CONVEXITY PROPERTIES FOR WEAK SOLUTIONS OF SOME DIFFERENTIAL EQUATIONS IN HILBERT SPACES

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1. In this work we obtain a simultaneous extension of Theorems 1.6 and 1.7 in Agmon and Nirenberg **(1)**, together with a partial extension of the result on backward unicity for parabolic equations by Lions and Malgrange **(4)**.

2. Let H be a Hilbert space. $(\cdot)$ and $|\ |$ are the notations for the scalar product and the norm in this space. Consider in H a family $B(t)$, $0 \leqslant t \leqslant T$, of closed linear operators with dense domain $D_{B(t)}$ (varying) with t. Let $L^2(0, T, H)$ be the space of Bochner square-integrable vector-valued functions with values in H. Our main result is the following

THEOREM 1. *Let $u(t)$ be a function defined for $0 \leqslant t \leqslant T$ and with values in H, with the following properties:*

(i) $u(t) \in L^2(0, T, H)$, $\dot{u}(t) = du/dt \in L^2(0, T, H)$; $u(t) \in D_{B(t)} \cap D_{B^*(t)}$ *for almost all* t, $0 \leqslant t \leqslant T$; $\dot{u}(t) - B(t)u = 0$, $0 \leqslant t \leqslant T$; $|u(t)| > 0$ *for* $0 \leqslant t \leqslant T$.

(ii) *The scalar function* $\operatorname{Re}(B(t)u(t),u(t))$ *is almost everywhere differentiable in* $[0, T]$, *and the derivative* $d[\operatorname{Re}(B(t)u(t), u(t))]/dt$ *is integrable in every interval* $\alpha \leqslant t \leqslant \beta$, *such that* $0 < \alpha < \beta < T$.

(iii) *There exist a constant* $k \geqslant 0$ *and an increasing twice continuously differentiable function* $\omega(t)$, $0 \leqslant t \leqslant T$, *such that the inequality*

$$\operatorname{Re}[d(B(t)u(t), u(t))/dt] \geqslant \tfrac{1}{2}|(B(t) + B^*(t))u(t)|^2 + (\ddot{\omega}/\dot{\omega}) \operatorname{Re}((B(t) - k)u, u) \tag{2.1}$$

holds, almost everywhere in $0 < t < T$.

Then, if (i)–(iii) *are fulfilled, the function* $\log|e^{-kt}u(t)|$ *is a convex function of* $s = \omega(t)$.

Proof of Theorem 1. We use the following (known) criterion of convexity:

LEMMA 1. *Let $f(t)$ be a continuous scalar function on $0 \leqslant t \leqslant T$, with the property that*

$$\int_0^T f(t)\mu(t)dt \geqslant 0$$

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for any $\mu(t) \geqslant 0$, *with compact support in* $]0, T[$, *and of the class* $C^2[0, T]$. *Then* $f(t)$ *is convex on* $[0, T]$.

Now, we shall prove the convexity of $f(t) = \log[e^{-2kt}|u(t)|^2]$ as a function of $s = \omega(t)$. Let $C_0^2(a, b)$ denote the class of functions twice continuously differentiable, with compact support in (a, b). We observe that the class of positive functions $\mu(t) \in C_0^2(0, T)$ is mapped by the transformation

$$\mu(t) \to M(s) \tag{2.2}$$

defined by $M(\omega(t)) = \mu(t)$ on the class of positive functions $M(s) \in C_0^2(\omega(0), \omega(T))$. Hence, we have to prove, according to the lemma, putting $t = \omega^{-1}(s)$, the relation

$$\int_{\omega(0)}^{\omega(T)} \log \exp[-2k\omega^{-1}(s)]|u(\omega^{-1}(s))|^2 \frac{d^2M(s)}{ds^2} \geqslant 0 \tag{2.3}$$

for every non-negative $M(s)$ in $C_0^2(\omega(0), \omega(T))$. Substituting $s = \omega(t)$, we deduce from (2.3) that the relation

$$\int_0^T \log[e^{-2kt}|u(t)|^2] \frac{\dot{\omega}\ddot{\mu} - \dot{\mu}\ddot{\omega}}{\dot{\omega}^2} dt \leqslant 0 \tag{2.4}$$

must hold, for any non-negative $\mu(t)$ in $C_0^2(0, T)$, where the dot indicates differentiation with respect to t. Now, we write $e^{-2kt}|u(t)|^2 = q(t)$, and follow essentially the calculation of (**1**, pp. 137–138).

