-module of finite type, define $F(N, Q, W; t) = \sum_{k=0}^{\infty} a_k t^k$, where a_k is the multiplicity of W as a composition factor in N_k. In this notation, if N is $\mathbf{F}_{p^n}[x_0, \ldots, x_{n-1}]$ (or $\mathbf{F}_{p^n}[x_0, \ldots, x_{n-1}] \otimes \mathrm{E}[y_0, \ldots, y_{n-1}]$), then $F(N, C, R_j; t)$ is the Poincaré series of the weight j summand in N. A classical theorem of Molien gives a formula for $F(N, Q, W; t)$ when $K = \mathbf{C}$, $N = \mathbf{C}[x_0, \ldots, x_{n-1}]$, and $Q \subseteq \mathrm{GL}_n(\mathbf{C})$ ([S], 2.1). In our case, we have the following.

THEOREM 5.1. *Let $[\overline{X}]$ and $[\overline{Y}]$ denote $[x_0, \ldots, x_{n-1}]$ and $[y_0, \ldots, y_{n-1}]$, respectively, then*

$$F(\mathbf{F}_{p^n}[\overline{X}], C, R_j; t) = \frac{1}{(p^n - 1)} \sum_{l=0}^{p^n-2} \left(\frac{\tilde{\omega}^{-lj}}{\prod_{k=0}^{n-1}(1 - \tilde{\omega}^{lp^k} t)} \right), \text{ and}$$

$$F(\mathbf{F}_{p^n}[\overline{X}] \otimes \mathrm{E}[\overline{Y}], C, R_j; t) = \frac{1}{(p^n - 1)} \sum_{l=0}^{p^n-2} \left(\frac{\tilde{\omega}^{-lj} \prod_{k=0}^{n-1}(1 + \tilde{\omega}^{lp^k} t)}{\prod_{k=0}^{n-1}(1 - \tilde{\omega}^{lp^k} t^2)} \right),$$

where $\tilde{\omega}$ is a primitive $(p^n - 1)$-st root of unity in $\mathbf{C}$.

PROOF. These follow exactly as in the classical case since $(p, |C|) = 1$ and $\mathbf{F}_{p^n}$ is a splitting field for C. (Recall that in $\mathbf{F}_{p^n}[\overline{X}] \otimes \mathrm{E}[\overline{Y}]$ we take $deg(x_k) = 2$ and $deg(y_k) = 1$.) ■

The above formulas also give the Poincaré series for the $Y_n(j)$'s since the series for the weight summands in $\mathbf{F}_p[\overline{X}] \otimes \mathrm{E}[\overline{Y}]$ (resp. $\mathbf{F}_2[\overline{X}]$, if $p = 2$) and $\mathbf{F}_{p^n}[\overline{X}] \otimes \mathbf{F}_{p^n}[\overline{Y}]$ (resp. $\mathbf{F}_{2^n}[\overline{X}]$) are the same.

REMARK 5.2. The Poincaré series for the indecomposable summands $X'_{(\lambda)}$ are given by $F(N, \mathrm{GL}_n(\mathbf{Z}/p), S'_{(\lambda)}; t)$, for N either $\mathbf{F}_2[t_0, \ldots, t_{n-1}]$ or $\mathbf{F}_p[t_0, \ldots, t_{n-1}] \otimes \mathrm{E}[u_0, \ldots, u_{n-1}]$ ([Mi1], 1.6). These are known for only a few cases: $n = 2, p = 2$ ([MP]); $n = 2$, p odd ([H] or [C]); $n = 3, p = 2$ ([Mi1] or [C]); $n = 3$, p odd ([C]); $n = 4, p = 2$ ([C]); for $S'_{(\lambda)}$ a twisted Steinberg representation ([Mi2], [MP]); and for $S'_{(\lambda)}$ *close* to the Steinberg representation ([CW]).

https://doi.org/10.4153/CJM-1992-006-8 Published online by Cambridge University Press

6. **Examples.** Recall from Section 1, that a complete decomposition of the space $B(\mathbf{Z}/p)^n_+$ is given by $\bigvee_{(\lambda)} a_{(\lambda)} X_{(\lambda)}$, where $(\lambda) = (\lambda_1, \ldots, \lambda_n)$, $0 \le \lambda_i \le (p-1)$, and a partial decomposition is given by $\bigvee_{(\lambda)} a_{(\lambda)} X'_{(\lambda)}$, where $(\lambda) = (\lambda_1, \ldots, \lambda_n)$, $0 \le \lambda_i \le (p-1)$, $\lambda_n \le p-2$. (The $X'_{(\lambda)}$ decompose as in Proposition 1.7.) The Selick and Campbell decompositions are given here in terms of the $X'_{(\lambda)}$ for some small cases. These can be determined either from Theorem 4.6 or by comparing Poincaré series.

