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A COMBINED ADAPTIVE CONTROL PARAMETRIZATION AND HOMOTOPY CONTINUATION TECHNIQUE FOR THE NUMERICAL SOLUTION OF BANG–BANG OPTIMAL CONTROL PROBLEMS

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Abstract

We present an efficient computational procedure for the solution of bang–bang optimal control problems. The method is based on a well-known adaptive control parametrization method, which is one of the direct methods for numerical solution of optimal control problems. First, the adaptive control parametrization method is reviewed and then its advantages and disadvantages are illustrated. In order to resolve the need for a priori knowledge about the structure of optimal control and for resolving the sensitivity to an initial guess, a homotopy continuation technique is combined with the adaptive control parametrization method. The present combined method does not require any assumptions on the control structure and the number of switching points. In addition, the switching points are captured accurately; also, efficiency of the method is reported through illustrative examples.

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Keywords and phrases: bang–bang optimal control, switching point, numerical methods, control parametrization method, homotopy continuation technique.

1. Introduction

Optimal control has been of great interest for many decades. Because of the complexity of most applications, optimal control problems are most often solved numerically [3, 27, 42, 44, 46]. In optimal control fields, a classical topic is the bang–bang type of control problems. Bang–bang control, where the input control jumps from one boundary to another, is the optimal strategy to solve a wide range of control problems in some of the well-known areas of application, such as

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industrial robots [18], aerospace engineering [43], cranes [17], applied physics [23] and biological systems [24, 25]. When the control has lower and upper bounds, and appears linearly in the objective function and the dynamical equations, the optimal control is often characterized by bang–bang type control [4]. Because of the difficulty in obtaining switching points and an optimal solution, the numerical approximation of the bang–bang optimal control problem has received considerable attention. Consequently, various algorithms for optimal bang–bang control have been reported, some of which are referred to below.

Kaya and Noakes [20] developed the switching time computation method. They found a feasible, but not necessarily optimal, concatenation of constant-input arcs from an initial point to a target point for a given number of switchings. They also developed the time-optimal switching algorithm [21] for time-optimal switching control of general nonlinear systems with a single control input. Their algorithm needed a feasible bang–bang solution to start, which was typically obtained by using the switching time computation method.

A time-scaling transformation technique [26] (also called the control parametrization enhancing technique) was developed for solving optimal discrete-valued control problems which are more general than the optimal bang–bang control problems. With this transformation, the original problem with variable control switching points is transformed into an ordinary optimal control problem with known and fixed switching points. Thus, the transformed problem can be easily solved using many existing optimal control methods.

Yu et al. [47] proposed a new computational method in which, by introducing new controls and applying an equivalent transformation, the original problem becomes a standard optimal control problem subject to equality and inequality constraints. Then, an exact penalty method is employed to solve the transformed problem.

Lin et al. [30] presented a computational method which is based on a piecewise-constant approximation of the control. This approximation scheme is accurate, but tends to lead to an approximate nonlinear programming problem which is very difficult to solve numerically. Accordingly, they presented a new technique [31], in order to solve difficulties of their old method [30], by introducing a novel procedure for transforming the approximate problem into a new problem that is easier to solve.

A modified pseudo-spectral scheme was given by Shamsi [41] for obtaining a bang–bang optimal solution to optimal control problems. The method obtains accurate solutions to bang–bang optimal control problems, and can capture switching points very accurately. However, the reader is referred to other works [28, 33, 35, 45], which deal with the computation of bang–bang optimal control problems. Note that some of these algorithms require the number of switching points and structure of the optimal control, some of which are sensitive to an initial guess.

The aim of this paper is to present an algorithm for the solution of bang–bang optimal control problems, such that the sensitivity to the initial guess is resolved, and does not require a priori knowledge of the optimal control solution. For this purpose, a well-known control parametrization method is combined with a homotopy

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continuation technique to derive a unified method for accurate and efficient solution of bang–bang optimal control problems.

In the control parametrization methods [8, 13], the control functions are approximated by an appropriate function with finitely many unknown parameters, and the state functions are computed by integrating the state equations, using an explicit numerical integration such as the Runge–Kutta method. Thereby, the problem is converted into a mathematical programming problem. Note that the control parametrization methods belong to the category of direct methods for numerical solution of optimal control problems. Direct methods have been used extensively in a variety of trajectory optimization problems. Their advantage over indirect methods, which rely on solving the necessary conditions derived from Pontryagin's minimum principle [4], is their wider radius of convergence to an optimal solution. Therefore, they do not need a good initial guess for the control function. Moreover, they do not need an initial guess for co-state variables. Furthermore, since the necessary conditions do not have to be derived, the direct methods can be quickly used to solve practical trajectory optimization problems which are large in dimension and include nonlinear terms.

In this paper, an adaptive control parametrization (ACP) method is used, in which the control functions are parametrized with their switching points. More precisely, each control function is discretized with a piecewise-constant function which takes only two discrete values, that is, the lower and upper bounds of the control function, and discontinuous points are considered unknown. Consequently, the optimal control problem is transformed into a nonlinear programming problem (NLP). It is noted that this control parametrization method has been previously used in the switching time computation method [20] and the time-optimal switching algorithm [21]. By using this ACP method, accurate results can be obtained. However, the method is sensitive to an initial guess. To overcome this sensitivity, a homotopy continuation technique is combined with the method in such a way that the consideration of the optimal control structure as well as the sensitivity to the initial guess are resolved.

