Canad. Math. Bull. Vol. **23** (3), 1979

A CHARACTERISATION OF LOCALLY COMPACT AMENABLE SUBSEMIGROUPS

BY
JAMES C. S. WONG[1]

ABSTRACT. In this paper, we prove that if S is a locally compact semigroup and T a locally compact Borel measurable subsemigroup of S, then T has a topological left invariant mean if and only if there is a topological left T-invariant mean M on S such that $M(\chi_T)=1$, where χ_T is the characteristic functional of T in S.

§1. **Introduction.** Let S be a locally compact topological semigroup and $M(S)$ the Banach algebra of all bounded regular Borel measures on S with total variation norm and convolution as multiplication. Also let T be a locally compact Borel subsemigroup of S. We obtain a necessary and sufficient condition for $M(T)^*$ to have a topological left invariant mean in terms of the characteristic functional of T in the dual $M(S)^*$ (see below for definitions). It is a topological analogue as well as an extension of a result of M. M. Day for semigroups (see Day [1, Theorem 2, page 518] and also Wilde and Witz [7] for a slightly extended version). This criterion of amenability of subsemigroups of locally compact semigroups is used in Wong [12], [13] and Day [2].

§2. **Notations and terminologies.** Let S be a locally compact semigroup, $M(S)$ its measure algebra and $M_0(S)$ the probability measures in $M(S)$. As in Wong [10], a linear functional $M\in M(S)^{**}$ is called a mean on $M(S)^*$ (or on S) if $M(F)\geq 0$ whenever $F\geq 0$ in $M(S)^*$ and $M(1)=1$ where $1\in M(S)^*$ is defined by $1(\mu)=\int d\mu=\mu(S)$, $\mu\in M(S)$. M is called topological left invariant if $M(\mu\odot F)=M(F)$ for any $F\in M(S)^*$ and $\mu\in M_0(S)$ where $\mu\odot F(\nu)=F(\mu*\nu)$, $\nu\in M(S)$. M is left invariant if $M(\varepsilon_a\odot F)=M(F)$ for any $F\in M(S)^*$ and $a\in S$. Here ε_a is the Dirac measure at a.

Let T be a locally compact Borel subsemigroup of S. The characteristic functional of T in S, denoted by χ_T is defined by

$$\chi_T(\mu)=\int_S \xi_T\,d\mu=\mu(T),\qquad \mu\in M(S)$$

Received by the editors May 18, 1978 and, in revised form, February 20, 1979.
[1] Research supported by the National Research Council of Canada, Grant No. A8227.
AMS Subject Classification (Mathematical Reviews 1973), 22.05, 22.20 and 42.50.
Key Words and Phrases. Locally compact semigroups and subsemigroups, topological left invariant means, measure algebras, characteristic functionals.

where ξ_T is the characteristic function of T in S. Clearly $\mathscr{X}_T \in M(S)^*$. A mean M on $M(S)^*$ is called topological left T-invariant if $M(\mu \odot F) = M(F)$ for any $F \in M(S)^*$ and $\mu \in M_0(S)$ such that $\mu(T') = 0$. It is left T-invariant if $M(\varepsilon_a \odot F) = M(F)$ for any $F \in M(S)^*$ and $a \in T$.

§3. **Some technical lemmas.** For clarity and simplification in the proofs of the main results, we collect some technical details in the following lemmas.

LEMMA 3.1.

(1) *If $\mu \in M(S)$ and μ_T is the restriction of μ to the Borel sets in T, then $\mu_T \in M(T)$ and $\int_T f\, d\mu_T = \int_S \bar{f}\, d\mu$ for any bounded Borel measurable function f on T where $\bar{f}$ is defined on S by $\bar{f}(x) = f(x)$, $x \in T$ and $\bar{f}(x) = 0$, $x \notin T$.*

(2) *If $\mu, \nu \in M(S)$ with $|\mu|(T') = 0$ and $x \in S$, then $\|(\mu * \nu)_T - \mu_T * \nu_T\| \le \int |\nu|(E(x))\, d|\mu|(x)$ where $E(x) = \{y \in S : y \notin T \text{ and } xy \in T\} \subset T'$.*

(3) *If $\mu, \nu \in M(S)$ with $|\mu|(T') = |\nu|(T') = 0$, then $(\mu * \nu)_T = \mu_T * \nu_T$.*

(4) *If S is a group, T a subgroup of S and $\mu, \nu \in M(S)$ with $|\mu|(T') = 0$, then $(\mu * \nu)_T = \mu_T * \nu_T$.*

Proof.

