On Sylvester's Dialytic Method of Elimination.

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Consider any two algebraic equations, which for simplicity we shall take to be a cubic

$$ax^3 + bx^2 + cx + d = 0 \qquad \text{.........................(1)}$$

whose roots are y_1, y_2, y_3, and a quadratic

$$\alpha x^2 + \beta x + \gamma = 0 \qquad \text{...........................(2)}$$

whose roots are x_1, x_2. The equation which is obtained by eliminating x between these two equations represents the condition that the two equations should have a root in common: it must therefore be equivalent to the equation

$$(x_1 - y_1)(x_1 - y_2)(x_1 - y_3)(x_2 - y_1)(x_2 - y_2)(x_2 - y_3) = 0 \qquad \text{.......(3)}$$

The result of eliminating x between the two equations (1) and (2) is however given by the well-known dialytic method of Sylvester in the form $D = 0$, where D denotes the determinant

$$\begin{vmatrix} 0 & a & b & c & d \\ a & b & c & d & 0 \\ 0 & 0 & \alpha & \beta & \gamma \\ 0 & \alpha & \beta & \gamma & 0 \\ \alpha & \beta & \gamma & 0 & 0 \end{vmatrix}$$

I do not remember to have seen anywhere a direct proof that the equation (3) is equivalent to the equation $D = 0$, and the purpose of the present note is to supply such a proof.

We have

$$(x_1 - x_2) D = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ x_1^4 & x_1^3 & x_1^2 & x_1 & 1 \\ x_2^4 & x_2^3 & x_2^2 & x_2 & 1 \end{vmatrix} \cdot \begin{vmatrix} 0 & a & b & c & d \\ a & b & c & d & 0 \\ 0 & 0 & \alpha & \beta & \gamma \\ 0 & \alpha & \beta & \gamma & 0 \\ \alpha & \beta & \gamma & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & a & 0 & 0 & \alpha \\ a & b & 0 & \alpha & \beta \\ b & c & \alpha & \beta & \gamma \\ ax_1^3 + bx_1^2 + cx_1 + d & ax_1^4 + bx_1^3 + cx_1^2 + dx_1 & 0 & 0 & 0 \\ ax_2^3 + bx_2^2 + cx_2 + d & ax_2^4 + bx_2^3 + cx_2^2 + dx_2 & 0 & 0 & 0 \end{vmatrix}$$

https://doi.org/10.1017/S0013091500036026 Published online by Cambridge University Press

$$= \begin{vmatrix} ax_1^3 + bx_1^2 + cx_1 + d & x_1 (ax_1^3 + bx_1^2 + cx_1 + d) \\ ax_2^3 + bx_2^2 + cx_2 + d & x_2 (ax_2^3 + bx_2^2 + cx_2 + d) \end{vmatrix} \cdot \begin{vmatrix} 0 & 0 & \alpha \\ 0 & \alpha & \beta \\ \alpha & \beta & \gamma \end{vmatrix}$$

Therefore

$$D = \alpha^3 (ax_1^3 + bx_1^2 + cx_1 + d)(ax_2^3 + bx_2^2 + cx_2 + d).$$

Since

$$ax^3 + bx^2 + cx + d \equiv a(x - y_1)(x - y_2)(x - y_3),$$

we have therefore

$$D = \alpha^3 a^2 (x_1 - y_1)(x_1 - y_2)(x_1 - y_3)(x_2 - y_1)(x_2 - y_2)(x_2 - y_3).$$

The equation $D = 0$ is therefore equivalent to the equation (3), which was to be proved.