The homotopy continuation technique is a well-known procedure in numerical analysis [1, 11, 38]. The idea is to solve a given difficult problem by starting from the solution of a somewhat related but easier problem. For this purpose, the given problem is embedded into a one-parameter family of subproblems that is obtained from the deformation of an easier problem into the given problem. The solution data of the problem serves as an initial guess for the next problem in the family, and this process continues until the solution to the desired problem is reached. The homotopy continuation methods are particularly suitable for highly nonlinear problems, for which initial solution estimates are difficult to obtain. They have been successfully applied to solve polynomial and nonlinear systems of equations, boundary-value problems and several physical and engineering problems [19, 39].

Since the 1990s, homotopy continuation methods have been successfully applied to solving optimal control problems. Bulirsch et al. [5] used the multiple-shooting method and a homotopy strategy for solving the abort landing in wind-shear problems.

Initially, they solved an unconstrained problem, and then they gradually activated the constraints until the original problem was solved. Ehtamo et al. [10] applied a continuation method for solving the minimum time for optimal control problems. Bertrand and Epenoy [2] developed a perturbation approach, called the continuation smoothing technique, for solving bang–bang optimal control problems. They added a perturbed energy term in the objective function to yield a continuous optimal control. The perturbation parameter was then updated by a simple continuation procedure or through a homotopy method. In some cases [12, 15, 34], homotopic approaches were also used to solve low-thrust orbit transfer problems. Cerf et al. [7] proposed a novel approach, based on a continuation method, to initialize a shooting method for solving the high-thrust coplanar orbit transfer with fixed final time. Some other related computational works dealing with optimal control problems using homotopy continuation methods have been presented in the literature [14, 16, 43].

In this paper, a homotopy strategy is used together with the ACP method. For this purpose, the main optimal control problem is embedded into a family of optimal control problems, where the first problem is easy to solve. Each optimal control problem in this family is solved by the ACP method in such a way that the solution to the first optimal control problem is used to construct an initial guess for the next optimal control problem, and this process continues until the main optimal control problem is solved. Furthermore, the present technique can handle the changes of the switching structure of optimal controls during the homotopy procedure. Consequently, the switching structures of the optimal control in the first and main problems can be different. Therefore, a priori knowledge of the switching structure of the optimal control in the main problem is not required.

This paper is organized as follows. In Section 2, we introduce a formulation of the bang–bang optimal control problems with control appearing linearly. Section 3 describes the ACP method. In Section 4, the advantages and disadvantages of the ACP method are illustrated on a test example, while Section 5 is devoted to the formulation of the homotopy continuation technique. In Section 6, the proposed combined method is applied to two optimal control problems, and the advantages of our method are given. Finally, a brief discussion in Section 7 concludes the paper.

2. Bang–bang optimal control problems

The problem is to find a scalar control vector $u(t)$, the corresponding state vector $\mathbf{x}(t) = [x_1(t), \dots, x_p(t)]^T$ and possibly the terminal time t_f which minimizes the functional

$$J = \mathscr{M}(\mathbf{x}(t_f), t_f), \tag{2.1a}$$

subject to a system of p nonlinear differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), u(t), t) = \mathbf{f}_1(\mathbf{x}(t), t) + \mathbf{f}_2(\mathbf{x}(t), t)u(t), \quad 0 \le t \le t_f \tag{2.1b}$$

with the initial and terminal conditions

$$\mathbf{x}(0) = \mathbf{x}_0, \tag{2.1c}$$

$$\boldsymbol{\varphi}(\mathbf{x}(t_f), t_f) = \mathbf{0}, \tag{2.1d}$$

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together with the box constraints

$$-1 \le u(t) \le 1. \tag{2.1e}$$

Here, the state $\mathbf{x}$ is continuous and the bang–bang control has a finite number of switching points. The vector functions $\boldsymbol{\varphi} : \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^l$, $\mathbf{f} : \mathbb{R}^{p+2} \to \mathbb{R}^p$ and $\mathbf{f}_1, \mathbf{f}_2 : \mathbb{R}^{p+1} \to \mathbb{R}^p$ are assumed to be smooth functions of their arguments. The real function $\mathscr{M} : \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}$ is also assumed to be smooth.

It is worthwhile to note that according to Pontryagin's minimum principle, when the controls are bounded and appear linearly in the dynamic equations, then the nonsingular optimal control solution is bang–bang. This type of optimal control problem, which appears frequently in many applications, is also called the bang–bang optimal control problem.

In addition, a Mayer-type cost functional is considered. If the cost to be minimized is a Lagrange- or Bolza-type problem, then it can be converted to a Mayer-type problem [4].

3. An adaptive control parametrization (ACP) method

We define the control function $u(t)$ as

$$u(t) = \begin{cases} -1, & s_0 \le t \le s_1, \\ +1, & s_1 < t \le s_2, \\ \vdots & \\ (-1)^n, & s_{n-1} < t \le s_n, \end{cases} \tag{3.1}$$

where $0 = s_0 \le s_1 \le \cdots \le s_{n-1} \le s_n = t_f$ are switching points. Naturally, for bang–bang optimal control problems, the above definition is more adequate than other approximations, such as piecewise-linear or polynomial approximation. According to the above formulation, the control $u(t)$ is parametrized by a vector $\mathbf{s} = [s_1, \ldots, s_n]$, or $\mathbf{s} = [s_1, \ldots, s_{n-1}]$, when t_f is fixed. Consequently, we can denote the control function $u(t)$ in equation (3.1) by $u(t; \mathbf{s})$. Note that this parametrization has been used by Kaya and Noakes [20, 21], too. In this section, we illustrate the solution method based on this parametrization from our point of view, which is suitable for developing our algorithm.