(1) This is proved in Wong [11, Lemma 3.1] with S and G in place of T and S respectively.

(2) Let $\mu, \nu \in M(S)$ with $|\mu|(T') = 0$. For any bounded Borel measurable function f on T and $y \in T$, define the right translate f_y on T by $f_y(x) = f(xy)$, $x \in T$. Similarly for functions on S. Now if g denotes the function on T by $g(y) = \int_S \bar{f}(xy)\, d\mu(x)$, $y \in T$. Then

$$\int_T f(xy)\, d\mu_T(x) = \int_T f_y\, d\mu_T = \int_S (\bar{f_y})\, d\mu = \int_S (\bar{f})_y\, d\mu = g(y)$$

since $|\mu|(T') = 0$ and $(\bar{f_y}) = (\bar{f})_y$ on T. Hence

$$(*) \qquad \int_T f\, d\mu_T * \nu_T = \int_T \int_T f(xy)\, d\mu_T(x)\, d\nu_T(y) = \int_S \bar{g}(y)\, d\nu(y).$$

On the other hand,

$$(**) \qquad \int_T f\, d(\mu * \nu)_T = \int_S \bar{f}\, d\mu * \nu = \int_S \int_S \bar{f}(xy)\, d\mu(x)\, d\nu(y).$$

From (*) and (**), we have

$$\left| \int_T f\, d(\mu * \nu) - \int_T f\, d\mu_T * \nu_T \right|$$
$$= \left| \int_S \int_S \bar{f}(xy)\, d\mu(x)\, d\nu(y) - \int_S \bar{g}(y)\, d\nu(y) \right|$$

https://doi.org/10.4153/CMB-1980-042-6 Published online by Cambridge University Press

$$=\left|\int_S\left\{\int_S \bar{f}(xy)\,d\mu(x)-\bar{g}(y)\right\}d\nu(y)\right|$$

$$=\left|\int_{T'}\int_S \bar{f}(xy)\,d\mu(x)\,d\nu(y)\right| \quad \text{(definition of } g \text{ and } \bar{g}\text{)}$$

$$=\left|\int_S\int_{T'} \bar{f}(xy)\,d\nu(y)\,d\mu(x)\right|$$

$$\leq \|f\|\cdot\int_S |\nu|\,(E(x))\,d\,|\mu|\,(x)$$

because

$$\left|\int_{T'} \bar{f}(xy)\,d\nu(y)\right| \leq \|f\|\cdot|\nu|\,(E(x)).$$

Therefore

$$\|(\mu*\nu)_T-\mu_T*\nu_T\|\leq\int_S |\nu|\,(E(x))\,d\,|\mu|\,(x).$$

(3) This is a consequence of (2) since $|\nu|\,(E(x))\leq|\nu|\,(T')=0$ for any $x\in S$.

(4) If S is a group and T a subgroup of S then for any $x\in T$, $E(x)=\{y\in S: y\notin T \text{ and } xy\in T\}=\phi$. From (2), we have $\|(\mu*\nu)_T-\mu_T*\nu_T\|\leq \int_S|\nu|\,(E(x))\,d\,|\mu|\,(x)=\int_T|\nu|\,(E(x))\,d\,|\mu|\,(x)=0$ (since $|\mu|\,(T')=0$).

Lemma 3.2. *If $\tau\in M(T)$, define $\mu(B)=\tau(B\cap T)$ for any Borel set B in S, then $\mu\in M(S)$ and $|\mu|\,(T')=0$. (And so $\mu_T=\tau$). Moreover, $\tau\in M_0(T)$ iff $\mu\in M_0(S)$.*

Proof. This is proved in Wong [11, Lemma 3.3] except the last statement which is easily verified.

