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A GENERALIZATION OF A CONSTRUCTION DUE TO ROBINSON

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1. Introduction. A method for constructing the product of two Schur functions was stated, but not proved in the most general case, by Littlewood and Richardson [**1**] in 1934. This method, which came to be known as the Littlewood-Richardson rule, was later proved completely by Robinson [**2**] in 1938. In this proof, Robinson describes an operation on a finite sequence of positive integers. It is this operation, set in a more general context, that is the subject of this paper.

This operation is also of interest from a combinatorial point of view. It is closely connected with the constructions of Schensted and Schützenberger. This, however, is outside the scope of this paper; a fuller investigation into these aspects of Robinson's operation may be found in Thomas [**3**].

Robinson's operation acts on a finite sequence of positive integers. By successive applications of this operation, any sequence may be *reduced* to a *lattice permutation*. It is this derivation of a lattice permutation from the initial sequence that I term *Robinson's Construction*. The construction as described in this paper is rather more general than that used by Robinson in [**2**]. This leads to a situation in which there exists in general, more than one way to reduce a given sequence to a lattice permutation, yet the resulting lattice permutation is always the same. The main result of this paper is a proof of this uniqueness.

I would like to express my gratitude to the referee whose suggestions have significantly improved the presentation of the proof of Lemma 1.

2. Definitions. Consider a sequence of n positive integers $c(1)c(2) \ldots c(n)$, repetitions allowed. For each r such that $c(r) > 1$; define s_r to be the number of r_1 such that $c(r_1) = c(r)$ and $r_1 \leqq r$; and define t_r to be the number of r_2 such that $c(r_2) = c(r) - 1$ and $r_2 < r$. Now define the *index*, $i(r)$, of r to be $s_r - t_r$, for each r such that $c(r) > 1$. In other words, $i(r)$ is the number of integers in the sequence lying to the left of $c(r)$ which are equal to $c(r)$, including $c(r)$ itself, minus the number of integers lying to the left of $c(r)$ which are equal to $c(r) - 1$.

If $i(r) \leqq 0$ for all r (such that $c(r) > 1$) then the sequence is a *lattice permutation*. If a sequence $c(1) \ldots c(n)$ is not a lattice permutation, then it can be *reduced* to a lattice permutation by a construction due to Robinson [**2**] which will be described shortly.

Consider the sequence $c(1) \ldots c(n)$. For $k = 2, 3, \ldots\ldots$ define a number p_k by

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$$c(p_k) = k,$$
$$i(p_k) \geqq i(r) \quad \text{for all } r > p_k \text{ such that } c(r) = k,$$
$$i(p_k) > i(r) \quad \text{for all } r < p_k \text{ such that } c(r) = k.$$

In other words, for each $k \geqq 2$, p_k is the position in $c(1) \ldots c(n)$ of the first k (reading from the left) which has maximal index.

Example.

r	1	2	3	4	5	6	7	8	9
$c(r)$	3	4	2	2	3	1	4	2	4
$i(r)$	1	0	1	2	0	–	0	2	1

$p_2 = 4, \quad p_3 = 1, \quad p_4 = 9;$

$i(4) = 2, \quad i(1) = 1, \quad i(9) = 1.$

For each $k \geqq 2$, define the operation $R(k)$ on a sequence $c(1) \ldots c(n)$ as follows.
Find p_k.
If $i(p_k) \leqq 0$, then leave the sequence unaltered.
If $i(p_k) > 0$, then put $c(p_k)$, (at present equal to k) equal to $k - 1$.

This is *Robinson's operation*. From now on, we shall assume $i(p_k) > 0$ whenever we consider the application of the operator $R(k)$. Repeated applications of $R(k)$'s for suitable choices of k will *reduce* a sequence $c(1) \ldots c(n)$ to a lattice permutation. Such a reduction will be called a *reduction by Robinson's construction.*

Example. Consider the previous example. The given sequence can be reduced to a lattice permutation by successively applying $R(k)$ for $k = 3, 4, 2, 2, 4, 3, 2, 3, 4$. This is shown below.

```
3  4  2  2  3  1  4  2  4
↓                            R(3)
2  4  2  2  3  1  4  2  4
                        ↓    R(4)
2  4  2  2  3  1  4  2  3
         ↓                   R(2)
2  4  2  1  3  1  4  2  3
      ↓                      R(2)
2  4  1  1  3  1  4  2  3
   ↓                         R(4)
2  3  1  1  3  1  4  2  3
            ↓                R(3)
2  3  1  1  2  1  4  2  3
↓                            R(2)
1  3  1  1  2  1  4  2  3
   ↓                         R(3)
1  2  1  1  2  1  4  2  3
                  ↓          R(4)
1  2  1  1  2  1  3  2  3
```

3. Preliminaries. Consider the reduction just described. The choice of k each time we apply $R(k)$ is reasonably free, so there is generally more than one way to reduce a given sequence to a lattice permutation.