For instance, if $\mathbf{s} = [1, 2, 4]$, then

$$u(t; \mathbf{s}) = \begin{cases} -1, & 0 \le t \le 1, \\ +1, & 1 < t \le 2, \\ -1, & 2 < t \le 4 \end{cases}$$

has two switching points. In equation (3.1), it is possible that $s_i = s_{i+1}$ for some $i = 0, 1, \ldots, n-1$. In such cases, although s_i and s_{i+1} seem to be switching points,

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$u(t)$ does not switch at these points. For example, let $\mathbf{s} = [0, 2, 4, 4, 5]$; then

$$u(t; \mathbf{s}) = \begin{cases} -1, & 0 \le t \le 0, \\ +1, & 0 < t \le 2, \\ -1, & 2 < t \le 4, \\ +1, & 4 < t \le 4, \\ -1, & 4 < t \le 5. \end{cases}$$

Note that the cases $0 \le t \le 0$ and $4 < t \le 4$ can be deleted. So, we can reduce the equation of the above control function to the following equation with one switching point, that is:

$$u(t; \mathbf{s}) = \begin{cases} +1, & 0 \le t \le 2, \\ -1, & 2 < t \le 5. \end{cases}$$

To apply the method with control parametrization in equation (3.1), first, the following initial value problem (IVP) is considered:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), u(t; \mathbf{s}), t), & 0 \le t \le t_f, \\ \mathbf{x}(0) = \mathbf{x}_0. \end{cases} \tag{3.2}$$

This IVP is obtained by replacing $u(t)$ by $u(t; \mathbf{s})$ in the dynamic equation (2.1b). We assume that the IVP in equation (3.2) has a unique solution, which is denoted by $\mathbf{x}(t; \mathbf{s})$. To obtain the solution of the IVP in equation (3.2) for a given $\mathbf{s} = [s_0, s_1, \ldots, s_n]$, the interval $[0, t_f]$ is first divided into n segments $[s_{i-1}, s_i]$, $i = 1, \ldots, n$. We then solve dynamic equations with the initial condition $\mathbf{x}(0) = \mathbf{x}_0$ in the first segment $[s_0, s_1]$. Then, the obtained value at $t = s_1$ is considered as an initial condition for the dynamic equations in the second segment $[s_1, s_2]$, and this process is continued until the last segment $[s_{n-1}, s_n]$ is reached.

Note that $\mathbf{x}(s_n; \mathbf{s})$ is an approximation of $\mathbf{x}(t_f)$ and, to evaluate it at a given $\mathbf{s}$, we need to solve the IVP in equation (3.2). Hence, by the control parametrization in equation (3.1), the optimal control problem (2.1) is converted to the following NLP:

$$\begin{array}{ll} \text{Minimize} & J = \mathscr{M}(\mathbf{x}(s_n; \mathbf{s}), s_n), \\ \text{such that} & \boldsymbol{\varphi}(\mathbf{x}(s_n; \mathbf{s}), s_n) = \mathbf{0}, \\ \text{with the constraints} & s_i \le s_{i+1}, \quad i = 0, \ldots, n-1. \end{array} \tag{3.3}$$

By solving the NLP (3.3), an optimal solution $\mathbf{s}^*$ is obtained. Then, by using equation (3.1), the approximation $u(t; \mathbf{s}^*)$ is obtained for the optimal control of problem (2.1).

Note that, for solving the NLP (3.3), any well-developed optimization algorithm may be used. For instance, Kaya and Noakes [21] used a gradient projection method for solving the NLP (3.3) while Kaya et al. [22] applied a mathematical programming formulation method for the same problem. Apart from various methods and softwares developed for solving NLPs, in this paper we use the MATLAB function `fmincon` and we set this solver to use the `sqp` (sequential quadratic programming) algorithm. Furthermore, we use the default “off” in its options, which causes `fmincon` to estimate the gradients of the cost function and terminal constraints using finite differences.

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Obviously, if the analytical gradients are supplied, the efficiency of the solver will be increased, too. For more details, see the paper by Lin et al. [29]. Note that, in this solver, we can specify termination tolerance on the objective function value, tolerance on the constraint violation and termination tolerance on decision variables, by parameters `TolFun`, `TolCon` and `TolX`, respectively. Hence, by using these parameters, we can adjust the accuracy of the obtained solution.

Also, the MATLAB function `ode45` is used for solving the IVP in equation (3.2). This solver is based on an explicit Runge–Kutta (4, 5) formula, the Dormand–Prince pair [9]. In addition, `ode45` controls the error by two parameters `RelTol` and `AbsTol`. By using these parameters, we can adjust the relative error tolerance and the absolute error tolerance (see the book by Shampine et al. [40]).

4. Advantages and disadvantages of the ACP method

The ACP method has advantages and disadvantages, just as any numerical method. As we shall see in the following example, this method provides accurate results. However, a drawback of the ACP is that it requires an accurate initial guess for the switching points and, therefore, has a poor convergence radius. In the following example, we will apply the ACP method in solving a given problem.

