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CENTRALISERS ON RINGS AND ALGEBRAS

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In this paper we investigate identities related to centralisers in rings and algebras. We prove, for example, the following result. Let A be a semisimple H^*-algebra and let $T : A \to A$ be an additive mapping satisfying the relation $T(x^{m+n+1}) = x^m T(x) x^n$ for all $x \in A$ and some integers $m \geqslant 1$, $n \geqslant 1$. In this case T is a left and a right centraliser.

Throughout, R will represent an associative ring with centre $Z(R)$. Given an integer $n \geqslant 2$, a ring R is said to be n-torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$. Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ imples that either $a = 0$ or $b = 0$, and is semiprime in case $aRa = (0)$ imples $a = 0$. An additive mapping $T : R \to R$ is called a left centraliser in case $T(xy) = T(x)y$ holds for all $x, y \in R$. The concept appears naturally in C^*-algebras. In ring theory it is more common to work with module homomorphisms. Ring theorists would write that $T : R_R \to R_R$ is a homomorphism of a ring module R into itself. For a semiprime ring R all such homomorphisms are of the form $T(x) = qx$ where q is an element of the Martindale right ring to quotients Q_r (see Chapter 2 by Beidar and Martindale). In case R has the identity element $T : R \to R$ is a left centraliser if and only if T is of the form $T(x) = ax$ for some $a \in R$. An additive mapping $T : R \to R$ is called a left Jordan centraliser in case $T(x^2) = T(x)x$ holds for all $x \in R$. The definition of right centralizer and right Jordan centralizer should be self-explanatory. In case $T : R \to R$ is a left and right centraliser, where R is a semiprime ring with extended centroid C, then there exists an element $\lambda \in C$ such that $T(x) = \lambda x$ for all $x \in R$ (see [**2**, Theorem 2.3.2]).

Zalar [**12**] has proved that any left (right) Jordan centraliser on a 2-torsion free semiprime ring is a left (right) centraliser. Molnár [**7**] has proved that in case we have an additive mapping $T : A \to A$, where A is a semisimple H^*-algebra, satisfying the relation $T(x^3) = T(x)x^2$ (respecitvely $T(x^3) = x^2T(x)$) for all $x \in A$, then T is a left (right) centraliser. Let us recall that a semisimple H^*-algebra is a semisimple Banach *-algebra whose norm is a Hilbert space norm such that $(x, yz^*) = (xz, y) = (z, x^*y)$

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is fulfilled for all $x, y, z \in A$ (see [1]). The result of Benkovič and Eremita [3] states that in case we have a prime ring R and an additive mapping $T : R \to R$ satisfying the relation $T(x^n) = T(x)x^{n-1}$ for all $x \in R$, where $n \geqslant 2$ is a fixed integer, then T is a left centraliser in case $\operatorname{char}(R) = 0$ or $\operatorname{char}(R) \geqslant n$. Some results concerning centalisers on semiprime rings can be found in [3, 6] and [8, 9, 10, 11]. Let X be a real or complex Banach space and let $L(X)$ and $F(X)$ denote the algebra of all bounded linear operators on X and the ideal of all finite rank operators in $L(X)$, respectively. An algebra $A(X) \subset L(X)$ is said to be standard in case $F(X) \subset A(X)$. Let us point out that any standard algebra is prime, which is a consequence of Hahn-Banach theorem. We denote by X^* the dual space of a Banach space X and by I the identity operator on X.

It is our aim in this paper to prove the following result.

Theorem 1. *Let A be a semisimple H^*-algebra and let $T : A \to A$ be an additive mapping satisfying the relation*

$$T(x^{m+n+1}) = x^m T(x) x^n$$

for all $x \in A$ and some integers $m \geqslant 1$, $n \geqslant 1$. In this case T is a left and a right centraliser.

For the proof of the theorem above we need the result below which is of independent interest.

Theorem 2. *Let X be a Banach space over a real or complex field F and let $A(X) \subset L(X)$ be a standard operator algebra. Suppose there exists an additive mapping $T : A(X) \to L(X)$ satisfying the relation*

$$T(A^{m+n+1}) = A^m T(A) A^n$$

for all $A \in A(X)$ and some integers $m \geqslant 1$, $n \geqslant 1$. In this case T is of the form $T(A) = \lambda A$ for some $\lambda \in F$.