Observe that using (i), we can integrate by parts reducing (2.4) to

$$\int_0^T \frac{\dot{q}}{q}\frac{\dot{\mu}}{\dot{\omega}} dt \leqslant 0, \tag{2.5}$$

for any non-negative $\mu(t) \geqslant 0$ in $C_0^2(0, T)$. As we have, almost everywhere on $(0, T)$,

$$\dot{q} = 2e^{-2kt}\operatorname{Re}(Bu, u) - 2kq, \tag{2.6}$$

(2.5) becomes

$$\int_0^T \left[\frac{e^{-2kt}\operatorname{Re}(Bu, u) - kq}{\dot{\omega}q}\right] \dot{\mu}\, dt \leqslant 0, \tag{2.7}$$

for all non-negative $\mu(t)$ in $C_0^2(0, T)$. As μ has compact support, say $[\alpha, \beta]$, in $(0, T)$, we can apply (ii), integrate by parts once more, and obtain, on account of (2.6),

$$\int_0^T \mu(t)\left[\frac{d(e^{-2kt}\operatorname{Re}(Bu, u) - kq)/dt}{q\dot{\omega}} - \frac{2(e^{-2kt}\operatorname{Re}(Bu, u) - kq)^2}{q^2\dot{\omega}}\right] dt - \int_0^T \mu(t)\left[\frac{1}{q}\frac{\ddot{\omega}}{\dot{\omega}^2}(e^{-2kt}\operatorname{Re}(Bu, u) - kq\right] dt \geqslant 0 \tag{2.8}$$

for all non-negative μ in $C^2(0, T)$. Hence (2.3) follows if we prove that the

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coefficient of $\mu(t)$ is non-negative almost everywhere on $(0, T)$, or, using $q\dot{\omega} > 0$, that

$$(2.9)\quad \frac{d}{dt}(e^{-2kt}\operatorname{Re}(Bu, u) - kq) - \frac{2}{q}(e^{-2kt}\operatorname{Re}(Bu, u) - kq)^2 - \frac{\ddot{\omega}}{\dot{\omega}}(e^{-2kt}\operatorname{Re}(Bu, u) - kq) \geqslant 0$$

almost everywhere on $(0, T)$. But (2.9) equals

$$(2.10)\quad e^{-2kt}\frac{d}{dt}\operatorname{Re}(Bu, u) - \frac{2}{q}e^{-4kt}(\operatorname{Re}(Bu, u))^2 - \frac{\ddot{\omega}}{\dot{\omega}}(e^{-2kt}\operatorname{Re}(Bu, u) - kq)$$

$$= e^{-2kt}\frac{d}{dt}\operatorname{Re}(Bu, u) - \frac{1}{2q}e^{-4kt}((B + B^*)u, u)^2 - \frac{\ddot{\omega}}{\dot{\omega}}(e^{-2kt}\operatorname{Re}(Bu, u) - kq)$$

$$\geqslant e^{-2kt}\frac{d}{dt}\operatorname{Re}(Bu, u) - \tfrac{1}{2}e^{-2kt}|(B + B^*)u|^2 - \frac{\ddot{\omega}}{\dot{\omega}}(e^{-2kt}\operatorname{Re}(Bu, u) - kq) \geqslant 0$$

by (iii). This proves Theorem 1.

Remark 1. The theorem is obviously an extension of (**1**, Theorem 1.7), where $\omega(t) = e^{ct}$, and stronger derivability hypotheses on $u(t)$ seem to be assumed.

Remark 2. Our Theorem 1 is also an extension of (**1**, Theorem 1.6). In fact, we can derive from Theorem 1 the following

THEOREM 2. *Let A be a symmetric operator in the Hilbert space H, with dense domain. Suppose $u(t) \in L^2(0, T; H)$, $\dot{u}(t) \in L(0, T; H)$, $|u(t)| > 0$ on $0 \leqslant t \leqslant T$, $u(t) \in D_A$ for almost every t, $0 \leqslant t \leqslant T$, $\dot{u} - \gamma A u = 0$, a.e. on $(0, T)$; γ is a complex number. Then* $\log|u(t)|$ *is a convex function of t, $0 \leqslant t \leqslant T$.*

This result extends (**1**, Theorem 1.6), where $u(t)$ is supposed to be twice strongly continuously differentiable.

