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HOROCYCLIC CLUSTER SETS OF FUNCTIONS DEFINED IN THE UNIT DISC

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1. Introduction.

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Unless otherwise stated, $f: D \to W$ shall mean $f(z)$ is an arbitrary single-valued function defined in the open unit disc D: $|z| < 1$ and assuming values in the extended complex plane W. The unit circle $|z| = 1$ is denoted by Γ.

We assume the reader to be familiar with the rudiments of the theory of cluster sets. A general reference would be Noshiro [21] or Collingwood and Lohwater [9]. We shall use the following sets defined in [9, p. 207]:

$C(f, \zeta)$, the cluster set of f at ζ;

$C_{\mathscr{A}}(f, \zeta)$, the outer angular cluster set of f at ζ;

$C_{\Delta}(f, \zeta)$, the cluster set of f at ζ on a Stolz angle Δ at ζ;

$F(f)$, the set of Fatou points of f;

$I(f)$, the set of Plessner points of f;

$M(f)$, the set of Meier points of f;

$R(f, \zeta)$, the range of f at ζ.

We denote the cluster set of f at ζ on a chord χ at ζ by $C_{\chi}(f, \zeta)$. The principal chordal cluster set of f at ζ is defined to be

$$\Pi_{\chi}(f, \zeta) = \bigcap_{\chi} C_{\chi}(f, \zeta),$$

where the intersection is taken over all chords χ at ζ; and the inner angular cluster set of f at ζ is defined to be

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$$C_{\mathscr{B}}(f, \zeta) = \bigcap_{\Delta} C_{\Delta}(f, \zeta),$$

where the intersection is taken over all Stolz angles Δ at ζ. In addition we shall define the following sets:

Bagemihl defined and studied the majority of these sets in [3].

If $f: D \rightarrow W$, then a point $w \in W$ is a non-tangential cluster value of f at ζ provided there exists a sequence $\{z_n\}$ lying between two chords at ζ such that $\lim z_n = \zeta$ and $\lim f(z_n) = w$.

A circle internally tangent to Γ at a point $\zeta \in \Gamma$ is called a horocycle at ζ, and will be denoted by $h_r(\zeta)$, where r $(0 < r < 1)$ is the radius of the horocycle. The point ζ itself is not considered to be part of $h_r(\zeta)$. A point $w \in W$ is a horocyclic cluster value of f at ζ provided there exists a sequence $\{z_n\}$ lying between two horocycles at ζ such that $\lim z_n = \zeta$ and $\lim f(z_n) = w$. Our purpose is to examine the relationships between non-tangential and horocyclic cluster values of a function f in D. In particular, we shall compare (metrically and topologically) the sets of Fatou points, Plessner points and Meier points of f with their horocyclic analogues.

Section 2 deals with arbitrary single-valued functions in D. First it is shown (Theorem 2) that Collingwood's theorem concerning $K(f)$, f meromorphic in D, is true for f arbitrary in D. If one defines $K_{\omega}(f)$ as the horocyclic analogue of $K(f)$, then (Theorem 3) $K_{\omega}(f)$ is both residual and of measure 2π on Γ; i.e. the horocyclic analogue of Collingwood's theorem is true. Theorem 4 states that there exists a set residual and of measure 2π on Γ such that at each point ζ of the set, each non-tangential cluster value of f at ζ is a horocyclic cluster value of f at ζ relative to every pair of horocycles at ζ. An immediate corollary is that almost every (in the sense of Lebesgue) horocyclic Fatou point of f is a Fatou point of f, and almost every Plessner point of f is a horocyclic Plessner point of f. This had been shown by Bagemihl [3, Theorems 1 and 2] for meromorphic functions.

Littlewood [16] and Lohwater and Piranian [17, Theorem 9] have shown that not almost every Fatou point of f need be a horocyclic Fatou point of f even if f is holomorphic and bounded in D. Theorems 5 and 12 demonstrate the same result. In [10] it has been shown that not almost every horocyclic Plessner point of f need be a Plessner point of f even if f is holomorphic in D. For the function f in [10], each of the sets of Fatou points of f and horocyclic Plessner points of f has measure 2π. In Section 3 some further properties of points which are simultaneously Fatou points of f and horocyclic Plessner points of f are proved for f meromorphic in D.

The results of the preceding paragraph imply the non-existence of the following horocyclic analogues of Fatou's theorem [11] and Plessner's theorem [22]: If f is holomorphic and bounded in D, then almost every point of Γ is a horocyclic Fatou point of f; if f is meromorphic in D, then almost every point of Γ is either a horocyclic Fatou point of f or a horocyclic Plessner point of f. Moreover, in Section 4 a function f is constructed such that f is holomorphic in D, but the union of the sets of horocyclic Fatou points, horocyclic Plessner points and horocyclic Meier points of f has measure zero. The horocyclic behavior of this function is explained by the introduction of what we call the primary-tangential pre-Meier point. The explanation is a consequence of a theorem (Theorem 11) similar to the statement cited as the horocyclic analogue of Plessner's theorem. Specifically, if f is meromorphic in D, then almost every point of Γ is either a primary-tangential pre-Meier point of f or a horocyclic Plessner point of f. A theorem similar to the statement cited as the horocyclic analogue of Fatou's theorem is Theorem 10: If f is a normal meromorphic function in D, then almost every point of Γ is either a primary-tangential pre-Meier point of f or a point ζ at which $\Pi_{T_\omega}(f, \zeta) = W$.

It can be easily shown [3, Theorem 3] that if f is meromorphic in D, then almost every Meier point of f is a horocyclic Meier point of f. Section 5 is devoted to proving that not almost every horocyclic Meier point of f need be a Meier pointof f even if f is holomorphic and bounded in D.

To conclude the introduction we give a brief description of horocyclic notation and terminology.

Given a horocycle $h_r(\zeta)$ at a point $\zeta \in \Gamma$, the region interior to $h_r(\zeta)$ will be denoted by $\Omega_r(\zeta)$. The half of $h_r(\zeta)$ lying to the right of the radius at ζ as viewed from the origin will be denoted by $h_r^+(\zeta)$, and is called the right horocycle at ζ with radius r. The left horocycle is defined analogously. Also, $\Omega_r^+(\zeta)$ and $\Omega_r^-(\zeta)$ denote the right and left half, respectively, of $\Omega_r(\zeta)$.

Suppose that $0 < r_1 < r_2 < 1$ and that r_3 $(0 < r_3 < 1)$ is so large that the circle $|z| = r_3$ intersects both of the horocycles $h_{r_1}(\zeta)$ and $h_{r_2}(\zeta)$. We define the right horocyclic angle $H^+_{r_1,r_2,r_3}(\zeta)$ at ζ with radii r_1, r_2, r_3 to be

$$H^+_{r_1,r_2,r_3}(\zeta) = \text{comp}\,[\overline{\Omega^+_{r_1}(\zeta)}] \cap \Omega^+_{r_2}(\zeta) \cap \{z:\ |z| \geqslant r_3\},$$

where the bar denotes closure and "comp" denotes complement, both relative to the plane. The corresponding left horocyclic angle is denoted $H^-_{r_1,r_2,r_3}(\zeta)$. We write $H_{r_1,r_2,r_3}(\zeta)$ to denote a horocyclic angle at ζ without specifying whether it be right or left, or simply $H(\zeta)$ in the event r_1, r_2, r_3 are arbitrary.