EXAMPLE 6.1. *For $p = 2$:*

$$\begin{aligned}
Y_1(0) &\simeq X'_0,\\
Y_2(0) &\simeq X'_{0,0},\\
Y_2(1) \simeq Y_2(2) &\simeq X'_{1,0},\\
Y_3(0) &\simeq X'_{0,0,0} \vee 2X'_{1,1,0},\\
Y_3(1) \simeq Y_3(2) \simeq Y_3(4) &\simeq X'_{1,0,0} \vee X'_{1,1,0},\\
Y_3(3) \simeq Y_3(5) \simeq Y_3(6) &\simeq X'_{0,1,0} \vee X'_{1,1,0},\\
Y_4(0) &\simeq X'_{0,0,0,0} \vee 2X'_{1,0,1,0} \vee 4X'_{1,1,1,0},\\
Y_4(1) \simeq Y_4(2) \simeq Y_4(4) &\simeq Y_4(8)\\
&\simeq X'_{1,0,0,0} \vee X'_{1,1,0,0} \vee X'_{1,0,1,0} \vee 2X'_{0,1,1,0} \vee 4X'_{1,1,1,0},\\
Y_4(3) \simeq Y_4(6) \simeq Y_4(9) &\simeq Y_4(12)\\
&\simeq X'_{0,1,0,0} \vee X'_{1,1,0,0} \vee X'_{1,0,1,0} \vee X'_{0,1,1,0} \vee 5X'_{1,1,1,0},\\
Y_4(5) \simeq Y_4(10) &\simeq X'_{0,1,0,0} \vee 2X'_{1,1,0,0} \vee 2X'_{0,1,1,0} \vee 4X'_{1,1,1,0},\\
Y_4(7) \simeq Y_4(11) \simeq Y_4(13) &\simeq Y_4(14)\\
&\simeq X'_{0,0,1,0} \vee 2X'_{1,1,0,0} \vee X'_{1,0,1,0} \vee X'_{0,1,1,0} \vee 4X'_{1,1,1,0}.
\end{aligned}$$

The indecomposable summands with rank 1 K-theory are: X_0, X_1, $X_{0,1}$, $X_{0,0,1}$, and $X_{0,0,0,1}$.

EXAMPLE 6.2. *For $p = 3$:*

$$\begin{aligned}
Y_1(0) &\simeq X'_0,\\
Y_1(1) &\simeq X'_1,\\
Y_2(0) &\simeq X'_{0,0} \vee X'_{2,1},\\
Y_2(1) \simeq Y_2(3) &\simeq X'_{1,0},\\
Y_2(2) \simeq Y_2(6) &\simeq X'_{2,0} \vee X'_{2,1},\\
Y_2(4) &\simeq X'_{0,1} \vee X'_{2,0},\\
Y_2(5) \simeq Y_2(7) &\simeq X'_{1,1},\\
Y_3(0) &\simeq X'_{0,0,0} \vee X'_{1,1,1} \vee 3X'_{2,2,0},\\
Y_3(1) \simeq Y_3(3) \simeq Y_3(9) &\simeq X'_{1,0,0} \vee X'_{1,2,0} \vee 2X'_{2,1,1} \vee X'_{0,2,1} \vee 2X'_{2,2,1},\\
Y_3(2) \simeq Y_3(6) \simeq Y_3(18) &\simeq X'_{2,0,0} \vee X'_{2,1,0} \vee X'_{1,1,1} \vee 2X'_{2,2,0} \vee X'_{1,2,1},\\
Y_3(4) \simeq Y_3(10) \simeq Y_3(12) &\simeq X'_{2,0,0} \vee X'_{0,1,0} \vee X'_{2,1,0} \vee 2X'_{2,2,0} \vee 2X'_{1,2,1},\\
Y_3(5) \simeq Y_3(15) \simeq Y_3(19) &\simeq X'_{1,1,0} \vee X'_{1,2,0} \vee X'_{2,1,1} \vee X'_{2,0,1} \vee 2X'_{2,2,1},\\
Y_3(7) \simeq Y_3(11) \simeq Y_3(21) &\simeq X'_{1,1,0} \vee X'_{1,2,0} \vee X'_{2,1,1} \vee X'_{0,2,1} \vee 2X'_{2,2,1},
\end{aligned}$$