For example, another reduction of 3 4 2 2 3 1 4 2 4 could be

$$
\begin{array}{ccccccccccl}
3 & 4 & 2 & 2 & 3 & 1 & 4 & 2 & 4 & & \\
 & & & \downarrow & & & & & & & R(2) \\
3 & 4 & 2 & 1 & 3 & 1 & 4 & 2 & 4 & & \\
 & & \downarrow & & & & & & & & R(2) \\
3 & 4 & 1 & 1 & 3 & 1 & 4 & 2 & 4 & & \\
 & & & & \downarrow & & & & & & R(3) \\
3 & 4 & 1 & 1 & 2 & 1 & 4 & 2 & 4 & & \\
\downarrow & & & & & & & & & & R(3) \\
2 & 4 & 1 & 1 & 2 & 1 & 4 & 2 & 4 & & \\
 & & & & & & & & \downarrow & & R(4) \\
2 & 4 & 1 & 1 & 2 & 1 & 4 & 2 & 3 & & \\
 & & & & & & \downarrow & & & & R(4) \\
2 & 4 & 1 & 1 & 2 & 1 & 3 & 2 & 3 & & \\
\downarrow & & & & & & & & & & R(2) \\
1 & 4 & 1 & 1 & 2 & 1 & 3 & 2 & 3 & & \\
 & \downarrow & & & & & & & & & R(4) \\
1 & 3 & 1 & 1 & 2 & 1 & 3 & 2 & 3 & & \\
 & \downarrow & & & & & & & & & R(3) \\
1 & 2 & 1 & 1 & 2 & 1 & 3 & 2 & 3 & &
\end{array}
$$

yet the lattice permutation we end up with is the same as that obtained previously. It is this result which is expressed in the following theorem.

THEOREM. *All reductions of a sequence* $c(1) \ldots c(n)$ *to a lattice permutation produce the same resulting lattice permutation.*

Remarks. An immediate consequence of this theorem is that all reductions of a give sequence contain the same number of operations. Hence, in the above example, all reductions of 3 4 2 2 3 1 4 2 4 will contain nine operations. It also follows that if a reduction of a sequence contains $R(2)$ k_2 times, $R(3)$ k_3 times, etc., then all reductions of the sequence contain $R(2)$ k_2 times, $R(3)$ k_3 times, etc.

What the theorem does not tell us is the number of operations required to reduce a given sequence to a lattice permutation. One would like some method of discovering this number short of actually performing the whole reduction. In addition, one would also like some method of determining the number of different reductions of a given sequence which are possible. Both these problems seem particularly difficult. A possible approach could be to consider permutations of the integers $1, 2, \ldots, n$ instead of arbitrary sequences of positive integers. Robinson's construction then has close connections with constructions due to Schensted and Schützenberger. For a survey of these connections, the reader is referred to Thomas [**3**].

Before proceeding with the proof of the theorem, we shall need one or two preliminary results.

Firstly, consider the operation $R(k)$ on a sequence $c(1) \ldots c(n)$. We note the following changes in the indices of the numbers in the sequence.

(i) $i(r)$ is unchanged if $r < p_k$.

(ii) $i(r)$ is unchanged if $c(r) < k - 1$ or $c(r) > k + 1$.

(iii) If $c(r) = k - 1$ or $k + 1$, and $r > p_k$, then $i(r)$ is increased by 1.

(iv) If $c(r) = k$ and $r > p_k$, then $i(r)$ is decreased by 2.

There is one more point to be noted regarding the indices. Consider $c(p_k)$ and let $i_k = i(p_k) > 0$. Now define $(p - 1)_k$ as the largest solution for r of

$$\begin{cases} c(r) = k \\ r < p_k \end{cases} \qquad \text{(There are at least } i_k \text{ solutions).}$$

Then $i((p - 1)_k) = i_k - 1$, and $c(r) \neq k - 1$ for $(p - 1)_k < r < p_k$.