In the proof of Theorem 2 we shall use some ideas similar to those used in [7] and the following purely algebraic results proved by Brešar [4] and Zalar [12].

Theorem A. ([4, Theorem 2].) *Let R be a 2-torsion free prime ring. Suppose there exists an additive mapping $F : R \to R$ satisfying the relation $\big[[F(x), x], x\big] = 0$ for all $x \in R$. In this case $[F(x), x] = 0$ holds for all $x \in R$.*

Theorem B. ([12, Proposition 1.4].) *Let T be a 2-torsion free semiprime ring and let $T : R \to R$ be a left (right) Jordan centraliser. In this case T is a left (right) centraliser.*

Proof of Theorem 2: We have the relation

$$T(A^{m+n+1}) = A^m T(A) A^n. \tag{1}$$

Let us first consider the restriction of T on $F(X)$. Let A be from $F(X)$ and let $P \in F(X)$ be a projection with $AP = PA = A$. From the above relation one obtains $T(P) = PT(P)P$, which gives

$$T(P)P = PT(P) = PT(P)P. \tag{2}$$

Putting $A + P$ for A in the relation (1), we obtain

$$\begin{aligned}(3)\quad &\sum_{i=0}^{m+n+1} \binom{m+n+1}{i} T(A^{m+n+1-i}P^i) \\ &= \left(\sum_{i=0}^{m} \binom{m}{i} A^{m-i}P^i\right)(T(A)+B)\left(\sum_{i=0}^{n} \binom{n}{i} A^{n-i}P^i\right),\end{aligned}$$

where B stands for $T(P)$. Using (1) and rearranging the equation (3) in the sense of collecting together terms involving an equal number of factors of P we obtain:

$$\sum_{i=1}^{m+n} f_i(A,P) = 0, \tag{4}$$

where $f_i(A,P)$ stands for the expression of terms involving i factors of P. Replacing A by $A+2P$, $A+3P$, ..., $A+(m+n)P$ in turn in the equation (1), and expressing the resulting system of $m+n$ homogeneous equations in the variables $f_i(A,P)$, $i = 1, 2, \ldots, m+n$, we see that the coefficient matrix of the system is a van der Monde matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{m+n} \\ \vdots & \vdots & \vdots & \vdots \\ m+n & (m+n)^2 & \cdots & (m+n)^{m+n} \end{bmatrix}.$$

Since the detminant of the matrix is different from zero, it follows that the system has only a trivial solution.

In particular

$$\begin{aligned} &f_{m+n-1}(A,P) \\ &\quad = \binom{m+n+1}{m+n-1} T(A^2) - \binom{m}{m-2}\binom{n}{n} A^2B - \binom{m}{m}\binom{n}{n-2} BA^2 \\ &\qquad - \binom{m}{m-1}\binom{n}{n} AT(A)P - \binom{m}{m}\binom{n}{n-1} PT(A)A - \binom{m}{m-1}\binom{m}{n-1} ABA \\ &\quad = 0,\end{aligned}$$

and

$$
\begin{aligned}
&f_{m+n}(A,P)\\
&\quad=\binom{m+n+1}{m+n}T(A)-\binom{m}{m-1}\binom{n}{n}AB-\binom{m}{m}\binom{n}{n-1}BA-\binom{m}{m}\binom{n}{n}PT(A)P\\
&\quad=0.
\end{aligned}
$$

The above equations reduce to

$$
\begin{aligned}
(5)\quad (m+n+1)(m+n)T(A^2)&=m(m-1)A^2B+n(n-1)BA^2+2mnABA\\
&\quad+2mAT(A)P+2nPT(A)A,
\end{aligned}
$$

and

$$(6)\qquad (m+n+1)T(A)=mAB+nBA+PT(A)P.$$

Right multiplications of the relation (6) by P gives

$$(7)\qquad (m+n+1)T(A)P=mAB+nBA+PT(A)P.$$

Similarly one obtains

$$(8)\qquad (m+n+1)PT(A)=mAB+nBA+PT(A)P.$$

Combining (7) with (8) gives

$$T(A)P=PT(A),$$

which reduces the relations (5) to

$$
\begin{aligned}
(9)\quad (m+n+1)(m+n)T(A^2)&=m(m-1)A^2B+n(n-1)BA^2+2mnABA\\
&\quad+2mAT(A)+2nT(A)A,
\end{aligned}
$$