Define the right outer horocyclic angular cluster set of f at ζ to be

$$C_{\mathfrak{A}^+}(f,\zeta) = \bigcup_{H^+} C_{H^+}(f,\zeta),$$

and the right inner horocyclic angular cluster set of f at ζ to be

$$C_{\mathfrak{B}^+}(f,\zeta) = \bigcap_{H^+} C_{H^+}(f,\zeta),$$

where in each case H^+ ranges over all right horocyclic angles at ζ. The outer horocyclic angular cluster set of f at ζ is defined to be

$$C_{\mathfrak{A}}(f,\zeta) = C_{\mathfrak{A}^+}(f,\zeta) \cup C_{\mathfrak{A}^-}(f,\zeta),$$

and the inner horocyclic angular cluster set of f at ζ to be

$$C_{\mathfrak{B}}(f,\zeta) = C_{\mathfrak{B}^+}(f,\zeta) \cap C_{\mathfrak{B}^-}(f,\zeta).$$

Finally the right principal horocyclic cluster set of f at ζ is defined to be

$$\Pi^+_\omega(f,\zeta) = \bigcap_{0<r<1} C_{h_r^+}(f,\zeta),$$

while we define the principal horocyclic cluster set of f at ζ to be

$$\Pi_\omega(f,\zeta) = \Pi^+_\omega(f,\zeta) \cap \Pi^-_\omega(f,\zeta).$$

If $f: D \to W$, then a point $\zeta \in \Gamma$ is called a right horocyclic Fatou point of f with right horocyclic Fatou value $w \in W$ provided

$$C_{\mathfrak{A}^+}(f, \zeta) = \{w\};$$

ζ is called a right horocyclic Plessner point of f provided

$$C_{\mathfrak{B}^+}(f, \zeta) = W;$$

ζ is called a right horocyclic Meier point of f provided

$$\Pi_\omega^+(f, \zeta) = C(f, \zeta) \subset W,$$

where $\subset$ denotes proper inclusion. The sets of right horocyclic Fatou points, right horocyclic Plessner points and right horocyclic Meier points of f are denoted by $F_\omega^+(f)$, $I_\omega^+(f)$ and $M_\omega^+(f)$ respectively. One defines $F_\omega^-(f)$, $I_\omega^-(f)$ and $M_\omega^-(f)$ in an analogous manner.

The sets of horocyclic Fatou points, horocyclic Plessner points and horocyclic Meier points of $f: D \to W$ are denoted by $F_\omega(f)$, $I_\omega(f)$ and $M_\omega(f)$ respectively, and are defined as follows:

$$\zeta \in F_\omega(f) \text{ if } C_{\mathfrak{A}}(f, \zeta) \text{ is a singleton;}$$
$$\zeta \in I_\omega(f) \text{ if } C_{\mathfrak{B}}(f, \zeta) = W; \text{ i.e. } I_\omega(f) = I_\omega^+(f) \cap I_\omega^-(f);$$
$$\zeta \in M_\omega(f) \text{ if } \Pi_\omega(f, \zeta) = C(f, \zeta) \subset W; \text{ i.e. } M_\omega(f) = M_\omega^+(f) \cap M_\omega^-(f).$$

By an arc at a point $\zeta \in \Gamma$ we mean a continuous curve Λ: $z = z(t)$ $(0 \leqslant t < 1)$ such that $|z(t)| < 1$ for $0 \leqslant t < 1$ and $\lim_{t \to 1} z(t) = \zeta$.

A point $\zeta \in \Gamma$ is said to be an ambiguous point of $f: D \to W$ if there exist two arcs Λ_1 and Λ_2 at ζ such that

$$C_{\Lambda_1}(f, \zeta) \cap C_{\Lambda_2}(f, \zeta) = \phi.$$

Bagemihl's ambiguous point theorem [1, Theorem 2] states that f has at most enumerably many ambiguous points. Thus,

$$[F_\omega^+(f) \cap F_\omega^-(f)] - F_\omega(f)$$

must be an enumerable set for $f: D \to W$.

If S_1 and S_2 are subsets of Γ such that $S_1 - S_2$ and $S_2 - S_1$ are of first Baire category (we sometimes say that nearly every point of S_1 is a point of S_2 and nearly every point of S_2 is a point of S_1), then S_1 and S_2 are said to be topologically equivalent. If $\text{meas}\,[S_1 - S_2] = \text{meas}\,[S_2 - S_1] = 0$, then S_1 and S_2 are said to be metrically equivalent.

2. Cluster sets of arbitrary functions.

Let $\mathscr{D}(1)$ be an open connected subset of D such that $\overline{\mathscr{D}(1)} \cap \Gamma = \{1\}$. By $\mathscr{D}(\zeta)$ we shall mean the transform of $\mathscr{D}(1)$ under the rotation about the origin that sends 1 into ζ. The following lemma is quite similar to that of Collingwood [8, Theorem 2].

LEMMA 1. *Let* $f: D \to W$. *Then*

$$C_{\mathscr{D}(\zeta)}(f, \zeta) = C(f, \zeta)$$

for a residual G_δ *subset of* Γ.

Proof. Let D be the set of points $\zeta \in \Gamma$ for which the condition of the lemma does not hold. It suffices to prove that E is an F_σ set of first category.

Considering W to be the Riemann sphere, let $\{Q_p: p = 1, 2, \cdots\}$ be the enumerable collection of open spherical discs on W such that the boundary of Q_p is a circle whose center has rational coordinates and whose radius has rational length. Let $\frac{1}{2} Q_p$ denote the open spherical disc on W with the same center as Q_p and area one-half that of Q_p.

Given $\zeta \in E$, there exists a disc Q_p such that

$$C(f, \zeta) \cap \frac{1}{2} Q_p \neq \phi \text{ and } C_{\mathscr{D}(\zeta)}(f, \zeta) \cap \overline{Q_p} = \phi.$$

Hence we can find a positive integer m such that

$$\overline{f(\mathscr{D}(\zeta) \cap \alpha_m)} \cap Q_p = \phi,$$

where α_m is the annulus $1 - 1/m < |z| < 1$. Thus we may write

$$E = \bigcup_{m, p} E_{m,p},$$

where

$$\overline{f(\mathscr{D}(\zeta) \cap \alpha_m)} \cap Q_p = \phi \text{ and } C(f, \zeta) \cap \frac{1}{2} Q_p \neq \phi, \ \zeta \in E_{m,p}.$$

Since $\mathscr{D}(1)$ is open, one can easily prove that

$$f(\mathscr{D}(\zeta) \cap \alpha_m) \cap Q_p = \phi, \ \zeta \in \overline{E_{m,p}}.$$

Also, it is readily seen that

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$$C(f,\zeta) \cap \overline{\frac{1}{2} Q_p} \neq \phi, \quad \zeta \in \overline{E_{m,p}}.$$

Thus, $\overline{E_{m,p}} \subseteq E$ for all values of m and p. Hence we have

$$E = \bigcup_{m,p} E_{m,p} \subseteq \bigcup_{m,p} \overline{E_{m,p}} \subseteq E.$$

Thus, E is an F_σ subset of Γ.

We now show that each set $\overline{E_{m,p}}$ is nowhere dense, so that E is of first category. If $\overline{E_{m,p}}$ is dense on any open arc Γ^* of Γ, then, setting

$$\alpha_m^* = \bigcup_{\zeta \in \Gamma^*} \mathscr{D}(\zeta) \cap \alpha_m,$$

we have

$$\overline{f(\alpha_m^*)} \cap Q_p = \phi.$$

Since $\mathscr{D}(1)$ is connected, we obtain α_m if we allow the points ζ to range over Γ in the previous union. Also $\overline{\mathscr{D}(1)} \cap \Gamma = \{1\}$, so that no point of Γ^* is a frontier point of $\alpha_m - \alpha_m^*$. Thus, given any $\zeta \in \Gamma^*$, there exists a positive integer $N = N(\zeta)$ such that

$$\{z \in D\colon\ |z-\zeta| < 1/n\} \subset \alpha_m^*, \quad n \geqslant N.$$

Since $\overline{f(\alpha_m^*)} \cap Q_p = \phi$,

$$C(f,\zeta) \cap Q_p = \phi, \quad \zeta \in \Gamma^*.$$

This contradicts the fact that

$$C(f,\zeta) \cap \frac{1}{2} Q_p \neq \phi, \quad \zeta \in E_{m,p} \cap \Gamma^* \neq \phi.$$

This completes the proof.

The following conventions will be used throughout the remainder of this paper.

Given a point $\zeta \in \Gamma$, $\Delta_{n,r}(\zeta)$, or more simply $\Delta_{n,r}$, represents the Stolz angle at ζ such that $\Delta_{n,r}$ has aperture $\pi/2^n$, n a positive integer; and the bisector of $\Delta_{n,r}$ at ζ makes a rational angle r $(-\pi/2 < r < \pi/2)$ with the radius at ζ. If α_m is the annulus $1 - 1/m < |z| < 1$ and $1 - 1/m > \sin\left(|r| + \frac{\pi}{2^{n+1}}\right)$, then we set

$$\Delta_{n,r,m} = \Delta_{n,r} \cap \alpha_m.$$

Then for each point $\zeta \in \Gamma$, we define $\sum(\zeta)$ to be the enumerable collection

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of all such Stolz "triangles" $\Delta_{n,r,m}(\zeta)$ at ζ. When we wish to refer to this collection without specifying a point ζ, we write Σ.