4.1. Illustration of the ACP method on a test example Consider the following time-optimal control problem [6]:

$$\begin{aligned} \text{Minimize} \quad & J = t_f, \\ \text{such that} \quad & \dot{x}_1 = x_2, \\ & \dot{x}_2 = u, \\ & x_1(0) = 1, \ x_2(0) = 3, \\ & x_1(t_f) = 0, \ x_2(t_f) = 0, \\ & -1 \le u(t) \le +1. \end{aligned} \tag{4.1}$$

This problem has an analytical solution given by

$$\begin{aligned} x_1^*(t) &= \begin{cases} -0.5t^2 + 3t + 1, & t \le t_1, \\ 0.5t^2 - t_f t + 0.5t_f^2, & t > t_1, \end{cases} \\ x_2^*(t) &= \begin{cases} 3 - t, & t \le t_1, \\ t - t_f, & t > t_1, \end{cases} \\ u^*(t) &= \begin{cases} -1, & t \le t_1, \\ +1, & t > t_1, \end{cases} \end{aligned} \tag{4.2}$$

where $t_1 = 3 + \sqrt{5.5} \simeq 5.345\,207\,879\,911\,7$ and $t_f = 3 + 2\sqrt{5.5} \simeq 7.690\,415\,759\,823\,4$. With the knowledge of the structure of optimal control, we parametrize the control function by a vector $\mathbf{s} = [s_1, s_2]^T$. By applying the ACP method, the problem is

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TABLE 1. (Problem (4.1)): The calculated values of the switching and final times for various values of fmincon and ode45 parameters.

TolFun	TolCon	TolX	RelTol	AbsTol	t_1	t_f
1.0e-03	1.0e-03	1.0e-03	1.0e-03	1.0e-03	5.345 207 887 352 1	7.690 415 774 683 0
1.0e-04	1.0e-04	1.0e-04	1.0e-03	1.0e-03	5.345 207 887 352 1	7.690 415 774 683 0
1.0e-04	1.0e-04	1.0e-04	1.0e-04	1.0e-04	5.345 207 887 355 6	7.690 415 774 685 2
1.0e-05	1.0e-05	1.0e-05	1.0e-05	1.0e-05	5.345 207 879 911 7	7.690 415 759 823 4
Exact switching and final times					5.345 207 879 911 7	7.690 415 759 823 4

converted to the following NLP with two decision parameters s_1 and s_2:

$$\begin{aligned} \text{Minimize} \quad & J = s_2, \\ \text{such that} \quad & x_1(s_2;[s_1,s_2]) = 0, \\ & x_2(s_2;[s_1,s_2]) = 0, \\ & s_1 \geq 0, \ \ s_2 \geq s_1. \end{aligned} \tag{4.3}$$

Now, by solving it, the optimal values of s_1 and s_2 are obtained as approximations of switching and final times, respectively. In general, we must solve the NLP in equation (4.3) numerically and, for this purpose, we need to provide initial guesses for s_1 and s_2. Based on the chosen initial guesses, the NLP solver may converge or diverge.

When the NLP solver converges, accurate results will be obtained. To show the accuracy of the ACP method, the switching and final times are reported in Table 1 for some values of the parameters.

It is seen that the ACP method provides accurate results. However, with an arbitrary and improper initial guess, convergence may not occur. Specifically, if a very good initial guess to the optimal solution is not available, the optimization solver will fail to find a feasible solution. This drawback is shown in Figure 1. In this figure, we report the convergence region of the ACP method. In view of Figure 1, note that if the initial guess is far from the optimal solution, then the method may fail.

In summary, the ACP method provides accurate results; however, it has a poor convergence radius. Hence, we need a strategy to overcome this difficulty. For this reason, we use a homotopy continuation strategy, which is introduced in the next section. As will be shown, by using this strategy, solving the resulting NLP is independent of the initial guess. Before introducing this strategy, we discuss another feature of the ACP method, which is useful for developing the homotopy continuation strategy.

In the ACP method, there is no need to specify the exact number of switching points by the initial guess, and we can obtain the solution with an initial guess which contains more switching points. For example, we apply the ACP method on problem (4.1), with switching points $\mathbf{s} = [2, 4, 6, 8]$. With this initial guess, we consider three switching points, but we know that the optimal control has only one switching point. However, by this initial guess, we get the following optimal solution from the NLP in equation (4.3):

$$\mathbf{s}^* = [3.1680, 3.1681, 5.3472, 7.6925].$$

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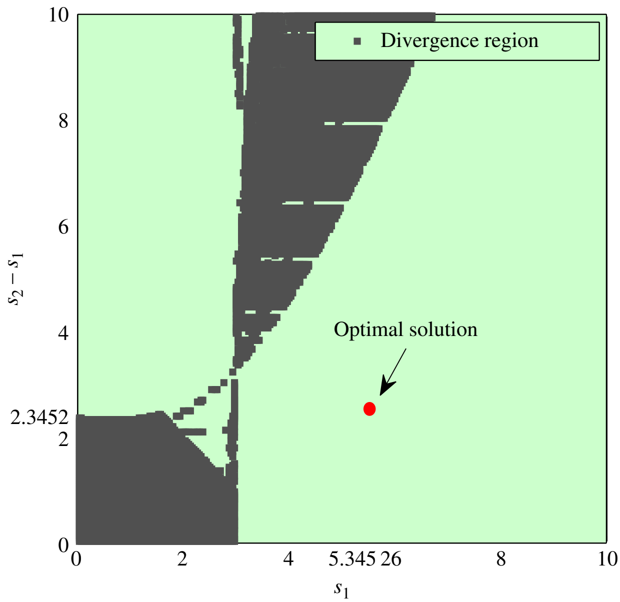

FIGURE 1. (Problem (4.1)): Convergence and divergence regions of the ACP method for various values of initial guesses $\mathbf{s} = [s_1, s_2]$. The divergence region is specified with a darker colour and is a region for which the optimization solver failed to find a feasible solution.