§4. Main results.

Theorem 4.1. *Let T be a locally compact Borel subsemigroup of a locally compact semigroup S, then the following statements are equivalent:*

(1) $M(T)^*$ has a topological left invariant mean

(2) There is a topological left T-invariant mean M on $M(S)^*$ such that $M(\chi_T)=1$.

Proof.

(2) implies (1).

Assume there is a topological left T-invariant mean M on $M(S)^*$ such that $M(\chi_T)=1$. For each $F\in M(T)^*$, define $F'\in M(S)^*$ by $F'(\mu)=F(\mu_T)$ where $\mu_T\in M(T)$ is defined in Lemma 3.1. Clearly, the map $F\to F'$ is linear, $F'\geq 0$ whenever $F\geq 0$ and $F'=1$ if $F=1$. Define $M_0\in M(T)^{**}$ by $M_0(F)=M(F')$. It is straightforward to verify that M_0 is a mean on $M(T)^*$. We shall prove that

https://doi.org/10.4153/CMB-1980-042-6 Published online by Cambridge University Press

M_0 is topological left invariant. (Using a modified topological version of Day's proof in Hewitt and Ross [5, §17.18 (*f*) page 238].)

Observe that any measure in $M_0(T)$ is of the form μ_T with $\mu \in M_0(S)$ and $\mu(T')=0$ by Lemma 3.2. Thus it is sufficient to prove that $M((\mu_T \odot F)') = M(\mu \odot F')$ for $F \in M(T)^*$ and *such* $\mu \in M_0(S)$. Now let $\nu \in M(S)$, $\nu \geq 0$. By Lemma 3.1 (2),

$$\|(\mu * \nu)_T - \mu_T * \nu_T\| \leq \|\mu\| \cdot |\nu| (T') = \nu(T').$$

So

$$\begin{aligned}(\mu_T \odot F)'(\nu) - (\mu \odot F')(\nu) &= F(\mu_T * \nu_T) - F((\mu * \nu)_T) \\ &\leq |F(\mu_T * \nu_T) - F((\mu * \nu)_T)| \\ &\leq \|F\| \cdot \|\mu_T * \nu_T - (\mu * \nu)_T\| \leq \|F\| \, \nu(T') \\ &= \|F\| \, \chi_{T'}(\nu).\end{aligned}$$

Hence $(\mu_T \odot F)' - \mu \odot F' \leq \|F\| \cdot \chi_{T'}$ and $M((\mu_T \odot F)') - M(\mu \odot F') \leq \|F\| \cdot M(\chi_{T'}) = 0$. (Since $M(\chi_T) = 1$ and $\chi_T + \chi_{T'} = 1$.) Similarly $M((\mu_T \odot F)') - M(\mu \odot F') \geq 0$. Consequently $M(\mu_T \odot F)') = M(\mu \odot F')$ and M_0 is topological left invariant.

(1) implies (2).

Assume $M(T)^*$ has a topological left invariant mean M_0. Define M on $M(S)^*$ by $M(G) = M(G/_T)$ where $G \in M(S)^*$ and $G/_T \in M(T)^*$ is defined by $G/_T(\mu_T) = G(\mu)$, $\mu \in M(S)$, $|\mu| (T') = 0$ (use Lemma 3.2). Clearly the map $G \to G/_T$ is linear, $G/_T \geq 0$ if $G \geq 0$, $G/_T = 1$ if $G = 1$ and $\chi_T/_T = 1$. Hence M is a mean on $M(S)^*$ with $M(\chi_T) = 1$. To prove that M is topological left T-invariant, let $\mu, \nu \in M_0(S)$, $\mu(T') = \nu(T') = 0$, then by Lemma 3.1,

$$\begin{aligned}(\mu \odot G)/_T(\nu_T) = (\mu \odot G)(\nu) &= G(\mu * \nu) = G/_T((\mu * \nu)_T) \\ &= G/_T(\mu_T * \nu_T) = \mu_T \odot G/_T(\nu_T).\end{aligned}$$

Hence $(\mu \odot G)/_T = \mu_T \odot G/_T$. (Note $M_0(T)$ spans $M(T)$.) Consequently, $M(\mu \odot G) = M_0((\mu \odot G)/_T) = M_0(\mu_T \odot G/_T) = M_0(G/_T) = M(G)$ since $\mu_T \in M_0(T)$ and M_0 is topological left invariant. This completes the proof.