Analogously, we define $\Sigma_\omega(\zeta)$ to be the enumerable collection of horocyclic angles $H_{r_1,r_2,r_3}(\zeta)$ at ζ with the radii r_1, r_2, r_3 rational.

Making use of the enumerability of Σ and Σ_ω we can prove

LEMMA 2. *Let $f: D \to W$. Then*

$$C_{\mathscr{B}}(f,\zeta) = C_{\mathfrak{B}}(f,\zeta) = C(f,\zeta)$$

for a residual G_δ subset of Γ.

Proof. For each $\Delta \in \Sigma$, we have $C_\Delta(f,\zeta) = C(f,\zeta)$ for a residual G_δ subset of Γ by Lemma 1. The intersection of these enumerably many residual G_δ sets is a residual G_δ subset E_1 of Γ such that

$$C(f,\zeta) = \bigcap_{\Delta\in\Sigma} C_\Delta(f,\zeta) = C_{\mathscr{B}}(f,\zeta), \;\; \zeta \in E_1.$$

Similarly, we can find a residual G_δ subset E_2 of Γ such that

$$C(f,\zeta) = \bigcap_{H\in\Sigma_\omega} C_H(f,\zeta) = C_{\mathfrak{B}}(f,\zeta), \;\; \zeta \in E_2.$$

Then $E_1 \cap E_2$ is the required subset of Γ.

THEOREM 1. (Bagemihl [3, Theorem 4]). *Let $f: D \to W$. Then the sets $I(f)$, $I^+_\omega(f)$, $I^-_\omega(f)$ and $I_\omega(f)$ are topologically equivalent.*

Proof. Since $C_{\mathfrak{B}}(f,\zeta) = C_{\mathfrak{B}^+}(f,\zeta) \cap C_{\mathfrak{B}^-}(f,\zeta)$ for each $\zeta \in \Gamma$, Lemma 2 implies that

$$C_{\mathscr{B}}(f,\zeta) = C_{\mathfrak{B}^+}(f,\zeta) = C_{\mathfrak{B}^-}(f,\zeta) = C_{\mathfrak{B}}(f,\zeta) = C(f,\zeta)$$

for a residual set of points $\zeta \in \Gamma$. This implies the desired result.

Remark 1. A further consequence of Lemma 2 is that if any one of the sets $I(f)$, $I^+_\omega(f)$, $I^-_\omega(f)$ or $I_\omega(f)$ is dense on an arc Γ^* of Γ (hence $C(f,\zeta) = W$ for each point $\zeta \in \Gamma^*$), then each of the four sets is residual on Γ^*.

Remark 2. (Bagemihl [3, Remark 3]). Let $f: D \to W$. Then the sets $F(f)$, $F^+_\omega(f)$, $F^-_\omega(f)$ and $F_\omega(f)$ are topologically equivalent. Since $C_{\mathscr{B}}(f,\zeta) \subseteq C_{\mathscr{A}}(f,\zeta)$, $C_{\mathfrak{B}^+}(f,\zeta) \subseteq C_{\mathfrak{A}^+}(f,\zeta)$, $C_{\mathfrak{B}^-}(f,\zeta) \subseteq C_{\mathfrak{A}^-}(f,\zeta)$ and $C_{\mathfrak{B}}(f,\zeta) \subseteq C_{\mathfrak{A}}(f,\zeta)$, Lemma 2 implies that

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$$C_{\mathscr{A}}(f,\zeta) = C_{\mathfrak{A}^+}(f,\zeta) = C_{\mathfrak{A}^-}(f,\zeta) = C_{\mathfrak{A}}(f,\zeta)$$

for a residual set of points $\zeta \in \Gamma$. The result now follows.

Remark 3. It need not be true that the sets $M(f)$ and $M_\omega(f)$ are topologically equivalent for $f: D \to W$. Let S be an enumerable everywhere dense subset of Γ. Define $f(z)$ in D by $f(0) = 0$ and

$$f(z) = 1 \text{ for } z \in h^+_{\frac{1}{2}}(\zeta),\ \zeta \in S,$$
$$f(z) = 0 \text{ for } z \in h^+_{\frac{1}{2}}(\zeta),\ \zeta \in \Gamma - S.$$

Since both S and $\Gamma - S$ are everywhere dense on Γ,

$$\Pi_x(f,\zeta) = C(f,\zeta) = \{0,1\},\ \zeta \in \Gamma.$$

However, $\Pi_\omega(f,\zeta) = \{0\}$ for $\zeta \in \Gamma - S$, and $\Pi_\omega(f,\zeta) = \{1\}$ for $\zeta \in S$. Thus $M(f) = \Gamma$, but $M_\omega(f) \cap \Gamma = \phi$. This example also shows that $M(f)$ and $M_\omega(f)$ need not be metrically equivalent for $f: D \to W$.

Definition 1. If $f: D \to W$, then $K(f)$ consists of those points $\zeta \in \Gamma$ for which $C_{\Delta_1}(f,\zeta) = C_{\Delta_2}(f,\zeta)$ for any pair of Stolz angles Δ_1 and Δ_2 at ζ.

Collingwood [7, Theorem 4a] has shown that $K(f)$ is both residual and of measure 2π on Γ for f meromorphic in D. It is a consequence of the following lemma that the same result holds for an arbitrary function f in D.

Lemma 3. *Let $f: D \to W$. Then at almost every and nearly every point $\zeta \in \Gamma$,*

$$C_{\mathscr{L}(\zeta)}(f,\zeta) \subseteq C_{\mathscr{B}}(f,\zeta)$$

where $\mathscr{L}(\zeta)$ is any set for which there exists a Stolz angle at ζ containing $\mathscr{L}(\zeta)$.

Proof. If E is the set of points $\zeta \in \Gamma$ for which the lemma fails to hold, then for each $\zeta \in E$ there exists a set $\mathscr{L}(\zeta)$ lying in the interior of a Stolz angle at ζ such that $C_{\mathscr{L}(\zeta)}(f,,\zeta) \not\subseteq C_{\Delta(\zeta)}(f,\zeta)$ for some (not necessarily the same) Stolz angle $\Delta(\zeta)$ at ζ. Then there exists a disc Q_p on the Riemann sphere W such that

$$C_{\mathscr{L}(\zeta)}(f,\zeta) \cap Q_p \neq \phi \text{ and } C_{\Delta(\zeta)}(f,\zeta) \cap \overline{Q_p} = \phi.$$

It is then possible to find a Stolz triangle $\Delta_{n,r,m}(\zeta) \in \sum(\zeta)$ such that $\overline{f(\Delta_{n,r,m}(\zeta))} \cap Q_p = \phi$. Thus we may write

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$$E = \bigcup_{n,r,m,p} E_{n,r,m,p},$$

where $\zeta \in E_{n,r,m,p}$ provided there exists at least one set $\mathscr{L}(\zeta)$ lying in a Stolz angle at ζ such that

$$C_{\mathscr{L}(\zeta)}(f,\zeta) \cap Q_p \neq \phi \text{ and } \overline{f(\varDelta_{n,r,m}(\zeta))} \cap Q_p = \phi.$$

Now suppose that some set $E_{n,r,m,p}$ has positive exterior measure. If $X \equiv E_{n,r,m,p}$, then

(1) $$\overline{f(\varDelta_{n,r,m}(\zeta))} \cap Q_p = \phi, \ \zeta \in \bar{X}.$$

Note that it is not necessarily true that $C_{\mathscr{L}(\zeta)}(f,\zeta) \cap Q_p \neq \phi$ for at least one set $\mathscr{L}(\zeta)$ lying in some Stolz angle at ζ for each $\zeta \in \bar{X}$.