and the relation (7) to

$$(10)\qquad (m+n)T(A)P=mAB+nBA.$$

Combining (10) with (6) gives

$$(11)\qquad T(A)=T(A)P.$$

From the above relation one can conclude that T maps $F(X)$ into itself. Further from (11), (10) reduces to

$$(12)\qquad (m+n)T(A)=mAB+nBA.$$

From this we can conclude that T is linear on $F(X)$. Further apply (12) we obtain

$$\begin{aligned}2mnABA &= n(mAB)A + mA(nBA)\\ &= n\big((m+n)T(A) - nBA\big)A + mA\big((m+n)T(A) - mAB\big)\\ &= (m+n)\big(nT(A)A + mAT(A)\big) - n^2BA^2 - m^2A^2B.\end{aligned}$$

We have therefore

$$2mnABA = (m+n)\big(nT(A)A + mAT(A)\big) - n^2BA^2 - m^2A^2B.$$

Applying (12) and the relation above to (9) we obtain

$$(m+n)T\big(A^2\big) = nT(A)A + mAT(A), \tag{13}$$

and multiplying by $(m+n)$ we obtain

$$(m+n)^2T\big(A^2\big) = n(m+n)T(A)A + mA(m+n)T(A).$$

Applying the above relation on both sides of (12) we obtain

$$(m+n)\big(mA^2B + nBA^2\big) = n(mAB + nBA)A + mA(mAB + nBA),$$

which reduces to

$$\big[[A,B],A\big] = 0. \tag{14}$$

Relation (12) gives $(m+n)\big[T(A),A\big] = mA[B,A] + n[B,A]A$. By the above relation one can replace $A[B,A]$ by $[B,A]A$ which gives

$$\big[T(A),A\big] = [B,A]A.$$

Then applying (14) we obtain $\Big[\big[T(A),A\big],A\Big] = \big[[B,A]A,A\big] = \big[[B,A],A\big]A = 0$. Thus we have

$$\Big[\big[T(A),A\big],A\Big] = 0,$$

for any $A \in F(X)$. We have therefore an additive mapping T which maps $F(X)$ into itself satisfying the relation above for any $A \in F(X)$. Since $F(X)$ is prime all the assumptions of Theorem A are fulfilled which means that

$$\big[T(A),A\big] = 0,$$

holds for any $A \in F(X)$. Applying this in (13), one obtains that $T\big(A^2\big) = T(A)A$ and $T\big(A^2\big) = AT(A)$ holds for all $A \in F(X)$. In other words, T is a left and a right

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Jordan centraliser on $F(X)$. By Theorem B it follows that T is a left and also a right centraliser of $F(X)$.

We intend to prove that there exists $C \in L(X)$, such that

$$T(A) = CA, \quad \text{for all} \quad A \in F(X). \tag{15}$$

For any fixed $x \in X$ and $f \in X^*$ we denote by $x \otimes f$ an operator from $F(X)$ defined by $(x \otimes f)y = f(y)x$, for all $y \in X$. For any $A \in L(X)$ we have $A(x \otimes f) = \big((Ax) \otimes f\big)$. Let us choose f and y such that $f(y) = 1$ and define $Cx = T(x \otimes f)y$. Obviously, C is linear. Using the fact that T is a left centraliser on $F(X)$ we obtain

$$(CA)x = C(Ax) = T\big((Ax) \otimes f\big)y = T\big(A(x \otimes f)\big)y = T(A)(x \otimes f)y = T(A)x, \; x \in X.$$

We have therefore $T(A) = CA$, for any $A \in F(X)$. Since T a right centraliser on $F(X)$ we obtain $C(AP) = T(AP) = AT(P) = ACP$, where $A \in F(X)$ and P is an arbitrary one-dimensional projection. We have therfore $[A, C]P = 0$. Since P is arbitrary one-dimensional projection it follows that $[A, C] = 0$, for any $A \in F(X)$. Using the closed graph theorem one can easily prove that C is continuous. Since C commutes with all operators from $F(X)$ one can conclude that $Cx = \lambda x$ holds for any $x \in X$ and some $\lambda \in F$, which together with the relation (15) gives that T is of the form

$$T(A) = \lambda A \tag{16}$$

any $A \in F(X)$ and some $\lambda \in F$.