For left invarinace, we can go a little further.

THEOREM 4.2. *Let T be a locally compact Borel subsemigroup of a locally compact semigroup S. Then the following statements are equivalent*:

(1) *$M(T)^*$ has a left invariant mean*

(2) *There is a left T-invariant mean M on $M(S)^*$ such that $M(\mathscr{X}_T) = 1$*

(3) *There is a left T-invariant mean M on $M(S)^*$ such that $M(\mathscr{X}_T) > 0$.*

Proof.

(1) implies (2).

This is the same as in the proof of Theorem 4.1 (1) implies (2). The only

https://doi.org/10.4153/CMB-1980-042-6 Published online by Cambridge University Press

required adjustment from topological left invariance to left invariance is to consider only Dirac measures instead of all probability measures in suitable places, and to observe that if $\mu = \varepsilon_x$, then $|\mu|(T') = 0$ iff $x \in T$.

(2) implies (3).

Obvious.

(3) implies (1).

This is essentially the same as in the proof of (2) implies (1) in Theorem 4.1 by specialising to Dirac measures. (Since we no longer have $M(\chi_{T'}) = 0$, we need a better upper bound for $(\mu_T \odot F)' - \mu \odot F'$ than $\|F\| \cdot \chi_{T'}$.) Proceed as in the proof of Theorem 4.1 (2) implies (1) except we define $M_0(F) = M(F')/M(\chi_T)$, until we come to the estimate of $\|(\mu * \nu)_T - \mu_T * \nu_T\|$. Putting $\mu = \varepsilon_x$, $x \in T$ we have if $\nu \in M(S)$, $\nu \geq 0$,

$$\|(\varepsilon_x * \nu)_T - (\varepsilon_x)_T * \nu_T\| \leq \nu(E(x))$$

by Lemma 3.1 (2) where $E(x) = \{y \in S : y \notin T \text{ and } xy \in T\} \subset T'$. Hence

$$(\varepsilon_x \odot F)' - \varepsilon_x \odot F' = ((\varepsilon_x)_T \odot F)' - \varepsilon_x \odot F' \leq \|F\| \cdot \chi_{E(x)}.$$

It remains to prove that $M(\chi_E) = 0$. Our method is to "transform" M into a mean on (bounded Borel measurable) functions on S and use Day's argument. (See e.g. Hewitt and Ross [5, §17.18 (f), page 238].) Let $BM(S)$ be the Banach space (with supremum norm) of all bounded Borel measurable (real-valued) functions on S. As shown in Wong [10], (although the space $CB(S)$ of all continuous bounded functions on S are considered there, the same can be said for $BM(S)$), the space $BM(S)$ can be embedded as a subspace in $M(S)^*$ by the linear isometry (into) $\varphi : BM(S) \to M(S)^*$ where $\varphi(f)(\mu) = \int f \, d\mu$, $f \in BM(S)$, $\mu \in M(S)$. φ is non-negative, satisfies $\varphi(1) = 1$ and commutes with left translations ($\varphi({}_af) = \varepsilon_a \odot \varphi(f)$ where we define ${}_af$ by ${}_af(x) = f(ax)$, $x \in S$). Moreover for any Borel set B in S, $\varphi(\xi_B) = \chi_B$. Now consider the adjoint $\varphi^* : M(S)^{**} \to BM(S)^*$ and let $m = \varphi^*(M)$, which is a left T-invariant mean on $BM(S)$ since M is a left T-invariant mean on $M(S)^*$. $m(\xi_T) = M(\chi_T) > 0$. Clearly $E = E(x) = \{y \in S : y \notin T \text{ and } xy \in T\}$ is Borel measurable in S and we have $M(\chi_E) = M(\varphi(\xi_E)) = \varphi^* M(\xi_E) = m(\xi_E) = 0$ by repeating the argument (of Day) in Hewitt and Ross [5, §17.18(f)] (where only the left T-invariance of the mean is used). Once $M(\chi_E)$ is shown to be zero the rest of the proof now is carried over from that of Theorem 4.1 (2) implies (1). This completes the proof.