Secondly, we need to consider what happens if we apply the operation $R(k)$ to a sequence and follow this by applying the operation $R(k')$ to the resulting sequence. We shall denote this operation by $R(k)R(k')$ and look at how it differs from the operation $R(k')R(k)$. Clearly $R(k)R(k') = R(k')R(k)$ if $k \neq k' - 1$ and $k \neq k' + 1$. The case $k = k' + 1$ needs slightly more careful attention. We shall prove that either $R(k)R(k + 1) = R(k + 1)R(k)$, or $R(k)R(k + 1)R(k + 1)R(k) = R(k + 1)R(k)R(k)R(k + 1)$.

It is clear that we need only prove the result for sequences of 1's, 2's, and 3's, and the operations $R(2)$ and $R(3)$. This greatly simplifies our notation.

LEMMA 1. *Let* $c(1) \ldots c(n)$ *be a sequence of the numbers* 1, 2, *and* 3 *in which* $i_2 > 0$ *and* $i_3 > 0$. *Then either*

$$R(2)R(3) = R(3)R(2), \quad \text{or} \quad R(2)R(3)R(3)R(2) = R(3)R(2)R(2)R(3).$$

Proof. There are two main cases to be considered, viz. $p_2 < p_3$ and $p_3 < p_2$. In fact, we shall only consider the case $p_3 < p_2$, the argument in the other case being virtually identical if one simply interchanges 2 and 3 throughout. So assume $p_3 < p_2$. Hence, $c(1) \ldots c(n)$ is of the form

$$\begin{matrix} & p_3 & & p_2 & \\ \cdots\cdots & 3 & \cdots\cdots & 2 & \cdots\cdots \end{matrix}$$

Recall that p_k is the first k in the sequence $c(1) \ldots c(n)$ of maximal index. For ease of notation, we shall print a number in boldface type in a sequence if it is the first number with maximal index.

So consider the sequence

$$\begin{array}{lccccc} & & & & X & \\ (1) & \cdots\cdots & \mathbf{3} & \cdots\cdots & \mathbf{2} & \cdots\cdots \\ & & i_3 & & i_2 & \end{array}$$

Case (i). All 3's after X in the sequence (1) have index $<i_3$. Apply $R(2)R(3)$ to (1).

$$\begin{array}{lcccccl} & & & & X & & \\ & \cdots\cdots & \mathbf{3} & \cdots\cdots & \mathbf{2} & \cdots\cdots & \text{All 3's after } X \text{ in the sequence have index } < i_3. \\ & & i_3 & & i_2 & & \\ \text{Apply } R(2) & & & & \downarrow & & \\ & \cdots\cdots & \mathbf{3} & \cdots\cdots & 1 & \cdots\cdots & \text{All 3's after } X \text{ have index } \leqq i_3. \\ & & i_3 & & & & \\ \text{Apply } R(3) & & \downarrow & & & & \\ & \cdots\cdots & 2 & \cdots\cdots & 1 & \cdots\cdots & \end{array}$$

Alternatively, apply $R(3)R(2)$ to the sequence (1).

$$\begin{array}{lcccccl} & & & & X & & \\ & \cdots\cdots & \mathbf{3} & \cdots\cdots & \mathbf{2} & \cdots\cdots & \\ & & i_3 & & i_2 & & \\ \text{Apply } R(3) & & \downarrow & & X & & \\ & \cdots\cdots & 2 & \cdots\cdots & \mathbf{2} & \cdots\cdots & X \text{ is still the first 2 of maximal index.} \\ & & & & i_2+1 & & \\ \text{Apply } R(2) & & & & \downarrow & & \\ & \cdots\cdots & 2 & \cdots\cdots & 1 & \cdots\cdots & \end{array}$$

Hence $R(2)R(3) = R(3)R(2)$ in this case.

Case (ii). There exists a 3 after X in the sequence (1) with index equal to i_3. Let Y be the first 3 after X with index i_3. We divide the sequence into four parts as shown below.

$$\begin{array}{lc|cc|cc|cc} & A & & B & & C & & D \\ & & Z & & X & & Y & \\ (2) & \cdots\cdots & \mathbf{3} & \cdots\cdots & \mathbf{2} & \cdots\cdots & 3 & \cdots\cdots \\ & & i_3 & & i_2 & & i_3 & \end{array}$$

We shall apply $R(2)R(3)R(3)R(2)$ to this sequence. A record will be kept of the maximal indices of the 2's and the 3's in each of these parts as the operations are applied.

