SEMI-ALGEBRAS IN $C(T)$

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Let $C(T)$ be the Banach algebra of all complex-valued continuous functions on the compact set T of all complex numbers with modulus one. As usual we may suppose that A is embedded in $C(T)$, where A is the disc algebra, i.e., the algebra of all complex-valued functions $f(\lambda)$ continuous for $|\lambda| \leqq 1$ and analytic for $|\lambda| < 1$. We set $M_\lambda = \{f \in A : f(\lambda) = 0\}$ and $M_\lambda^+ = \{f \in A : f(\lambda) \geqq 0\}$.

Following Bonsall [1], we call a subset S of $C(T)$ a *semi-algebra* if, whenever $f, g \in S$ and t is a non-negative number, we have $f+g \in S$, $fg \in S$ and $tf \in S$. In connection with the semi-algebra S, we consider the real subalgebra $S_b = S \cap (-S)$ of $C(T)$ and the complex subalgebra $S_c = S_b + iS_b$. It is convenient to let $e = e(\lambda)$ stand for the function identically one. Our theorem shows that all these items are intimately related.

THEOREM 1. *Let S be a semi-algebra in $C(T)$, where $-e \notin S$. Then either S_c is dense in $C(T)$ or no M_λ^+, with $|\lambda| < 1$, is properly contained in S.*

Proof. Suppose that S properly contains some M_λ^+, with $|\lambda| < 1$. Without loss of generality, we may take $\lambda = 0$ in the ensuing argument. We must show that S_c is dense in $C(T)$.

Consider the subalgebra

$$B = S_c + \mathbb{C}e, \tag{1}$$

where $\mathbb{C}$ is the field of complex numbers. Since S_c contains the maximal ideal M_0 of A, we get $B \supset A$. Hence, by Wermer's maximality theorem [5], the closure of B is either $C(T)$ or A.

If the closure of B is $C(T)$, there exist a sequence $\{p_n(\lambda)\}$ in S_c and a sequence $\{\alpha_n\}$ in $\mathbb{C}$ such that, in the metric of $C(T)$, $p_n(\lambda) + \alpha_n e(\lambda) \to \lambda^{-1}$. Notice that the functions $\lambda p_n(\lambda)$ and $\alpha_n \lambda$, as functions of λ, all lie in S_c and that, in $C(T)$, $\lambda p_n(\lambda) + \alpha_n \lambda \to e(\lambda)$. Therefore, by (1), the closure of S_c is the closure of B, which is here $C(T)$.

Our conclusion would then follow if we could show that the closure of B cannot be A. Suppose that the closure of B is A. By (1) and the fact that S_c contains the maximal ideal M_0 of A, we see that

$$A = S_c + \mathbb{C}e. \tag{2}$$

Next we show that $e \notin S_c$. For otherwise we could write $e = f + ig$, where f and g lie in S_b. Then we could write

$$-e = f^2 + g^2 - 2f.$$

Since the right side lies in $S_b \subset S$, we get a contradiction.

It now follows from (2) that S_c is a proper ideal in A containing M_0. Therefore $S_c = M_0$. Now take $g \in S$. The function $\lambda g(\lambda)$ lies in $S_b \subset M_0$ and is therefore an element of A vanishing

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at zero. Hence there exists $w \in A$ such that $\lambda g(\lambda) = \lambda w(\lambda)$, $|\lambda| = 1$. Therefore $g \in A$ and so $M_0^+ \subset S \subset A$, where $-e \notin S$.

We shall show from this that $S = M_0^+$. For let $v \in S$. First we show that $v(0) = -a, a > 0$, is impossible. For suppose otherwise and set $w = a^{-1}v$. Now M_0 is a maximal linear subspace of A; so there is a scalar λ and $f \in M_0$ such that $-e = f + \lambda w$. Evaluating at zero, we see that $\lambda = 1$, so that $-e = f + w \in S$, which is impossible. It follows that $v(0) = bi$, b real, $b \neq 0$ is impossible, for otherwise $v^2 \in S$ and $v^2(0) = -b^2$. Next we show that $v(0) = a + bi$ with a, b real, $a < 0$, $b \neq 0$ is impossible, for otherwise $w = -ae + v \in S$ and $w(0) = bi$. Next we rule out $v(0) = a + bi$, a, b real, $a > 0$, $b \neq 0$. For if this holds, then $v^n(0)$ must lie in the open left-hand plane for some positive integer and $v^n \in S$. By elimination we see finally that $v(0) \geqq 0$ or $v \in M_0^+$. Therefore $S = M_0^+$.

However this is in conflict with the hypothesis that S properly contains M_0^+ and the proof of the theorem is completed.

The choice $S = A$ shows that the requirement that $-e \notin S$ cannot be dropped from the hypothesis. Also, S_c may fail to be dense and, simultaneously, S can properly contain some M_λ^+, with $|\lambda| = 1$. For consider $g \in C(T)$, where $g \notin A$ and $g(1) = 0$. The semi-algebra S generated by M_1^+ and g properly contains M_1^+ and fails to contain $-e$, but has the property that S_c is at a distance of one from $-e$.

The following special case of Theorem 1 is, to the author, somewhat surprising.

COROLLARY 1. *Let $g \in C(T)$, where $g \neq 0$ and g vanishes on a subset T_0 of T of positive Lebesgue measure. Let λ be a complex number with $|\lambda| < 1$. If S is the semi-algebra generated by M_λ^+ and g, then S_c is dense in $C(T)$.*

Proof. A well-known theorem of F. and M. Riesz [**2**, p. 50] shows that $g \notin A$, so that S properly contains M_λ^+. The conclusion follows from Theorem 1 if we verify that $-e \notin S$. Suppose that $-e \in S$. Then there exists a finite subset $f_0, f_1, \ldots, f_n$ of M_λ^+ such that

$$-e = f_0 + \sum_{k=1}^{n} f_k g^k. \tag{3}$$

Notice that, from (3), $e + f_0$ is identically zero on T_0. The F. and M. Riesz theorem then gives $f_0 = -e$, which is impossible.

For a ring R with identity 1, Harrison [**4**] defines a *preprime* as a nonvoid set closed under addition and multiplication and not containing -1. He calls a maximal preprime a *prime*. Civin and White [**3**, p. 243] showed that, if P is a closed prime in a Banach algebra B with identity 1, then $1 \in P$ and P is a semi-algebra. If further, B is a complex and commutative Banach algebra, then $iP_b \subset P_b$ [**3**, Proposition 1.11]. They also point out [**3**, p. 245] that M_λ^+ with $|\lambda| < 1$ is not a prime in $C(T)$. By using Theorem 1, more can be shown along these lines.

COROLLARY 2. *Let S be closed semi-algebra in $C(T)$ where $-e \notin S$ and S contains some M_λ^+ with $|\lambda| < 1$. Then S is not a prime in $C(T)$.*

Proof. Suppose that S is a prime in $C(T)$. As noted above, this implies that $iS_b \subset S_b$. Consequently $S_c \subset S$, so that S_c cannot be dense in $C(T)$. Theorem 1 shows that S cannot properly contain any M_α^+ with $|\alpha| < 1$. Therefore $S = M_\lambda^+$. But in this situation the proof of Corollary 1 provides the existence of a preprime properly containing S. This is a contradiction.

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