Remarks.

(1) If the semigroups are discrete Theorems 4.1 and 4.2 imply the result of M. Day [1, Theorem 2] (see also Hewitt and Ross [5, §17.18(f)] and Wilde and Witz [8]).

(2) For topological left invariance, we are unable to relax the condition $M(\chi_T) = 1$ to $M(\chi_T) > 0$ because we cannot find an upper estimate $G \in M(S)^*$

for $(\mu_T \odot F)' - \mu \odot F'$ (where $\mu \in M_0(S)$, $\mu(T')=0$ and $F \in M(S)^*$) such that $M(G)=0$. We can no longer use $G = \chi_{T'}$, as in Theorem 4.1 since $M(\chi_{T'}) = 1 - M(\chi_T)$ need not be zero. For group and subgroups, $(\mu * \nu)_T = \mu_T * \nu_T$ (Lemma 3.1 (4)) and $\mu_T \odot F' = (\mu \odot F)'$ so that the condition $M(\chi_T)=1$ can be replaced by $M(\chi_T)>0$.

However, we shall present an even stronger result in the next section when T is open.

§5. **Special cases.**

5.1 Left Ideals. Let the subsemigroup T of S satisfy the condition (L): there is some $\nu \in M_0(S)$ such that $\nu(T')=0$ and $\mu * \nu(T')=0$ for all $\mu \in M_0(S)$. Then every topological left T-invariant mean M on $M(S)^*$ is topological left invariant (since $M(G) = M(\mu * \nu)\odot G) = M(\nu \odot(\mu \odot G)) = M(\mu \odot G)$ for all $\mu \in M_0(S)$, $G \in M(S)^*$. Consequently, by Theorem 4.1, $M(T)^*$ has a topological left invariant mean iff $M(S)^*$ has a topological left invariant mean M such that $M(\chi_T)=1$.

Notice that condition (L) is satisfied if there is some $a \in T$ such that $Sa \subset T$ (take $\nu = \varepsilon_a$) and a fortiori if T is a left ideal of S.

5.2 Right Ideals. If T satisfies the condition (R): there is some $\nu \in M_0(S)$, $\nu(T')=0$ such that $\nu * \mu(T')=0$ for all $\mu \in M_0(S)$. Then every topological left T-invariant mean M on $M(S)^*$ satisfies $M(\chi_T)=1$ (since $\nu \odot \chi_T(\mu) = \chi_T(\nu * \mu) = \nu * \mu(T) = 1-(\nu * \mu)(T') = 1$ and so $\nu \odot \chi_T = 1$). Consequently, by Theorem 4.1, $M(T)^*$ has a topological left invariant mean iff there is a topological left T-invariant mean on $M(S)^*$.

Conditions (R) is satisfied if there is some $a \in T$ such that $aS \subset T$ and a fortiori if T is a right ideal of S.

5.3. Groups and Subgroups.

Theorem 5.4. *Let T be a locally compact Borel subgroup of a locally compact group S, then the following statements are equivalent*:

(1) *There is a topological left invariant mean on $M(T)^*$*

(2) *There is a topological left T-invariant mean M on $M(S)^*$ such that $M(\chi_T)=1$*

(3) *There is a topological left T-invariant mean M on $M(S)^*$ such that $M(\chi_T)>0$.*

If in addition T is open, then each of the above statements is equivalent to

(4) *There is a topological left T-invariant mean on $M(S)^*$.*

Proof.

(1) implies (2)

This is a special case of Theorem 4.1.

https://doi.org/10.4153/CMB-1980-042-6 Published online by Cambridge University Press

(2) implies (3)

Obvious.