Hence, the optimal control is given by

$$u(t; \mathbf{s}) = \begin{cases} -1, & 0 \le t \le 3.1680, \\ +1, & 3.1680 < t \le 3.1681, \\ -1, & 3.1681 < t \le 5.3472, \\ +1, & 5.3472 < t \le 7.6925. \end{cases} \tag{4.4}$$

This control function is also plotted in Figure 2. Note that the obtained control in equation (4.4) is similar to the optimal control in equation (4.2) but a small arc, the second arc, appears in the obtained control (4.4). We call such arcs *artificial arcs*, which are produced because of the truncation error in NLP and IVP solvers. However, the length of these arcs depends on the values of those parameters of IVP and NLP solvers, by which the truncation error is controlled.

We note that when the given number of switching points in the initial guess is more than the exact number of switching points, then the artificial arcs appear in the solution. In this case, we can eliminate (prune) the artificial arcs to obtain the exact number of switching points. For this purpose, each artificial arc is merged into its previous and next arcs. In the sequel, we name this elimination operation as the *pruning procedure*. For instance, in control (4.4), the obtained switching point $\mathbf{s} = [3.1680, 3.1681, 5.3472, 7.6925]$ is converted to $\mathbf{s} = [5.3472, 7.6925]$ by the pruning procedure.

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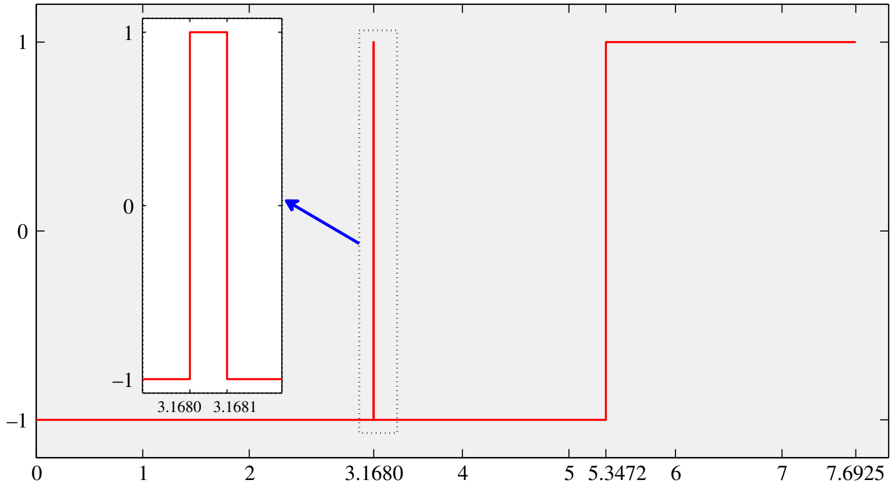

FIGURE 2. (Problem (4.1)): Control history obtained by the ACP method with the initial guess $\mathbf{s} = [2, 4, 6, 8]$.

5. Applying the homotopy continuation technique

We use the homotopy continuation technique for overcoming the difficulties of the ACP method. For this purpose, we define a set of bang–bang optimal control problems, which depend on a parameter $\lambda \in [0, 1]$ in such a way that they connect a simple bang–bang optimal control problem for $\lambda = 0$ to the desired problem for $\lambda = 1$. To this effect, at first we select a simple bang–bang problem for which we know the corresponding solution, or can find its solution easily. The only thing which should be considered is that it must have the same dimension as the desired problem.

Observe that among the bang–bang problems with a known solution, we can choose one as the starting problem, although it must be chosen in a way that is similar in nature to the desired problem. However, by using the idea for constructing the test example (4.1), we can construct the problem with every dimension as the starting problem. For example, consider the problem

$$\begin{array}{ll} \text{Minimize} & J = \mathscr{M}_s(\mathbf{x}(t_f), t_f), \\ \text{such that} & \dot{\mathbf{x}}(t) = \mathbf{f}_s(\mathbf{x}(t), u(t), t), \\ & \mathbf{x}(0) - \mathbf{a}_0 = \mathbf{0}, \\ & \boldsymbol{\varphi}_s(\mathbf{x}(t_f), t_f) = \mathbf{0}, \\ & -1 \le u(t) \le +1. \end{array}$$

Now, we construct the following homotopy problem to connect the starting problem with the desired bang–bang optimal control problem:

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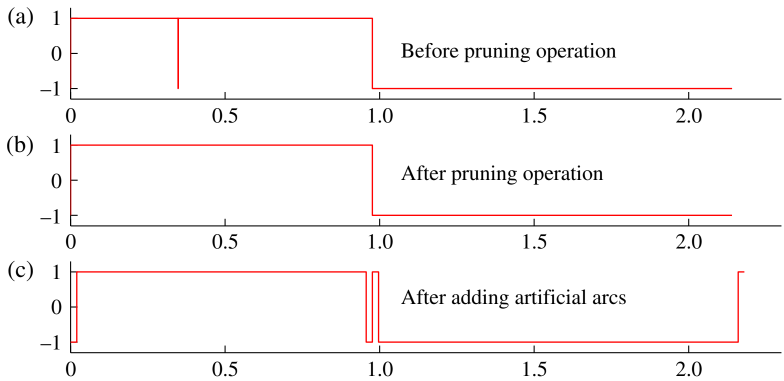

FIGURE 3. Pruning operation and adding artificial arcs.