We apply Theorem 1. Hypothesis (i) is obviously fulfilled. The non-trivial part in the proof is the verification of (ii). This result is a consequence of the following.

LEMMA 2. *Given*

(a) *a symmetric operator C with dense domain D_C in a Hilbert space H;*

(b) *a function $v(t) \in L^2(0, T; H)$, with $\dot{v}(t) \in L^2(0, T; H)$, belonging to D_C for almost every $t \in (0, T)$, such that $Cv \in L^2(0, T; H)$;*

(c) *a C^∞ scalar function $\zeta(t)$, defined for $t \geqslant 0$, such that $\zeta(t) = 0$ for $t \geqslant T - \delta$, $\zeta(t) = 1$ for $t \leqslant T - 2\delta$ $(\delta > 0)$.*

Then

$$(2.11)\qquad (C(\zeta v), \zeta v) = -2\operatorname{Re}\int_t^\infty (C(\zeta v), d(\zeta v)/dt)dt$$

for almost all $t \in (0, T)$.

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Remark. A similar formula is proved in (**3**, p. 136, formula (4.8)). The proof given there is easily adapted to our case.

Finally, one easily verifies (iii), taking $k = 0$ and $\omega(t) = t$, and Theorem 2 is proved.

We can, as a matter of fact, prove a further extension of Theorem 2 directly. However, it will no longer be a special case of Theorem 1. We state this result as follows.

THEOREM 2′. *Let A be a symmetric operator with dense domain in H. Suppose $u(t)$ is a strongly continuous function with values in H, defined for $0 < t < T$, $|u(t)| > 0$, and satisfying*

$$(2.12)\qquad -\int_0^T (u(t), \dot{\phi}(t))dt = \int_0^T (u(t), \bar{\gamma}A^*\phi(t))dt$$

for every $\phi(t) \in C_0^1(0, T; H)$, $\phi(t) \in D_{A^*}$, $0 \leqslant t \leqslant T$, $A^*\phi(t) \in L^2(0, T; H)$. Then, $\log|u(t)|$ is convex in t.

We indicate the proof briefly. Denote by $\tilde{A}$ the closure of A, and consider a sequence $\{\alpha_n(t)\}$ such that:

$$\alpha_n(t) \in C_0^\infty(-\infty, \infty), \quad \alpha_n = 0 \qquad \text{for } |t| > 1/n, \int \alpha_n\, dt = 1 \ (\alpha_n \to \delta).$$

The regularizations

$$(u * \alpha_n)(t) = \int_{-\infty}^{\infty} u(\tau)\alpha_n(t - \tau)d\tau$$

are well defined for $1/n < t < T - 1/n$. It is easy to prove that the $C^\infty(1/n, T - 1/n; H)$-valued functions, $(u*\alpha_n)(t)$, also belong to $D_{\tilde{A}}$ for every t, $1/n < t < T - 1/n$, and that

$$(2.13)\qquad d(u*\alpha_n)/dt = \gamma\tilde{A}(u*\alpha_n) \qquad \text{for } 1/n < t < T - 1/n.$$

It is known that this implies that $\log|u*\alpha_n|$ is convex in t for

$$1/n < t < T - 1/n.$$

As $n \to \infty$, $u*\alpha_n \to u(t)$, $0 < t < T$; hence

$$\lim_{n\to\infty} \log|u*\alpha_n| = \log|u(t)|, \qquad 0 < t < T.$$

As the limit of convex functions is convex, our result follows.

3. Our last application of Theorem 1 is a partial extension of a result by Lions and Malgrange (**4**).

Let us recall their notations and definitions. Consider two Hilbert spaces V and H, $V \subset H$ with continuous immersion, V dense in H. The symbols $((,))$ and $(,)$ denote the scalar products in V and H respectively, while $\| \ \|$ and $| \ |$ denote the corresponding norms.