(3) implies (1)

Let M be a topological left T-invariant mean on $M(S)^*$ such that $M(\chi_T)>0$. Define M_0 on $M(T)^*$ by $M_0(F)=M(F')/M(\chi_T)$, $F\in M(T)^*$ and $F'(\mu)=F(\mu_T)$, $\mu\in M(S)$. M_0 is a mean on $M(T)^*$. It is topological left invariant on $M(T)^*$ because $\mu_T \odot F'=(\mu\odot F)'$ for any $\mu\in M_0(S)$ with $\mu(T')=0$ (see Remark (2) after Theorem 4.2).

Assume now that T is also *open.*

(4) implies (1).

Let $\varphi: BM(S)\to M(S)^*$ be the linear isometry (into) defined in the proof of Theorem 4.2 (3) implies (1). If M is a left T-invariant mean on $M(S)^*$, then M induces a left T-invariant mean $m=\varphi^*(M)$ on $BM(S)$ and in particular on the right uniformly continuous functions $UCB_r(S)$ on the group S (see [4] for definition of $UCB_r(S)$). For any $f\in UCB_r(T)$, define $f'(x)=f(t_x)$ where $x\in S$, $t_x\in T$ and $x=t_x\tau(Tx)$ where $\tau(Tx)$ is an arbitrary but fixed element of the right coset Tx (see [5, Theorem 17.12]). Since T is *open* and $t_{ax}=at_x$ for $a\in T$ and $x\in S$, it follows that $f'\in UCB_r(S)$. Now define $m_0(f)=m(f')$, $f\in UCB_r(T)$, then m_0 is a left invariant mean on $UCB_r(T)$. Since T is group, m_0 is necessarily topological left invariant and there is a net ν_α in $M_0(S)$ such that $\|\nu*\nu_\alpha-\nu_\alpha\|\to 0$ for each $\nu\in m_0(T)$ (see [4, Theorems 2.4.2 and 2.4.3] and also [6]). This implies that $M(T)^*$ has a topological left invariant mean by [10, Theorem 3.1].

(1) implies (4)

Since (1) implies (3) which is formally stronger than (4).

Note. The construction of f' from f is due to Følner [3] who considers only discrete groups. Rickert adapts it to right uniformly continuous functions (See Rickert [7]).

References

1. M. M. Day, *Amenable semigroups*, Illinois J. Math. **1** (1957), 509–544.

2. M. M. Day, *Lumpy subsets in left-amenable locally compact semigroups*, Pacific J. Math. **62** (1976), 87–92.

3. E. Følner, *On groups with full Banach mean value*, Math. Scand. **3** (1955), 243–254.

4. F. P. Greenleaf, *Invariant means on topological groups*, Van Nostrand Methematical Studies No. **16,** Van Nostrand, New York, 1969.

5. E. Hewitt and K. A. Ross, *Abstract harmonic analysis I*, Springer-Verlag, Berlin, 1963.

6. A. Hulanicki, *Means and Følner condition on locally compact groups*, Studia Math. **27** (1966), 87–104.

7. N. Rickert, *Amenable groups and groups with the fixed point property*, Trans. Amer. Math. Soc. **127** (1967), 221–232.

8. C. Wilde and K. Witz, *Invariant means and the Stone-Cech compactification*, Pacific J. Math. **21** (1967), 577–586.

9. James C. S. Wong, *Invariant means on locally compact semigroups*, Proc. Amer. Math. Soc. **31** (1972), 39–45.

https://doi.org/10.4153/CMB-1980-042-6 Published online by Cambridge University Press

10. James C. S. Wong, *An ergodic property of locally compact amenable semigroups*, Pacific J. Math. **48** (1973), 615–619.

11. James C. S. Wong, *Absolutely continuous measures on locally compact semigroups*, Canadian Math. Bull. **18** (1975), 127–131.

12. James C. S. Wong, *Amenability and substantial semigroups*, Canadian Math. Bull. **19** (1976), 231–234.

13. James C. S. Wong, *a characterisation of topological left thick subsets in locally compact left amenable semigroups*, Pacific J. Math. **62** (1976), 295–303.

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF CALGARY
CALGARY, CANADA. T2N 1N4