$$\begin{aligned} \text{Minimize} \quad & J = \lambda \mathscr{M}(\mathbf{x}(t_f), t_f) + (1-\lambda)\mathscr{M}_s(\mathbf{x}(t_f), t_f), \\ \text{such that} \quad & \dot{\mathbf{x}}(t) = \lambda \mathbf{f}(\mathbf{x}(t), u(t), t) + (1-\lambda)\mathbf{f}_s(\mathbf{x}(t), u(t), t), \\ & \lambda(\mathbf{x}(0) - \mathbf{x}_0) + (1-\lambda)(\mathbf{x}(0) - \mathbf{a}_0) = \mathbf{0}, \\ & \lambda \boldsymbol{\varphi}(\mathbf{x}(t_f), t_f) + (1-\lambda)\boldsymbol{\varphi}_s(\mathbf{x}(t_f), t_f) = \mathbf{0}, \\ & -1 \le u(t) \le +1. \end{aligned} \tag{5.1}$$

For $\lambda = 0$, we have the starting problem and, for $\lambda = 1$, we get the desired problem (2.1). Using the solution of the starting problem as an initial guess, we try to solve the desired problem for $\lambda = 1$. This sequential approach is called *discrete continuation* [15, 39].

For every $\lambda \in [0, 1]$, the homotopy problem (5.1) is a bang–bang optimal control problem, but the switching structure of these problems may be changed, when λ is changed from the starting problem to the desired problem. So, the switching structure of the starting problem can be different from the desired problem. Consequently, we do not have a priori knowledge of the structure of the control function in the desired problem. On the other hand, the present method should be able to handle the various changes of the switching structure during the homotopy procedure. For this purpose, we consider the following strategy.

Assume that $\mathbf{s}_k$ is the associated solution of the homotopy problem (5.1) for $\lambda = \lambda_k$. For this situation, in the classical homotopy continuation methods, $\mathbf{s}_k$ is considered as an initial guess for the homotopy problem with $\lambda = \lambda_{k+1}$. However, in this technique, we make two changes in $\mathbf{s}_k$ before considering it as an initial guess for the next problem. First, we begin the pruning operation on $\mathbf{s}_k$. Second, we add an artificial arc of length ϵ before and after each of the switching points. The value of ϵ should be chosen greater than all the parameters of `fmincon`. These changes, which must be applied to $\mathbf{s}_k$ before considering it as an initial guess for the next problem, are shown graphically in Figure 3. Note that the purpose of adding the artificial arcs is to enable the method to change its structure for finding the optimal solution.

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6. Illustrative examples

We give two examples to demonstrate the applicability and accuracy of our method. In these examples, we solve the final NLP (3.3) by the MATLAB function `fmincon` and solve the system of ODEs in equation (3.2) by the MATLAB function `ode45`.

6.1. Example 1 (Van der Pol oscillator) Consider the time-optimal control of a Van der Pol oscillator [35]. The control problem is to minimize the final time t_f subject to

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -x_1 - (x_1^2 - 1)x_2 + u,$$

together with the following constraints on the control:

$$-1 \leq u \leq 1.$$

We consider the following two cases for initial and terminal state constraints.

CASE I. $\mathbf{x}(0) = [1, 1]^T$ and $\mathbf{x}(t_f) = [0, 0]^T$.

CASE II. $\mathbf{x}(0) = [-0.4, 0.6]^T$ and $\mathbf{x}(t_f) = [0.6, 0.4]^T$.

Maurer and Osmolovskii [35] obtained the following results for optimal control in Cases I and II, respectively:

$$\text{Case I:} \quad u^*(t) = \begin{cases} -1, & 0 \leq t \leq 0.723\,003\,73, \\ +1, & 0.723\,003\,73 < t \leq 3.095\,202\,34, \end{cases}$$

$$\text{Case II:} \quad u^*(t) = \begin{cases} +1, & 0 \leq t \leq 0.158\,320\,137\,6, \\ -1, & 0.158\,320\,137\,6 < t \leq 1.254\,074\,73. \end{cases}$$

In each case, there is one switching point, but the structures of the optimal controls are different from each other.

Here, we solve this problem by using the ACP method together with the homotopy continuation technique. Also, we use the optimal control problem (4.1) as the starting problem. In Figure 4, the control functions obtained by the proposed continuation strategy for $\lambda = 0$ to $\lambda = 1$ are given, and the corresponding states are shown in Figure 5.

Note that the proposed continuation method can capture the solution of the desired problem when the structure of the starting problem is different (Case II), as well as when the structure of the starting problem is identical (Case I).

To show the accuracy of our method, the switching and final times are reported in Tables 2 and 3 for some values of the parameters. As is shown (in Tables 2 and 3), the results are compared with the approximated solution of Maurer and Osmolovskii [35], which was obtained by Oberle and Grimm [37] by using the code BNDSCO. Observe that the present method provides an accurate solution of bang–bang optimal control problems, and can capture the switching points accurately.