$$Y_3(8) \simeq Y_3(20) \simeq Y_3(24) \simeq X'_{0,2,0} \vee X'_{2,1,0} \vee X'_{1,1,1} \vee 2X'_{2,2,0} \vee X'_{1,2,1},$$
$$Y_3(13) \simeq X'_{1,1,0} \vee X'_{0,0,1} \vee 3X'_{2,2,1},$$
$$Y_3(14) \simeq Y_3(16) \simeq Y_3(22) \simeq X'_{0,2,0} \vee 2X'_{2,1,0} \vee 2X'_{2,2,0} \vee X'_{1,0,1} \vee X'_{1,2,1},$$
$$Y_3(17) \simeq Y_3(23) \simeq Y_3(25) \simeq X'_{0,1,1} \vee 2X'_{1,2,0} \vee X'_{2,1,1} \vee X'_{2,0,1} \vee 2X'_{2,2,1},$$

The indecomposable summands with rank 1 K-theory are: $X_0, X_1, X_2, X_{0,2}, X_{1,1}, X_{0,0,2}$, and $X_{0,1,1}$.

EXAMPLE 6.3. *For $p = 5$:*

$$Y_1(0) \simeq X'_0,$$
$$Y_1(1) \simeq X'_1,$$
$$Y_1(2) \simeq X'_2,$$
$$Y_1(3) \simeq X'_3,$$

$$Y_2(0) \simeq X'_{0,0} \vee X'_{2,3} \vee X'_{4,2},$$
$$Y_2(1) \simeq Y_2(5) \simeq X'_{1,0} \vee X'_{3,3},$$
$$Y_2(2) \simeq Y_2(10) \simeq X'_{2,0} \vee X'_{4,1} \vee X'_{4,3},$$
$$Y_2(3) \simeq Y_2(15) \simeq X'_{3,0} \vee X'_{3,2},$$
$$Y_2(4) \simeq Y_2(20) \simeq X'_{2,3} \vee X'_{4,0} \vee X'_{4,2},$$
$$Y_2(6) \simeq X'_{0,1} \vee X'_{2,0} \vee X'_{4,3},$$
$$Y_2(7) \simeq Y_2(11) \simeq X'_{1,1} \vee X'_{3,0},$$
$$Y_2(8) \simeq Y_2(16) \simeq X'_{2,1} \vee X'_{4,0} \vee X'_{4,2},$$
$$Y_2(9) \simeq Y_2(21) \simeq X'_{3,1} \vee X'_{3,3},$$
$$Y_2(12) \simeq X'_{0,2} \vee X'_{2,1} \vee X'_{4,0},$$
$$Y_2(13) \simeq Y_2(17) \simeq X'_{1,2} \vee X'_{3,1},$$
$$Y_2(14) \simeq Y_2(22) \simeq X'_{2,2} \vee X'_{4,1} \vee X'_{4,3},$$
$$Y_2(18) \simeq X'_{0,3} \vee X'_{2,2} \vee X'_{4,1},$$
$$Y_2(19) \simeq Y_2(23) \simeq X'_{1,3} \vee X'_{3,2}.$$

The indecomposable summands with rank 1 K-theory are: X_0, X_1, X_2, X_3, X_4, $X_{0,4}$, $X_{1,3}$, $X_{2,2}$, and $X_{3,1}$.

REFERENCES

[AGM] J. F. Adams, J. H. Gunawardena, and H. R. Miller, *The Segal conjecture for elementary abelian p-groups*, Topology **24**(1985), 435–460.

[A] J. Aguadé, *The cohomology of GL_2 of a finite field*, Arch. Math. **34**(1980), 509–516.