It remains to prove that the above relation holds for any $A \in A(X)$ as well. Let us introduce $T_1 : A(X) \to L(X)$ by $T_1(A) = \lambda A$ and consider $T_0 = T - T_1$. The mapping T_0 is, obviously additive and satisfies the relation (1). Besides, T_0 vanishes on $F(X)$. It is our aim to prove that T_0 vanishes on $A(X)$ as well. Let $A \in A(X)$, let P be a one-dimensional projection and let $S = A + PAP - (AP + PA)$. Note that S can be written in the form $S = (I - P)A(I - P)$, where I denotes the identity operator on X. Since, obviously, $S - A \in F(X)$, we have $T_0(S) = T_0(A)$. Besides, $SP = PS = 0$. We have therefore the relation

$$T_0\big(A^{m+n+1}\big) = A^m T_0(A) A^n, \tag{17}$$

for all $A \in A(X)$. Applying the above relation we obtain

$$\begin{aligned} S^m T_0(S) S^n = T_0\big(S^{m+n+1}\big) &= T_0\big(S^{m+n+1} + P\big) = T_0\big((S + P)^{m+m+1}\big) \\ &= (S + P)^m T_0\big(S + (S^m + P)\big) T_0(S)(S^n + P) \\ &= S^m T_0(S) S^n + P T_0(S) S^n + S^m T_0(S) P + P T_0(S) P. \end{aligned}$$

We have therefore

(18) $$PT_0(S)S^n + S^mT_0(A)P + PT_0(A)P = 0.$$

Multiplying the above relation from both sides by P we obtain

(19) $$PT_0(A)P = 0,$$

which reduces (18) to

(20) $$PT_0(A)S^n + S^mT_0(A)P = 0.$$

Right multiplication by P then gives

(21) $$S^mT_0(A)P = 0.$$

We intend to prove that

(22) $$S^{m-1}T_0(A)P = 0.$$

Putting $A+B$ for A, where $B \in F(X)$, in (21) and using the fact that T_0 vanishes on $F(X)$, we obtain

$$(S_1S^{m-1} + SS_1S^{m-2} + \cdots + S^{m-1}S_1)T_0(A)P = 0,$$

where S_1 stands for $(I-P)B(I-P)$ (see [5]). The substitution $T(A)PB$ for B in the above relation gives because of (19)

$$(T_0(A)PBS^{m-1} + ST_0(A)PBS^{m-2} + \cdots + S^{m-1}T(A)PB)T_0(A)P = 0.$$

Multiplying from the left side by S^{m-1} and applying (21) we obtain

$$(S^{m-1}T_0(A)P)B(S^{m-1}T_0(A)P) = 0,$$

for all $B \in F(X)$. Then it follows $S^{m-1}T_0(A)P = 0$ by the primeness of $F(X)$, which proves (22).

Now, (21) implies (22), one can conclude by induction that $ST_0(A)P = 0$, which gives

$$AT_0(A)P - PAT_0(A)P = 0,$$

because of (19). Then putting $A+B$ for A, where $B \in F(X)$, we obtain $0 = (A+B)T_0(A)P - P(A+B)T_0(A)P = BT_0(A)P - PBT_0(A)P$. We have therefore proved that

$$BT_0(A)P - PBT_0(A)P = 0$$

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holds for all $A \in A(X)$ and all $B \in F(X)$. The substitution $T_0(A)PB$ for B in the above relation gives, because of (19), $(T_0(A)P)B(T_0(A)P) = 0$, for all $B \in F(X)$. Thus it follows $T_0(A)P = 0$ by the primeness of $F(X)$. Since P is an arbitrary one-dimensional projection, one can conclude that $T_0(A) = 0$, for any $A \in A(X)$, which completes the proof of the theorem. ∎

PROOF OF THEOREM 1: The proof goes through using the same arguments as in the proof of the Theorem of [**7**], with the exception that one has to use Theorem 2 instead of the Lemma in [**7**].