If

(2) $$G = \bigcup_{\zeta \in \bar{X}} \varDelta_{n,r,m}(\zeta),$$

then G is composed of finitely many open simply connected subregions $G_1, \cdots, G_N$ of D. There are only finitely many such subregions because $\Gamma - \bar{X}$ contains only finitely many arcs with length exceeding a fixed number between 0 and 2π. As in [23, p. 220], we conclude that each subregion G_k $(1 \leqslant k \leqslant N)$ has a rectifiable Jordan curve J_k $(1 \leqslant k \leqslant N)$ as boundary.

Now $X \cap J_k$ must be of positive exterior measure for at least one curve J_k. Also the tangent to J_k at almost every point of $\Gamma \cap J_k$ coincides with the tangent to Γ. Consequently, there exist points of X belonging to $\Gamma \cap J_k$ at which the tangent to J_k coincides with the tangent to Γ. At any such point each Stolz angle at the point has a terminal portion (i.e. a Stolz triangle at ζ) contained in G_k. Thus there exist points $\zeta \in X$, such that $C_{\mathscr{L}(\zeta)}(f,\zeta) \subseteq \overline{f(G_k)}$ for every set $\mathscr{L}(\zeta)$ at ζ which is contained in a Stolz angle at ζ. By (1) and (2),

$$\overline{f(G_k)} \cap Q_p = \phi.$$

However, according to the definition of X, we must have $C_{\mathscr{L}(\zeta)}(f,\zeta) \cap Q_p \neq \phi$ for at least one set $\mathscr{L}(\zeta)$ lying in some Stolz angle at ζ for every $\zeta \in X$ which is inconsistent with the previous statement. Hence each set $E_{n,r,m,p}$, and consequently E, has measure zero.

It is evident that our proof needs only minor modifications to establish that each set $E_{n,r,m,p}$, and consequently E, is of first category.

THEOREM 2. *Let* $f: D \to W$. *Then* $K(f)$ *is both residual and of measure* 2π *on* Γ.

Proof. At each point $\zeta \in \Gamma - K(f)$ there exists a Stolz angle $\Delta(\zeta)$ such that $C_{\Delta(\zeta)}(f, \zeta) \nsubseteq C_{\mathscr{B}}(f, \zeta)$. Lemma 3 implies that $\Gamma - K(f)$ is of measure zero and first category.

DEFINITION 2. If $f: D \to W$, then $K_\omega(f)$ consists of those points $\zeta \in \Gamma$ for which $C_{H_1}(f, \zeta) = C_{H_2}(f, \zeta)$ for any pair of horocyclic angles H_1 and H_2 at ζ.

Remark 4. A most crucial line of reasoning in the proof of Lemma 3 was that each Jordan curve J_k was rectifiable so that the tangent to J_k coincided with the tangent to Γ at almost every point $\zeta \in \Gamma \cap J_k$; and consequently, at almost every point $\zeta \in \Gamma \cap J_k$, each Stolz angle at ζ had a terminal portion interior to G_k.†

For a fixed horocyclic angle $H_{r_1, r_2, r_3}(\zeta)$ and a closed set $P \subset \Gamma$, define

$$G^\omega = \bigcup_{\zeta \in P} H_{r_1, r_2, r_3}(\zeta).$$

By [3, Lemma 1], G^ω is composed of finitely many simply connected subregions $G_1^\omega, \cdots, G_N^\omega$ having as their respective boundaries the rectifiable Jordan curves $J_1^\omega, \cdots, J_N^\omega$. Hence the tangent to $J_k^\omega (1 \leqslant k \leqslant N)$ at almost every point $\zeta \in \Gamma \cap J_k^\omega$ coincides with the tangent to Γ. However, this does not imply that at almost every point $\zeta \in \Gamma \cap J_k^\omega$, each horocyclic angle H has a terminal portion which lies in G_k^ω, because the tangent to H at ζ also coincides with the tangent to Γ at ζ. But if we can verify that this latter statement is true, then by virtually the same proof as of Lemma 3 we can obtain a horocyclic analogue of Lemma 3 (see Lemma 6).

LEMMA 4. *Let* P *be a perfect nowhere dense subset of* $[0,1]$. *For almost every point* $p \in P$, *if* $\{(a_n, b_n)\}$ *is any sequence of open intervals in* $[0,1] - P$ *converging to* p, *then*

$$|a_n - p|/(b_n - a_n) \text{ tends to } +\infty.$$

† If $S \subset D$ such that $\bar{S} \cap \Gamma = \{\zeta\}$, then $S' \subseteq S$ is called a terminal portion of S if $S' \cap D - \alpha_m = \phi$ and $S' \cap \alpha_p = S \cap \alpha_p$, where $p \geqslant m > 0$.

Proof. According to Hobson [12, p. 194], the metric density exists and is unity at almost every point $p \in P$. Let $p \in P$ be such a point, and suppose the sequence $\{(a_n, b_n)\}$ converges to p from the right. Then by the definition of metric density

$$\lim_{n \to +\infty} \frac{\text{meas}\,(P \cap (p, b_n))}{\text{meas}\,(p, b_n)} = 1 \tag{3}$$

and

$$\lim_{n \to +\infty} \frac{\text{meas}\,(P \cap (p, a_n))}{\text{meas}\,(p, a_n)} = 1. \tag{4}$$

Let $x_n = \text{meas}\,(P \cap (p, b_n))$, $y_n = a_n - p$ and $z_n = b_n - a_n$. Then (3) implies

$$\frac{x_n}{y_n + z_n} \to 1$$

and, since $P \cap (p, b_n) = P \cap (p, a_n)$, (4) implies

$$\frac{x_n}{y_n} \to 1.$$

Since $x_n > 0$, $y_n > 0$ and $z_n > 0$, these conditions imply that

$$\frac{z_n}{y_n} \to 0; \text{ i.e. } \frac{y_n}{z_n} \to +\infty.$$

Thus $(a_n - p)/(b_n - a_n) \to +\infty$ and in general, $|a_n - p|/(b_n - a_n) \to +\infty$.

LEMMA 5. *Let P be a perfect nowhere dense subset of Γ and set*

$$G^{\omega} = \bigcup_{\zeta \in P} H_{r_1, r_2, r_3}(\zeta),$$

where H_{r_1, r_2, r_3} is a fixed horocyclic angle. Then at almost every point $\zeta \in P$ each disc $\Omega_r(\zeta)$ $(0 < r < 1)$ has a terminal portion lying interior to G^{ω}.

Proof. Without explicitly going through all the details we note that it is possible, by means of a bilinear mapping $L(z)$, to transfer the setting of our lemma from D to the upper half-plane and arrive at an equivalent formulation. We now give this formulation in a somewhat extensive form.

Let P be a perfect nowhere dense set on the finite interval I of the real axis, and let the two circles (take $(a_n, b_n) \subset I - P$)

$$C_1\colon\ (x - a_n)^2 + (y - R)^2 = R^2 \text{ and } C_2\colon\ (x - b_n)^2 + (y - r)^2 = r^2 \tag{5}$$

have radii satisfying

(6) $$0 < R_1 \leqslant r \leqslant R_2 < R_3 \leqslant R \leqslant R_4.$$

We choose r and R in this fashion because the two horocycles $h_{r_1}(\zeta)$ and $h_{r_2}(\zeta)$ forming part of $H_{r_1,r_2,r_3}(\zeta)$, and hence part of the boundary of G^{ω}, would be mapped by $L(z)$, as ζ ranges over $P \subset \Gamma$, onto circles of the form (5) whose radii satisfy a condition of the form (6).