Let t be a real variable, $0 \leqslant t \leqslant T$; for every such t a sesqui-linear form $a(t, u, v)$ is defined, continuous on $V \times V$, which we shall suppose, less generally than in **(4)**, to be symmetric:

$$a(t, u, v) = a(\overline{t, v, u}), \qquad u, v \in V. \tag{3.1}$$

Moreover, as in **(2)**, we assume that

$$a(t, u, v) \in C^1[0, T], \qquad u, v \in V, \tag{3.2}$$

and there are two positive numbers λ and α such that

$$a(t, v, v) + \lambda|v|^2 \geqslant \alpha||v||^2 \qquad \text{for all } v \in V. \tag{3.3}$$

We remark that these relations readily imply that

$$|a(t, u, v)| \leqslant M||u||\;||v||, \qquad |\dot{a}(t, u, v)| \leqslant M||u||\;||v||$$

for $u, v \in V$, $0 \leqslant t \leqslant T$, where M is a positive constant.

The form $a(t, u, v)$ defines an (unbounded) linear operator $A(t)$ in H through

$$(A(t)u, v) = a(t, u, v); \qquad t \in [0, T], u \in D_{A(t)}, v \in H \tag{3.4}$$

where

$$D_{A(t)} = \{u \in V, |a(t, u, v)| \leqslant C_u|v| \; v \in V\}.$$

From (3.1) it follows that $A(t)$ is self-adjoint in H. Its domain $D_{A(t)}$ is not constant; but it is easily seen that

$$D_{(A(t)+\lambda)^{1/2}} = V. \tag{3.5}$$

We indicate how one can derive from our Theorem 1 the following result.

THEOREM 3. *Let $u(t) \in L^2(0, T; V)$, $\dot{u}(t) \in L^2(0, T; H)$, $u(t) \in D_{A(t)}$ for almost every $t \in [0, T]$, $\dot{u}(t) + A(t)u(t) = 0$. Then, if $|u(t)| > 0$, $0 \leqslant t \leqslant T$, the function $\log[e^{-kt}|u(t)|]$ is a convex function of $s = e^{ct}$, for some positive constants k and c.*

Remark. S. G. Krein **(2)** announced a similar theorem, but he assumes that $D_{A(t)}$ is constant for $0 \leqslant t \leqslant T$; this is less general than (3.5).

We shall show how Theorem 3 follows from Theorem 1, where $\omega(t) = e^{ct}$ with some $c > 0$ to be determined.

We need a preliminary result which is only slightly different from **(4**, Lemma 2.2**)**.

LEMMA 3. *Assuming the hypotheses of Theorem 3, the function $\dot{u}(t)$ belongs to $L^2(\alpha, \beta; V)$ for $0 < \alpha < \beta < T$.*

The proof is an easy adaptation of that of **(4**, Lemma 2.2**)**.

Next, apply Theorem 1, and verify that the conditions (i)–(iii) are satisfied for $B(t) = -A(t)$. Consider the scalar function

$$\mathrm{Re}(B(t)u, u) = -a(t, u, u).$$

By Lemma 3, its derivative almost everywhere is

$$d[\mathrm{Re}(B(t)u,u)]/dt = -\dot{a}(t, u, u) - a(t, \dot{u}, u) - a(t, u, \dot{u}).$$

We have

$$|a(t, u, u)| \leqslant M\|u(t)\|^2 \in L^1(0, T),$$
$$|a(t, \dot{u}, u)| \leqslant M\|\dot{u}\| \, \|u\| \in L^1(\alpha, \beta) \quad \text{for } 0 < \alpha < \beta < T.$$

Finally, we have to prove that $c > 0$, $k > 0$ can be chosen such that for $\omega(t) = e^{ct}$,

$$-\dot{a}(t, u, u) - 2\,\mathrm{Re}[a(t, u, \dot{u})] = -\dot{a}(t, u, u) + 2\,\mathrm{Re}[a(t, u, Au)]$$
$$= -\dot{a}(t, u, u) + 2|A(t)u|^2 \geqslant 2|Au|^2 - c\,\mathrm{Re}((A(t) + k)u, u).$$

This is equivalent to

$$\dot{a}(t, u, u) \leqslant c(a(t, u, u) + k|u|^2),$$

which follows from the facts that $|\dot{a}(t, u, u)| < M\|u\|^2$ and $a(t, u, u) + \lambda|u|^2 \geqslant \alpha\|u\|^2$, with some $c, k > 0$.

Remark. The above convexity property is valid for positive-norm solutions $u(t)$. But, with an obvious argument, it implies the backward unicity for all solutions, which means, as is well known, that if $u(T) = 0$, then $u(t) = 0$, $0 \leqslant t \leqslant T$.

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