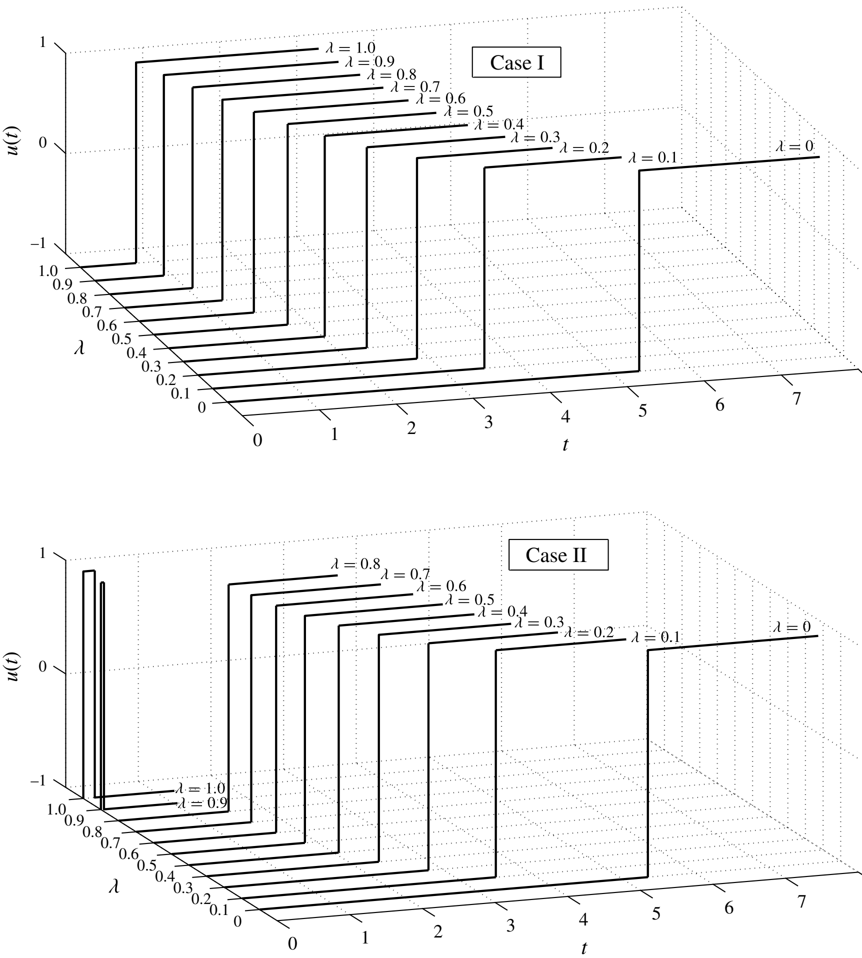

FIGURE 4. (Van der Pol oscillator): Control histories obtained by using homotopy continuation technique.

TABLE 2. (Van der Pol oscillator in Case I): The calculated values of switching and final times, for various values of fmincon and ode45 parameters, by the present method and the method in the paper [35].

TolFun	TolCon	TolX	RelTol	AbsTol	t_1	t_f
1.0e-04	1.0e-04	1.0e-04	1.0e-03	1.0e-03	0.723 003 688 914 9	3.095 202 486 425 8
1.0e-09	1.0e-09	1.0e-09	1.0e-09	1.0e-09	0.723 003 741 913 1	3.095 202 339 474 9
1.0e-11	1.0e-11	1.0e-11	1.0e-11	1.0e-14	0.723 003 742 133 5	3.095 202 340 462 8
1.0e-14	1.0e-14	1.0e-14	1.0e-11	1.0e-14	0.723 003 742 133 5	3.095 202 340 462 8
Results of [35]					0.723 003 73	3.095 202 34

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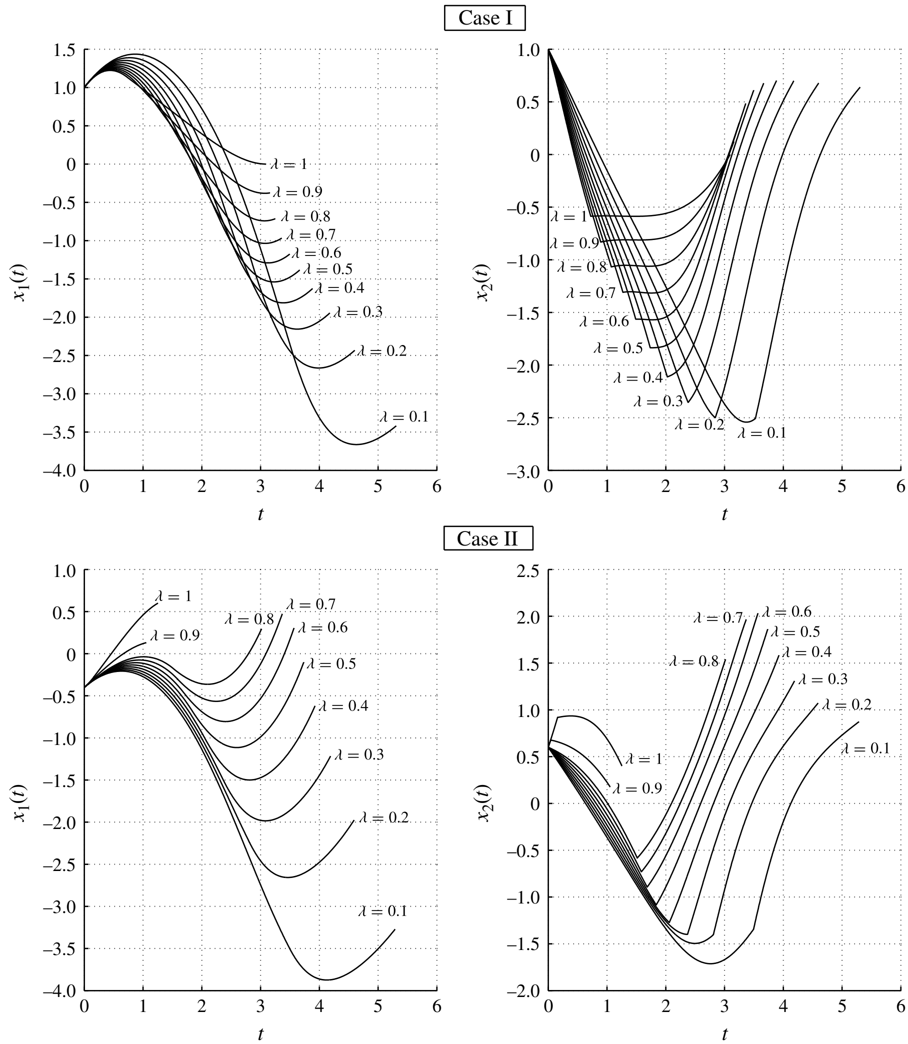

FIGURE 5. (Van der Pol oscillator): State histories obtained by using homotopy continuation technique.