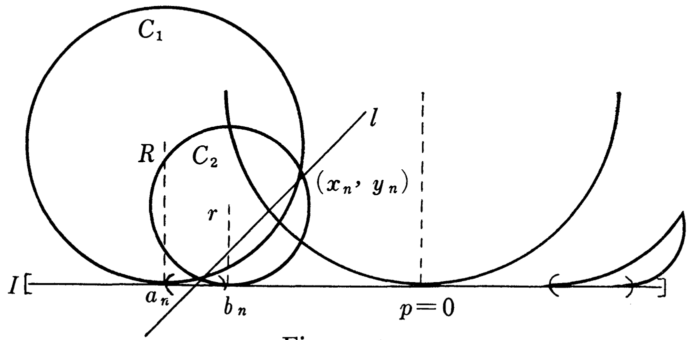

Figure 1

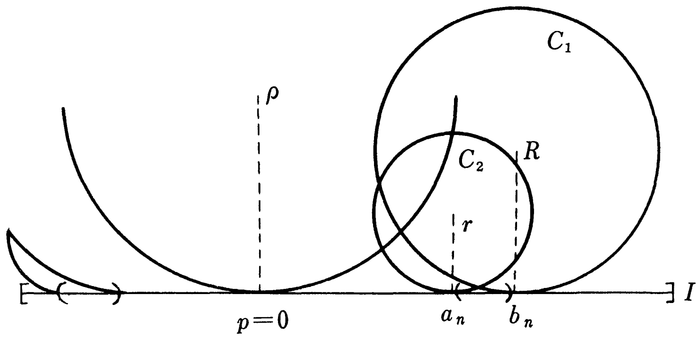

Figure 2

At the left and right endpoints of each interval in $I-P$ construct circles C_1 and C_2 respectively (see Figure 1). In the proof it shall become apparent that we could choose C_1 to be at the right endpoint and C_2 at the left endpoint of each interval in $I-P$ (see Figure 2). These two situations correspond to the choice of $H_{r_1,r_2,r_3}(\zeta)$ as a left and right horocyclic angle, respectively.

Our ultimate goal is to prove:

(7) At almost every point $p \in P$, for any sequence $\{(a_n, b_n)\}$ of arcs in $I-P$

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converging to p, the point $(x_n, y_n) \in C_1 \cap C_2$ (see Figure 1) lies interior to any given circle tangent to the x-axis at p for at most finitely many values of n.

Our method of proof will be to show that the condition on p in (7) is satisfied at each point $p \in P$ at which Lemma 4 holds. Since Lemma 4 holds for almost every point $p \in P$, (7), and hence our lemma, will be established.

Suppose to the contrary that there exists a point $p \in P$ at which Lemma 4 holds and the condition on p in (7) fails to be true. Without loss of generality we may assume that $p = 0$. Hence, we are assuming that there exists a circle C: $x^2 + (y - \rho)^2 = \rho^2\ (0 < \rho < +\infty)$ and a sequence $\{(a_n, b_n)\}$ in $I - P$ converging to $p = 0$ for which $|a_n|/(b_n - a_n) \to +\infty$, but the point $(x_n, y_n) \in C_1 \cap C_2$ lies interior to C for infinitely many values of n; i.e.

(8) $$x_n^2 + (y_n - \rho)^2 < \rho^2 \text{ for infinitely many } n.$$

Also, since $|a_n|/(b_n - a_n) \to +\infty$ and $sgn(a_n) = sgn(b_n)$,

(9) $$|b_n + a_n|/(b_n - a_n) \to +\infty.$$

Consider the radical axis l of C_1 and C_2 passing through $C_1 \cap C_2$. The equation for l is given by

$$(x - a_n)^2 + (y - R)^2 - R^2 - [(x - b_n)^2 + (y - r)^2 - r^2] = 0,$$

or

$$x = \frac{R - r}{b_n - a_n} y + \frac{b_n + a_n}{2}.$$

Hence,

(10) $$x_n = \frac{R - r}{b_n - a_n} y_n + \frac{b_n + a_n}{2}.$$

Solving (10) simultaneously with the equation of C_1 in (5) for y_n, we have

$$\left(\frac{R - r}{b_n - a_n} y_n + \frac{b_n + a_n}{2} - a_n\right)^2 + (y_n - R)^2 = R^2.$$

This can be rewritten as

$$(R - r)^2 \frac{y_n}{(b_n - a_n)^2} + \frac{(b_n - a_n)^2}{y_n} = R + r - y_n.$$

Since $y_n \to 0^+$ we immediately have

$$y_n = o((b_n - a_n)^2),$$

and hence,

(11) $\qquad y_n < K(b_n - a_n)^2, \ K > 0,$ for all sufficiently large n.

Now we show that (8) is impossible. Substituting (10) in (8) yields

(12) $$\left(\frac{R-r}{b_n - a_n}\right)^2 y_n + (R-r)\left(\frac{b_n + a_n}{b_n - a_n}\right) + \left(\frac{b_n + a_n}{2}\right)^2 \frac{1}{y_n} + y_n < 2\rho.$$

The left-hand side of (12) is greater than

$$(R-r)\left(\frac{b_n + a_n}{b_n - a_n}\right) + \left(\frac{b_n + a_n}{2}\right)^2 \frac{1}{y_n},$$

and by (6) and (11), this expression is greater than

$$(R_3 - R_2)\left(\frac{b_n + a_n}{b_n - a_n}\right) + \left(\frac{b_n + a_n}{2}\right)^2 \frac{1}{K(b_n - a_n)^2}$$
$$= \frac{b_n + a_n}{b_n - a_n}\left[R_3 - R_2 + \frac{1}{4K}\,\frac{b_n + a_n}{b_n - a_n}\right].$$

By (9) this latter expression tends to $+\infty$ so that (12), and hence (8), can hold for at most finitely many values of n, which is a contradiction. Thus our lemma is proved.

LEMMA 6. *Let $f: D \to W$. Then at almost every and nearly every point $\zeta \in \Gamma$,*

$$C_{\mathscr{H}(\zeta)}(f, \zeta) \subseteq C_{\mathfrak{B}}(f, \zeta)$$

where $\mathscr{H}(\zeta)$ is any set for which there exists a disc $\Omega_r(\zeta)$ at ζ containing $\mathscr{H}(\zeta)$.

Proof. As stated in Remark 4, the proof of Lemma 3 with only minor modifications can be used here. We replace Stolz angles by horocyclic angles, the region G by a region G^{ω} and apply Lemma 5 where needed.

THEOREM 3. *Let $f: D \to W$. Then $K_\omega(f)$ is both residual and of measure 2π on Γ.*

Proof. At each point $\zeta \in \Gamma - K_\omega(f)$ there exists a horocyclic angle $H(\zeta)$ such that $C_{H(\zeta)}(f, \zeta) \nsubseteq C_{\mathfrak{B}}(f, \zeta)$. Lemma 6 implies that $\Gamma - K_\omega(f)$ is of measure zero and first category.

COROLLARY 1. *Let* $f: D \to W$. *Then the sets* $F^+_\omega(f)$, $F^-_\omega(f)$ *and* $F_\omega(f)$ *are metrically equivalent, and the sets* $I^+_\omega(f)$, $I^-_\omega(f)$ *and* $I_\omega(f)$ *are metrically equivalent.*

Proof. If ζ belongs to at least one of the sets $F^+_\omega(f)$, $F^-_\omega(f)$, $F_\omega(f)$, but not to all of them, then $C_{H_1}(f,\zeta) \neq C_{H_2}(f,\zeta)$ for some pair of horocyclic angles H_1 and H_2 at ζ. By Theorem 3, the set of such points $\zeta \in \Gamma$ is of measure zero. This proves the first part of Corollary 1, and the proof of the second part is identical.

Remark 5. Lemma 2 affords some additional information concerning $K(f)$ and $K_\omega(f)$. The relation

$$C_\Delta(f,\zeta) = C_H(f,\zeta) = C(f,\zeta)$$

holds at nearly every point $\zeta \in K(f) \cap K_\omega(f)$ for any Stolz angle Δ at ζ and any horocyclic angle H at ζ.

THEOREM 4. *Let* $f: D \to W$. *Then at almost every and nearly every point* $\zeta \in \Gamma$,

$$C_\Delta(f,\zeta) \subseteq C_H(f,\zeta)$$

for each Stolz angle Δ *at* ζ *and each horocyclic angle* H *at* ζ.

Proof. If ζ is a point where the condition fails to hold, then $C_{\Delta(\zeta)}(f,\zeta) \not\subseteq C_{\mathfrak{B}}(f,\zeta)$ for some Stolz angle $\Delta(\zeta)$ at ζ. Lemma 6 implies the desired result.

We can now generalize two results of Bagemihl [3, Theorems 1 and 2].

COROLLARY 2. *Let* $f: D \to W$. *Then almost every horocyclic Fatou point of* f *is a Fatou point of* f, *and almost every Plessner point of* f *is a horocyclic Plessner point of* f.