Max. 2 index	$\leqq i_2 - 1$	$\leqq i_2 - 1$	$\leqq i_2$	$\leqq i_2$
Max. 3 index	$\leqq i_3 - 1$	$\leqq i_3$	$\leqq i_3 - 1$	$\leqq i_3$
		Z	X	Y
(3)	$\cdot\cdot\cdot\cdot\cdot$	$\mathbf{3}\cdot\cdot\cdot\cdot\cdot$	$\mathbf{2}\cdot\cdot\cdot\cdot\cdot$	$3\cdot\cdot\cdot\cdot\cdot$
		i_3	i_2	i_3

Apply $R(2)$

	$\leqq i_2 - 1$	$\leqq i_2 - 1$	$\leqq i_2 - 2$	$\leqq i_2 - 2$
	$\leqq i_3 - 1$	$\leqq i_3$	$\leqq i_3$	$\leqq i_3 + 1$
		Z	X	Y
(4)	$\cdot\cdot\cdot\cdot\cdot$	$3\cdot\cdot\cdot\cdot\cdot$	$1\cdot\cdot\cdot\cdot\cdot$	$\mathbf{3}\cdot\cdot\cdot\cdot\cdot$
		i_3		$i_3 + 1$

Apply $R(3)$

	$\leqq i_2 - 1$	$\leqq i_2 - 1$	$\leqq i_2 - 2$	$\leqq i_2 - 1$
	$\leqq i_3 - 1$	$\leqq i_3$	$\leqq i_3$	$\leqq i_3 - 1$
		Z	X	Y
(5)	$\cdot\cdot\cdot\cdot\cdot$	$\mathbf{3}\cdot\cdot\cdot\cdot\cdot$	$1\cdot\cdot\cdot\cdot\cdot$	$2\cdot\cdot\cdot\cdot\cdot$
		i_3		

Apply $R(3)$

	$\leqq i_2 - 1$	$\leqq i_2$	$\leqq i_2 - 1$	$\leqq i_2$
	$\leqq i_3 - 1$	$\leqq i_3 - 2$	$\leqq i_3 - 2$	$\leqq i_3 - 3$
		Z	X	Y
(6)	$\cdot\cdot\cdot\cdot\cdot$	$2\cdot\cdot\cdot\cdot\cdot$	$1\cdot\cdot\cdot\cdot\cdot$	$2\cdot\cdot\cdot\cdot\cdot$

Now apply $R(2)$. This causes a 2 with index i_2 in part B of the sequence to be changed to a 1. Denote this 2 by 2^+. We can show that 2^+ is in part B of the sequence as follows.

Let 2^* denote the 2 immediately preceding X in sequence (3). The index of 2^* is $i_2 - 1$, and there are no 1's between 2^* and X. The index of 2^* remains $i_2 - 1$ in sequences (4) and (5). In sequence (6), the further right of 2^* and Z has index i_2, (since there are no 1's between 2^* and X). Let 2^+ be the first 2 in part B with index i_2. We can see from sequence (6) that 2^+ is the first 2 in the whole sequence of maximal index.

Now consider $R(3)R(2)R(2)R(3)$.

$\leqq i_2 - 1$	$\leqq i_2 - 1$	$\leqq i_2$	$\leqq i_2$
$\leqq i_3 - 1$	$\leqq i_3$	$\leqq i_3 - 1$	$\leqq i_3$
A	B	C	D
	Z	X	Y
$\cdot\ \cdot\ \cdot\ \cdot\ \cdot$	$\mathbf{3} \cdot \cdot \cdot \cdot \cdot \cdot \cdot$	$\mathbf{2} \cdot \cdot \cdot \cdot \cdot \cdot$	$3 \cdot \cdot \cdot \cdot \cdot \cdot$
	i_3	i_2	i_3

Apply $R(3)$

$\leqq i_2 - 1$	$\leqq i_2$	$\leqq i_2 + 1$	$\leqq i_2 + 1$
$\leqq i_3 - 1$	$\leqq i_3 - 2$	$\leqq i_3 - 3$	$\leqq i_3 - 2$
$\cdot\ \cdot\ \cdot\ \cdot\ \cdot$	$2 \cdot 2^{+} \cdot \cdot \cdot$	$\mathbf{2} \cdot \cdot \cdot \cdot \cdot$	$3 \cdot \cdot \cdot \cdot \cdot$
	i_2	$i_2 + 1$	$i_3 - 2$