In the proof of Theorem 2 (the relation (13)) we met an additive mapping $T : F(X) \to F(X)$ satisfying the relation

$$(m+n)T(A^2) = mAT(A) + nT(A)A$$

for all $A \in F(X)$. In the case $m = n$ this reduces to $2T(A^2) = T(A)A + AT(A)$. Vukman [**7**] has proved that when we have an additive mapping $T : R \to R$, where R is an arbitrary 2-torsion free semiprime ring, satisfying the relation $2T(x^2) = T(x)x + xT(x)$ for all $x \in R$, then T is a left and right centraliser. These observations lead to the following conjecture.

CONJECTURE 1. Let m and $n, m \neq -n$ be some nonzero integers and let R be a semiprime ring with suitable torsion restrictions. Suppose there exists an additive mapping $T : R \to R$ satisfying the relation

$$(m+n)T(x^2) = mxT(x) + nT(x)x$$

for all $x \in R$. In this case T is a left and right centraliser.

Our last result is related to conjecture above.

THEOREM 3. *Let m and $n, m \neq -n$, be some nonzero integers and let R be a $|mn|$ and $|m+n|$-torsion free semiprime ring. Suppose there exists and additive mapping $T : R \to R$ satisfying the relation*

$$(m+n)T(xy) = mxT(y) + nT(x)y, \tag{23}$$

for all pairs $x, y \in R$. In this case T is a left and a right centraliser.

PROOF: We have the relation

$$(m+n)T(xy) = mxT(y) + nT(x)y, \tag{23}$$

for all pairs $x, y \in R$. We compute the expression $(m+n)^2T(xyx)$ in two ways. First applying the relation above

$$\begin{aligned}(m+n)^2T(x(yx)) &= m(m+n)xT(yx) + n(m+n)T(x)yx \\ &= mx(myT(x) + nT(y)x) + n(m+n)T(x)yx, \quad x, y \in R.\end{aligned}$$

Thus we have

$$(24)\qquad (m+n)^2T(xyx) = m^2xyT(x) + mnxT(y)x + mnT(x)yx + n^2T(x)yx,$$

for x, $y \in R$. On the other hand using (23)

$$\begin{aligned}(m+n)^2T\big((xy)x\big) &= m(m+n)xyT(x) + n(m+n)T(xy)x \\ &= m(m+n)xyT(x) + n\big(mxT(y) + nT(x)y\big)x, \quad x, y \in R.\end{aligned}$$

Thus we have

$$(25)\quad (m+n)^2T(xyx) = m^2xyT(x) + mnxyT(x) + mnxT(y)x + n^2T(x)yx; x,\ y \in R.$$

Subtracting the relation (25) from (24) we obtain $mn\big(T(x)yx = xyT(x)\big) = 0$, for all pairs x, $y \in R$, which reduces to

$$T(x)yx - xyT(x) = 0, \quad x,\ y \in R$$

since we have assumed that R is $|mn|$-torsion free. Putting in the above relation first yx for y then multiplying from the right side by x and subtracting the relations so obtained one from another we obtain $xy\big[T(x), x\big] = 0$, for all pairs x, $y \in R$. From this one obtains easily $\big[T(x), x\big]y\big[T(x), x\big] = 0$, for all pairs x, $y \in R$. Hence it follows

$$(26)\qquad \big[T(x), x\big] = 0,\ x \in R$$

by the semiprimeness of R. The substitution $y = x$ in (23) gives

$$(m+n)T\big(x^2\big) = mxT(x) + nT(x)x,\ x \in R.$$

By (26) one can then replace $xT(x)$ by $T(x)x$ which gives $(m+n)T\big(x^2\big) = (m+n)T(x)x$ for all $x \in R$. Since we have assumed that R is $|m+n|$-torsion free, it follows that $T\big(x^2\big) = T(x)x$ holds for all $x \in R$. Of course, we also have $T\big(x^2\big) = xT(x)$, for all $x \in R$. In other words, T is a left and right Jordan centraliser. By Theorem B T is a left and a right centraliser. The proof of the theorem is complete. ∎

We conclude with the following conjecture.

Conjecture 2. Let R be a semiprime ring with suitable torsion restrictions and let $T : R \to R$ be an additive mapping satisfying the relation

$$T\big(x^{m+n+1}\big) = x^mT(x)x^n$$

for all $x \in R$ and some integers $m \geqslant 1, n \geqslant 1$. In this case T is a left and right centraliser.

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