Proof. If $\zeta \in F_\omega(f)$, then there exists a horocyclic angle $H(\zeta)$ at ζ and a point $w_\zeta \in W$ such that $C_{H(\zeta)}(f,\zeta) = \{w_\zeta\}$. From Theorem 4 we conclude that $C_{\mathscr{A}}(f,\zeta) = \{w_\zeta\}$ for almost every point $\zeta \in F_\omega(f)$; i.e. almost every point of $F_\omega(f)$ is a point of $F(f)$.

If $\zeta \in I(f)$, then $C_{\mathscr{B}}(f,\zeta) = W$. According to Theorem 4, $C_{\mathscr{B}}(f,\zeta) \subseteq C_{\mathfrak{B}}(f,\zeta)$ for almost every point $\zeta \in \Gamma$. Thus $C_{\mathfrak{B}}(f,\zeta) = W$ for almost every point $\zeta \in I(f)$, which is the desired conclusion.

3. The set $F(f) \cap I_\omega(f)$.

The following example, a special case of an example of Lohwater and Piranian [17, Theorem 9], shows that $F(f)$ and $F_\omega(f)$ need not be metrically equivalent.

THEOREM 5. *There exists a function $B(z)$ holomorphic and bounded in D such that the set of horocyclic Fatou points of $B(z)$ has measure zero.*

Proof. The Blaschke product

$$B(z) = \prod_{n=1}^{\infty} \frac{(\rho_n)^{2^n} + (z)^{2^n}}{1 + (\rho_n z)^{2^n}}, \quad \rho_n = 1 - (n^2 2^n)^{-1}, \quad n = 1, 2, \cdots,$$

has zeros at the points

$$z_{n,k} = \rho_n e^{i(2k-1)2^{-n}\pi}, \quad n = 1, 2, \cdots; \; k = 1, 2, \cdots, 2^n.$$

In [10] it is shown that for each point $\zeta \in \Gamma$ and each horocycle $h_r(\zeta)$ $(0 < r < 1)$ at ζ, there exist sequences of these zeros lying interior to $\Omega_r^+(\zeta)$ and $\Omega_r^-(\zeta)$. Thus, for each $\zeta \in \Gamma$,

$$(13) \qquad 0 \in C_{\Omega_r^+(\zeta)}(B, \zeta) \; (0 < r < 1) \text{ and } 0 \in C_{\Omega_r^-(\zeta)}(B, \zeta) \; (0 < r < 1).$$

It is well-known [24, p. 94] that a Blaschke product has a Fatou value of modulus one at almost every point $\zeta \in \Gamma$. Take $\zeta \in F(B)$ such that B has Fatou value α, $|\alpha| = 1$, at ζ. If ζ is a right horocyclic Fatou point of B, then the right horocyclic Fatou value must be 0 because a result of Lindelof [6, p. 42] states that the right horocyclic Fatou value of B at ζ must equal

$$C_{\Omega_r^+(\zeta)}(B, \zeta) \qquad (0 < r < 1),$$

and, from (13), 0 belongs to each such cluster set. Thus,

$$C_{\Omega_r^+(\zeta)}(B, \zeta) = \{0\} \qquad (0 < r < 1).$$

However, this contradicts the fact that $C_{\Delta(\zeta)}(B, \zeta) = \{\alpha\}$ for each Stolz angle $\Delta(\zeta)$ at ζ. Thus the set of right horocyclic Fatou points of B is of measure zero. By Corollary 1, $F_\omega(f)$ has measure zero, and the proof is complete.

To show that $I(f)$ and $I_\omega(f)$ need not be metrically equivalent, we cite the following theorem proven in [10].

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THEOREM 6. *There exists a function $f(z)$ holomorphic in D such that every point of Γ is a horocyclic Plessner point of f and almost every point of Γ is a Fatou point of f.*

The following corollary is interesting in view of Plessner's theorem [22] and Meier's theorem [18, Theorem 5].

COROLLARY 3. *There exists a function $f(z)$ holomorphic in D such that almost every point of Γ is a Fatou point of f and nearly every point of Γ is a Plessner point of f.*

Proof. By Theorem 1, $I(f)$ and $I_\omega(f)$ are topologically equivalent. Since every point $\zeta \in \Gamma$ is a point of $I_\omega(f)$, the result follows.

Theorem 6 shows that $F(f) \cap I_\omega(f)$ may be large metrically even if f is holomorphic in D. However, for $f: D \to W$, $F(f) \cap I_\omega(f)$ must be of first category by Theorem 1.

An arc Λ_ω at $\zeta \in \Gamma$ is said to be an admissible tangential arc at ζ if there exists a sequence $\{H_{r_1^{(n)}, r_2^{(n)}, r_3^{(n)}}(\zeta)\}$ of nested right or of nested left horocyclic angles at ζ with $\lim_{n\to\infty} [r_2{}^{(n)} - r_1{}^{(n)}] = 0$, each term of which contains some terminal subarc of Λ_ω.

We now define

$$\Pi_{T_\omega}(f, \zeta) = \bigcap_{\Lambda_\omega} C_{\Lambda_\omega}(f, \zeta),$$

where the intersection is taken over all admissible tangential arcs Λ_ω at ζ.

THEOREM 7. *If $f(z)$ is meromorphic in D, then*

$$\Pi_{T_\omega}(f, \zeta) \cup R(f, \zeta) = W$$

for each point $\zeta \in F(f) \cap I_\omega(f)$ with the possible exception of at most enumerably many such points.

Proof. If ζ is a point of $F(f) \cap I_\omega(f)$ such that $\Pi_{T_\omega}(f, \zeta) \cup R(f, \zeta) \subset W$, then either $W - [\Pi_{T_\omega}(f, \zeta) \cup R(f, \zeta)]$ is the Fatou value of f at ζ or there exists a value $w \notin \Pi_{T_\omega}(f, \zeta) \cup R(f, \zeta)$ different from the Fatou value of f at ζ. We assert that in either case, ζ is an ambiguous point of f. Bagemihl's ambiguous point theorem [1, Theorem 2] then implies the desired result.

In the first case $C_\chi(f, \zeta) \cap C_{\Lambda_\omega}(f, \zeta) = \phi$ for each chord χ at ζ and some admissible tangential arc Λ_ω at ζ, so that ζ is an ambiguous point of f.

In the second case there must be an admissible tangential arc Λ_ω at ζ such that $w \notin C_{\Lambda_\omega}(f,\zeta)$. Let χ be a chord at ζ disjoint from Λ_ω, and join the endpoints of χ and Λ_ω by means of a Jordan arc J^* so that $\{\zeta\} \cup \Lambda_\omega \cup J^* \cup \chi$ is a Jordan curve. Let G denote the interior of this Jordan curve and set $J = \Lambda_\omega \cup J^* \cup \chi$. Since Λ_ω is an admissible tangential arc at ζ, G must contain at least one right or left horocyclic angle at ζ. Thus $C_G(f,\zeta) = W$. Since w is not the Fatou value of f at ζ and $w \notin C_{\Lambda_\omega}(f,\zeta)$, $w \notin C_J(f,\zeta)$. Moreover, $w \notin R_G(f,\zeta)$, because $w \notin R(f,\zeta)$. Hence

$$w \in [C_G(f,\zeta) - C_J(f,\zeta)] \cap \operatorname{comp} R_G(f,\zeta),$$

so that by the Gross-Iversen theorem [9, p. 101], there exists an arc Λ at ζ such that $C_\Lambda(f,\zeta) = \{w\}$. Hence, ζ is an ambiguous point of f, and the theorem is proved.

COROLLARY 4. *If $f(z)$ is holomorphic in D, then*

$$\infty \in \Pi_{T_\omega}(f,\zeta)$$

for each point $\zeta \in F(f) \cap I_\omega(f)$ with the possible exception of at most enumerably many such points.

We now prove that Corollary 4 is no longer true if we replace $F(f) \cap I_\omega(f)$ by $I_\omega(f)$.