Apply $R(2)$

$\leqq i_2 - 1$	$\leqq i_2$	$\leqq i_2 - 1$	$\leqq i_2 - 1$
$\leqq i_3 - 1$	$\leqq i_3 - 2$	$\leqq i_3 - 2$	$\leqq i_3 - 1$
$\cdot\ \cdot\ \cdot\ \cdot\ \cdot$	$2 \cdot \mathbf{2}^{+} \cdot \cdot \cdot$	$1 \cdot \cdot \cdot \cdot \cdot$	$3 \cdot \cdot \cdot \cdot \cdot$
	i_2		$i_3 - 1$

Apply $R(2)$

$\leqq i_2 - 1$	$\leqq i_2 - 1$	$\leqq i_2 - 3$	$\leqq i_2 - 3$
$\leqq i_3 - 1$	$\leqq i_3 - 1$	$\leqq i_3 - 1$	$\leqq i_3$
$\cdot\ \cdot\ \cdot\ \cdot\ \cdot$	$2 \cdot 1 \cdot \cdot \cdot$	$1 \cdot \cdot \cdot \cdot \cdot$	$\mathbf{3} \cdot \cdot \cdot \cdot \cdot$
			i_3

Now apply $R(3)$ and change Y from a 3 to a 2. The resulting sequence is the same as that obtained previously. This completes the proof of the lemma.

4. Proof of theorem. Firstly, given a sequence $c = c(1) \ldots c(n)$, we define its *degree*, $\deg(c)$, to be $\sum_{r=1}^{n} c(r)$.

Apply $R(k)$ to $c(1) \ldots c(n)$ (assuming $i_k > 0$) and denote the resulting sequence by $c^* = c^*(1) \ldots c^*(n)$. We have $\deg(c^*) = \deg(c) - 1$. We may note at this point, that if a sequence $c = c(1) \ldots c(n)$ is reduced to a lattice permutation $\bar{c} = \bar{c}(1) \ldots \bar{c}(n)$ then the number of operations in the reduction is $\deg(c) - \deg(\bar{c})$. We shall now proceed with the proof of the theorem by using induction on the degree of c.

Assume the theorem to be true for all non-lattice permutations of degree less than N. Now consider a non-lattice permutation $c = c(1) \ldots c(n)$ of degree N.

(7) Suppose $R(k)$ and $R(k')$ $(k \neq k')$ are the first operations in any pair of

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possible reductions of c. We wish to show that both these reductions produce the same lattice permutation.

(8) First assume that $k' \neq k - 1$ or $k + 1$.

Apply $R(k)$ to c. The result is non-lattice of degree $N - 1$. So, by the induction hypothesis, all reductions of this resulting sequence are the same, so choose one that starts with $R(k')$.

Alternatively, apply $R(k')$ to c. The result is non-lattice of degree $N - 1$. So, by the induction hypothesis, all reductions of this sequence are the same, so choose one that starts with $R(k)$.

But $R(k)R(k') = R(k')R(k)$ and the resulting sequence after applying $R(k)R(k')$ to c is of degree $N - 2$. If the result is a lattice permutation, then we have no more to prove; if it is non-lattice, then we apply the induction hypothesis to say that all reductions are the same.

Hence we have proved that the results of pairs of reductions satisfying (7) and (8) are the same.

We now consider the case $k' = k + 1$. (The case $k' = k - 1$ is identical and need not be considered separately). We assume that there are reductions of c that can start with either $R(k)$ or $R(k + 1)$.

When applied to c, we know from Lemma 1 that $R(k)R(k + 1)$ could be the same as $R(k + 1)R(k)$. If this is the case, then the proof follows through exactly as above with $k' = k + 1$.

If this is not the case, we know that

$$R(k)R(k + 1)R(k + 1)R(k) = R(k + 1)R(k)R(k)R(k + 1)$$

when applied to c. So apply $R(k)$ to c. The result is non-lattice of degree $N - 1$. By the induction hypothesis, all reductions of this are the same, so choose one that starts with $R(k + 1)R(k + 1)R(k)$.

Alternatively, apply $R(k + 1)$ to c. Follow by choosing a reduction of the resulting sequence which starts with $R(k)R(k)R(k + 1)$.

Now $R(k)R(k + 1)R(k + 1)R(k) = R(k + 1)R(k)R(k)R(k + 1)$ when applied to c and the result is of degree $N - 4$. If the resulting sequence is a lattice permutation, then we have no more to prove; if it is not a lattice permutation, we apply the induction hypothesis which says that all reductions of this resulting sequence are the same and the proof of the theorem is complete.

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