[CS] H. E. A. Campbell and P. S. Selick, *Polynomial algebras over the Steenrod algebra*, Comment. Math. Helv., **65**(1990) 171–180.

[C] D. Carlisle, *The modular representation theory of* GL(n,p)*, and applications to topology*. Ph.D. thesis, University of Manchester, 1985.

[CK] D. Carlisle and N. J. Kuhn, *Subalgebras of the Steenrod algebra and the action of matrices on truncated polynomial algebras*, J. Algebra **121**(1989), 370–387.

[CW] D. P. Carlisle and G. Walker, *Poincaré series for the occurrence of certain modular representations of* GL(n,p) *in the symmetric algebra*, preprint (1989).

[Ca] G. Carlsson, *Equivariant stable homotopy and Segal's Burnside ring conjecture*, Ann. of Math. (2)**120** (1984), 189–224.

[Co] F. Cohen, *Splitting certain suspensions via self-maps*, Ill. J. Math. **20**(1976), 336–347.

[CR1] C. W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*. Wiley, 1962.

[CR2]______, *Methods of Representation Theory*, vol. 1, Wiley, 1981.

[D] H. Davenport, *Bases for finite fields*, J. London Math. Soc. **43**(1968), 21–39.

[H] J. C. Harris, *Stable splittings of classifying spaces*. Ph.D. thesis, University of Chicago, 1985.

[HK] J. C. Harris and N. J. Kuhn, *Stable decompositions of classifying spaces of finite abelian p-groups*, Math. Proc. Camb. Phil. Soc. **103**(1988), 427–449.

[HB] B. Huppert and N. Blackburn, *Finite Groups, II*. Springer-Verlag, 1982.

[JK] G. James and A. Kerber, *The Representation Theory of the Symmetric group*. Encyclopedia of Math. and its Applications, **16**, Addison-Wesley, 1981.

[K] N. J. Kuhn, *The Morava K-theory of some classifying spaces*, Trans. Amer. Math. Soc. **304**(1987), 193–205.

[LS] J. Lannes and L. Schwartz, *Sur la structure des $\mathcal{A}$-modules instable injectifs*, Topology **28**(1989), 153–169.

[LZ1] J. Lannes and S. Zarati, *Sur les $\mathcal{U}$-injectives*, Ann. Scient. Ec. Norm. Sup. **19**(1986), 303–333.

[LZ2]______, *Sur les foncteurs dérivés de la déstabilisation*, Math. Z. **194**(1987), 25–59.

[M] J. P. May, *Stable maps between classifying spaces*, Amer. Math. Soc. Cont. Math. **37**(1985), 121–129.

[Mi] H. R. Miller, *The Sullivan conjecture on maps from classifying spaces*, Ann. of Math. **120**(1984), 39–87.

[Mi1] S. A. Mitchell, *Splitting $B(\mathbf{Z}/p)^n$ and BT_n via modular representation theory*, Math. Z. **189**(1985), 1–9.

[Mi2]______, *Finite complexes with $A(n)$-free cohomology*, Topology **24**(1985), 227–248.

[MP] S. A. Mitchell and S. B. Priddy, *Stable splittings derived from the Steinberg module*, Topology **22**(1983), 285–298.

[N] G. Nishida, *Stable homotopy type of classifying spaces of finite groups*, Algebraic and Topological Theories (1985), 391–404.

[S] R. P. Stanley, *Invariants of finite groups and their applications to combinatorics*, Bull. Amer. Math. Soc. (1)**3**(1979), 475–511.

[T] P. A. Tucker, *On the reduction of induced representations of finite groups*, Amer. J. Math. **84**(1962), 400–420.

[W] C. M. Witten, *Self-maps of classifying spaces of finite groups and classification of low-dimensional Poincaré duality spaces*. Ph.D. Thesis, Stanford University, 1978.

[Wo] R. M. W. Wood, *Splitting $\Sigma(\mathbf{C}P^\infty \times \cdots \times \mathbf{C}P^\infty)$ and the action of Steenrod squares Sq^i on the polynomial ring $\mathbf{F}_2[x_1, \ldots, x_n]$*, in *Algebraic Topology Barcelona* 1986, Lecture Notes in Math. **1298** Springer 1987, 237–255.

Department of Mathematics
University of Toronto
Toronto, Ontario M5S 1A1
(harris@math.toronto.edu)