THEOREM 8. *Let P be a perfect nowhere dense subset of Γ. Then there exists a function $f(z)$ holomorphic in D such that almost every point of P is a point of $I_\omega(f)$, and $\Pi_{T_\omega}(f,\zeta) = \{0\}$ for each point $\zeta \in P$ with at most enumerably many exceptions.*

Proof. Set

$$T = \bigcup_{\zeta \in P} h^{\dagger}_{\frac{1}{2}}(\zeta).$$

Then T is a tress in the sense of Bagemihl and Seidel [4, Definition 1], and there exists a function $f(z)$ holomorphic in D such that

$$C_{h^{\dagger}_{\frac{1}{2}}(\zeta)}(f,\zeta) = \{0\} \tag{14}$$

for each point $\zeta \in P$ [4, Corollary 1].

If $\operatorname{meas}[P \cap F(f)] > 0$, then, since $C_{h^{\dagger}_{\frac{1}{2}}(\zeta)}(f,\zeta) = \{0\}$ for each point $\zeta \in P \cap F(f)$, f must have 0 as Fatou value at each point $\zeta \in P \cap F(f)$ with the possible exception of at most enumerably many ambiguous

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points. But this is impossible by Priwalow's theorem [9, Theorem 8.1]. Hence almost every point of P is a point of $I(f)$ by Plessner's theorem. By Corollary 2, almost every point of P is a point of $I_\omega(f)$. By (14), $\Pi_{T_\omega}(f,\zeta)=\{0\}$ at any point of P which is not an ambiguous point of f. This completes the proof of the theorem.

Remark 6. By [21, Remark, p. 74], it is not possible to construct the function $f(z)$ of Theorem 8 to have both a right and a left horocycle at almost every point $\zeta \in P$ on which f is bounded.

Remark 7. Theorem 4 states that $C_{\mathscr{B}}(f,\zeta) \subseteq C_{\mathfrak{B}}(f,\zeta)$ for almost every point $\zeta \in \Gamma$ for $f: D \to W$. It is a consequence of Theorem 8 that even if f is holomorphic in D, then it need not be true that $\Pi_x(f,\zeta) \subseteq \Pi_\omega(f,\zeta)$ for almost every point $\zeta \in \Gamma$.

If f is holomorphic in D, then, by applying the Gross-Iversen theorem, one sees that

$$\infty \in \Pi_x(f,\zeta) \cup \Pi_\omega(f,\zeta)$$

for each point $\zeta \in I(f) \cup I_\omega(f)$ with the possible exception of at most enumerably many ambiguous points. Thus, for the function $f(z)$ in Theorem 8, $\infty \in \Pi_x(f,\zeta)$ and $\infty \notin \Pi_\omega(f,\zeta)$ for almost every point $\zeta \in P$ since almost every point of P is a point of $I_\omega(f)$.

It is an open question whether $\Pi_x(f,\zeta) \subseteq \Pi_\omega(f,\zeta)$ for nearly every point $\zeta \in \Gamma$ if $f(z)$ is meromorphic in D.

4. Horocyclic cluster sets of meromorphic functions.

THEOREM 9. *There exists a function $f(z)$ holomorphic in D such that almost every point of Γ is a Fatou point of f, but*

$$\text{meas}\,[F_\omega(f) \cup M_\omega(f) \cup I_\omega(f)] = 0.$$

Proof. For the Blaschke product $B(z)$ of Theorem 5, almost every point $\zeta \in \Gamma$ is a Fatou point of B with Fatou value of modulus one. By a theorem of Lusin [12, p. 192], this set of Fatou points of B contains a set S of measure 2π such that $S=\bigcup_n S_n$, where $S_1 \subset S_2 \subset \cdots \subset S_n \subset S_{n+1} \subset \cdots \subset \Gamma$ and each S_n is a perfect nowhere dense set.

By essentially the same method as used in [10], it is possible to construct a function $g(z)$ holomorphic in D such that $g(z)$ is bounded on the

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disc $\Omega_{\frac{1}{2}}(\zeta)$ for every point $\zeta \in S$; and for each point $\zeta \in \Gamma$, there exists a sequence $\{z_n\} \subset D$ converging to ζ for which $\mathscr{R} g(z_n) \to +\infty$ and $|B(z_n)| \geqslant \frac{1}{2}$. If we set $f(z) = B(z) e^{g(z)}$, then the latter property of $g(z)$ implies that $\infty \in C(f, \zeta)$ for each point $\zeta \in \Gamma$. The former property of $g(z)$ implies that $f(z)$ is bounded on $\Omega_{\frac{1}{2}}(\zeta)$ for each point $\zeta \in S$. Hence the set $M_\omega(f) \cup I_\omega(f)$ is of measure zero, while the set of Fatou points of f has measure 2π by Plessner's theorem.

Let $\zeta \in \Gamma$ be a point at which $f(z)$ has a non-zero Fatou value and $f(z)$ is bounded on $\Omega_{\frac{1}{2}}(\zeta)$. The set of such points has measure 2π since it contains all points of S. Since the zeros of $B(z)$ are zeros of $f(z)$,

$$0 \in C_{\Omega_r^+(\zeta)}(f, \zeta)\ (0 < r < 1) \text{ and } 0 \in C_{\Omega_r^-(\zeta)}(f, \zeta)\ (0 < r < 1).$$

By the same argument as in Theorem 5, the point ζ cannot be a right horocyclic Fatou point of f. Thus $F_\omega(f)$ has measure zero.

We now indicate how to modify the method in [10] in order to construct the function $g(z)$. For each $n = 1, 2, \cdots$, define

$$G_n = \Big(\bigcup_{\zeta \in S_n} \Omega_{\frac{1}{2}}(\zeta)\Big) \cup \{z : |z| < \rho_n\},$$

where $\frac{1}{2} < \rho_1 < \rho_2 < \cdots < \rho_n < \rho_{n+1} < \cdots < 1$ and $\rho_n \to 1$. Also, for each $n = 1, 2, \cdots$, let Z_n be a finite subset of $D - \overline{G_n}$ chosen as follows:

(1) in each component of $D - \overline{G_1}$ having area in the range $[\pi/2^n, \pi/2^{n-1})$, choose a point z in $D - \overline{G_n}$ at which $|B(z)| \geqslant \frac{1}{2}$ (recall that $B(z)$ has radial limit of modulus one on a dense set of radii);

(2) in each component of $D - \overline{G_2}$ having area in the range $[\pi/2^{n+1}, \pi/2^n)$ choose a point z in $D - \overline{G_n}$ at which $|B(z)| \geqslant \frac{1}{2}$;

$\vdots$

(n) in each component of $D - \overline{G_n}$ having area in the range $[\pi/2^{2n-1}, \pi/2^{2n-2})$ choose a point z at which $|B(z)| \geqslant \frac{1}{2}$.

It is easily proven that the collection $\bigcup_n Z_n$ has Γ as its derived set, so that for each $\zeta \in \Gamma$ there exists a sequence $\{z_{n_k}\}$ converging to ζ where $z_{n_k} \in Z_{n_k}$.

For the function $t(z)$ defined on the sets T_n we substitute the function $\tau(z)$ defined on the sets Z_n by $\tau(z)=n$, $z\in Z_n$, $n=1,2,\cdots$. Also, we define

$$F_n=\overline{G}_n\cup\Big(\bigcup_{1\leqslant j\leqslant n}Z_j\Big),\quad n=1,2,\cdots,$$

so that each F_n is a compact set with connected complement. We obtain by induction a sequence of polynomials $\{p_n(z)\}$ converging (uniformly on compact subsets of D) to a function $g(z)$ holomorphic in D such that $g(z)$ is bounded in G_n, $n=1,2,\cdots$. Since $\Omega_{\frac{1}{2}}(\zeta)$ is a subset of G_n for each $\zeta\in S_n$ $(n=1,2,\cdots)$, $g(z)$ is bounded on $\Omega_{\frac{1}{2}}(\zeta)$ for each $\zeta\in S_n$ $(n=1,2,\cdots)$ as required.

The sequence $\{p_n(z)\}$ also satisfies

$$|p_n(z)-\tau(z)|<2^{-n},\quad z\in\bigcup_{1\leqslant j\leqslant n}Z_j,$$

$$|g(z)-p_n(z)|<2^{-n},\quad z\in D_{\rho_n}.$$

Thus,

$$\lim_{\substack{z\to\zeta\in\Gamma\\ z\in\bigcup_n Z_n}}|g(z)-\tau(z)|=0.$$

Hence for each point $\zeta\in\Gamma$ there exists a sequence $\{z_{n_k}\}$ converging to ζ, $z_{n_k}\in Z_{n_k}$, such that

$$\lim_{k\to\infty}|\mathscr{R}g(z_{n_k})-\tau(z_{n_k})|=\lim_{k\to\infty}|\mathscr{R}g(z_{n_k})-n_k|=0.$$

The function $g(z)$ has the required properties, and the theorem is proved.