TABLE 3. (Van der Pol oscillator in Case II): The calculated values of switching and final times, for various values of fmincon and ode45 parameters, by the present method and the method in the paper [35].

`TolFun`	`TolCon`	`TolX`	`RelTol`	`AbsTol`	t_1	t_f
1.0e-04	1.0e-04	1.0e-04	1.0e-03	1.0e-03	0.158 320 135 838 1	1.254 074 730 983 0
1.0e-09	1.0e-09	1.0e-09	1.0e-09	1.0e-09	0.158 320 142 224 3	1.254 074 729 223 3
1.0e-11	1.0e-11	1.0e-11	1.0e-11	1.0e-14	0.158 320 142 287 4	1.254 074 729 336 4
1.0e-14	1.0e-14	1.0e-14	1.0e-11	1.0e-14	0.158 320 142 287 4	1.254 074 729 336 4
Results of [35]					0.158 320 137 6	1.254 074 73

https://doi.org/10.1017/S1446181114000261 Published online by Cambridge University Press

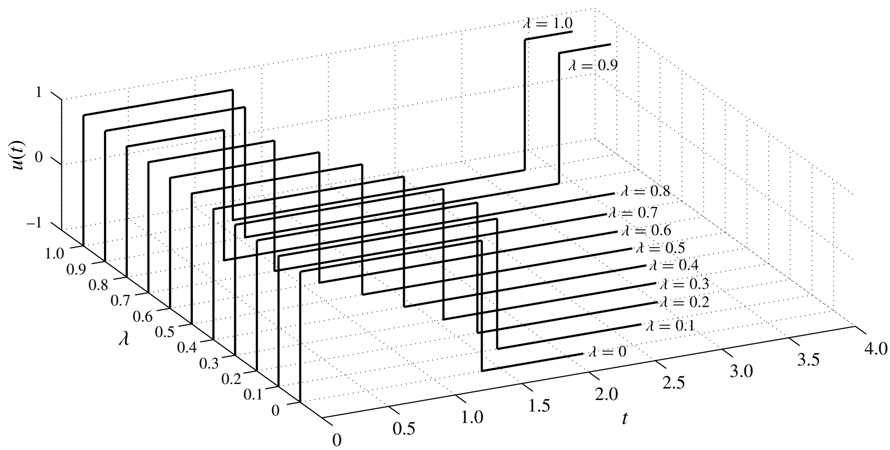

FIGURE 6. (Rayleigh problem): Control history obtained by using homotopy continuation technique.

TABLE 4. (Rayleigh problem): The calculated values of switching and final times, for various values of fmincon and ode45 parameters, by the present method and the method in the paper [35].

TolFun	TolCon	TolX	RelTol	AbsTol	t_1	t_2	t_f
1.0e-04	1.0e-04	1.0e-04	1.0e-04	1.0e-04	1.120 577 920 972 4	3.310 159 255 002 8	3.668 301 929 402 8
1.0e-09	1.0e-09	1.0e-09	1.0e-09	1.0e-09	1.120 506 474 052 9	3.310 047 021 111 5	3.668 173 389 052 0
1.0e-11	1.0e-11	1.0e-11	1.0e-11	1.0e-14	1.120 506 761 102 4	3.310 046 930 628 6	3.668 173 388 632 0
1.0e-14	1.0e-14	1.0e-14	1.0e-11	1.0e-14	1.120 506 761 102 4	3.310 046 930 628 7	3.668 173 388 632 0
Results of [35]					1.120 506 58	3.310 046 98	3.668 173 39

6.2. Example 2 (Rayleigh problem) Consider the Rayleigh problem given by Maurer and Osmolovskii [35] and Navabi et al. [36]. This is a free terminal-time problem, where the system is described by

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -x_1 + (1.4 - 0.14x_2^2)x_2 + 4u,$$

with $\mathbf{x}(0) = [-5, -5]^T$, and the desired final state $\mathbf{x}(t_f) = [0, 0]^T$. The constraints on the control are $-1 \leq u \leq 1$. Our objective is to determine a control u that minimizes

$$J = t_f.$$

Here, we solve this problem by using the proposed combined method with the starting problem (4.1). In Table 4, the control switching times and the objective function for this problem are given, and the control function is shown in Figure 6. Note that the number of switching points in the starting problem is one, whereas, in the desired problem (in optimal control), we have two switching points. Therefore, our method is capable of adding a switching point to reach the optimal solution.

7. Conclusion

In the present work, an efficient and simple control parametrization method combined with the homotopy continuation technique has been developed for the solution of bang–bang optimal control problems. The present method provides an accurate solution of bang–bang optimal control problems, and can capture switching points accurately. In addition, this method does not require a priori knowledge about the optimal control solution and, during the homotopy continuation procedure, the structure and optimal number of switching points are captured. Furthermore, the artificial arcs, which have been produced because of the truncation error in NLP and IVP solvers, can be eliminated during the continuation procedure. It is worthwhile to note that these artificial arcs may be avoided by including a total variation term in the objective function [32]. Further research in the usage of the time-scaling transformation technique [32] to improve the efficiency of the presented method will be interesting.

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