To determine the horocyclic behavior of the function $f(z)$ of Theorem 9, we begin with the definition of a normal meromorphic function in the unit disc D due to Noshiro [20].

Definition 3. Let $f(z)$ be a meromorphic function in D. Denote by $z'=L(z)$ an arbitrary one-to-one conformal mapping of D onto itself. The function $f(z)$ is called normal in D if the family of functions $\{f(L(z))\}$ is normal in the sense of Montel, where convergence is defined in terms of the spherical metric.

Lemma (Bagemihl [3, Lemma 4]). *If $f(z)$ is a normal meromorphic function in D and $\zeta\in K_\omega(f)$, then*

$$\Pi_{T_\omega}(f,\zeta) = C_{\mathfrak{A}}(f,\zeta).$$

Remark 8. A meromorphic function assuming each of three values only finitely often in D is normal in D (see [19, pp. 125–123] or [15, p. 54]). If f is meromorphic in D and ζ is a horocyclic Meier point of f, then $C(f,\zeta) \subset W$. Thus f is normal on each disc $\Omega_r(\zeta)$ $(0<r<1)$. From this and the lemma of Bagemihl just cited, one can prove that

$$\Pi_{T_\omega}(f,\zeta) = C(f,\zeta) \subset W$$

at each horocyclic Meier point of a meromorphic function f.

DEFINITION 4. The primary-tangential cluster set of f at ζ is defined to be

$$C_\Omega(f,\zeta) = \overline{\bigcup_{0<r<1} C_{\Omega_r(\zeta)}(f,\zeta)}.$$

The term "primary-tangential" is used to differentiate this cluster set from similar cluster sets wherein tangential approach of higher order is used.

Remark 9. It is evident that

$$C_{\mathfrak{B}}(f,\zeta) \subseteq C_{\mathfrak{A}}(f,\zeta) \subseteq C_\Omega(f,\zeta)$$

for every point $\zeta \in \Gamma$. By Lemma 6,

$$C_\Omega(f,\zeta) \subseteq C_{\mathfrak{B}}(f,\zeta)$$

at almost every point $\zeta \in \Gamma$. Thus, at almost every point $\zeta \in \Gamma$,

$$C_{\mathfrak{A}}(f,\zeta) = C_{\mathfrak{B}}(f,\zeta) = C_\Omega(f,\zeta).$$

DEFINITION 5. A point $\zeta \in \Gamma$ is said to be a primary-tangential pre-Meier point of $f: D \to W$ provided

$$\Pi_{T_\omega}(f,\zeta) = C_\Omega(f,\zeta) \subset W.$$

The term "pre-Meier" is used because the condition

$$C_{h_r^-}(f,\zeta) = C_{h_{r'}^+}(f,\zeta) \subset W \ (0<r<1;\ 0<r'<1)$$

is fulfilled at each primary-tangential pre-Meier point of f, and this is a necessary condition that a point $\zeta \in \Gamma$ be a horocyclic Meier point of f. If it is also true that $C_\Omega(f,\zeta) = C(f,\zeta) \subset W$, then the point ζ is in fact a horocyclic Meier point of f.

Each horocyclic Meier point of a function f meromorphic in D is a primary-tangential pre-Meier point of f because of Remark 8. An example can be easily constructed to show that the word "meromorphic" cannot be omitted.

Although a horocyclic analogue of Fatou's theorem does not exist, we can prove

THEOREM 10. *If $f(z)$ is a normal meromorphic function in D, then almost every point $\zeta \in \Gamma$ is either a primary-tangential pre-Meier point of f or a point at which $\Pi_{T_\omega}(f, \zeta) = W$.*

Proof. By Remark 9, $C_{\mathfrak{A}}(f, \zeta) = C_{\Omega}(f, \zeta)$ almost everywhere on Γ. Since $K_\omega(f)$ is of measure 2π, Bagemihl's lemma implies that

$$\Pi_{T_\omega}(f, \zeta) = C_{\Omega}(f, \zeta)$$

for almost every point $\zeta \in \Gamma$. The theorem now follows from the fact that at every point $\zeta \in \Gamma$, either $C_{\Omega}(f, \zeta) \subset W$ or $C_{\Omega}(f, \zeta) = W$.

Applying Theorem 10 to the holomorphic bounded function $B(z)$ in Theorem 5, we see that the set of primary-tangential pre-Meier points of B has measure 2π and the set of horocyclic Fatou points of B has measure zero.

Although a horocyclic analogue of Plessner's theorem does not exist, we can prove

THEOREM 11. *If $f(z)$ is meromorphic in D, then almost every point $\zeta \in \Gamma$ is either a primary-tangential pre-Meier point of f or a horocyclic Plessner point of f.*

Proof. At a point $\zeta \in \Gamma - I_\omega(f)$, $C_{\mathfrak{B}}(f, \zeta) \subset W$. By Theorem 3 and Remark 9, for almost every point $\zeta \in \Gamma - I_\omega(f)$,

$$\zeta \in K_\omega(f) \text{ and } C_{\mathfrak{B}}(f, \zeta) = C_{\mathfrak{B}}(f, \zeta) = C_{\Omega}(f, \zeta) \subset W. \tag{15}$$

Let the point $\zeta \in \Gamma - I_\omega(f)$ satisfy (15), and let Λ_ω be an admissible tangential arc at ζ. Then there exists a disc $\Omega_{r_0}(\zeta)$ at ζ containing Λ_ω. Since $C_{\Omega}(f, \zeta) \subset W$, $f^*(z)$, the restriction of $f(z)$ to $\Omega_{r_0}(\zeta)$, is a normal meromorphic function in $\Omega_{r_0}(\zeta)$ by Remark 8. Furthermore, $\zeta \in K_\omega(f)$ implies that $\zeta \in K_\omega(f^*)$, where the meaning of $K_\omega(f^*)$ is the natural one. Bagemihl's lemma applied to the function $f^*(z)$ implies that

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$$C_{\Lambda_\omega}(f,\zeta) = C_{\Lambda_\omega}(f^*,\zeta) = C_{\Omega_{r_0}(\zeta)}(f^*,\zeta) = C_{\Omega_{r_0}(\zeta)}(f,\zeta) = C_{\Omega}(f,\zeta),$$

where the last equality follows because $C_{\mathfrak{B}}(f,\zeta) = C_{\Omega}(f,\zeta)$. Since Λ_ω was an arbitrary admissible tangential arc, $\Pi_{T_\omega}(f,\zeta) = C_{\Omega}(f,\zeta)$. Thus almost every point $\zeta \in \Gamma - I_\omega(f)$ is a primary-tangential pre-Meier point of f, and the theorem is proved.

Theorem 11 implies that for the function $f(z)$ in Theorem 9 almost every point $\zeta \in \Gamma$ is a primary-tangential pre-Meier point of f, but meas $[F_\omega(f) \cup M_\omega(f) \cup I_\omega(f)] = 0$.

Since no primary-tangential pre-Meier point of a function is a Plessner point of the function, Plessner's theorem implies that almost every primary-tangential pre-Meier point of a meromorphic function $f(z)$ is a Fatou point of $f(z)$. Since meas$[F(f) \cap I_\omega(f)] = 2\pi$ for the function $f(z)$ of Theorem 6, the converse is not true.

Finally we point out that for a meromorphic function $f(z)$ almost every point of $F_\omega(f) \cup M_\omega(f)$ is a primary-tangential pre-Meier point of f. This follows from Theorem 11 and the fact that no point of $F_\omega(f) \cup M_\omega(f)$ is a point of $I_\omega(f)$. The function $f(z)$ in Theorem 9 shows that the converse need not be true.

5. The set $F(f) \cap M_\omega(f)$.

In the proof of our final theorem, we shall need

Remark 10. Let $c \subset D$ be the arc of a circle C orthogonal to Γ (i.e. $c = D \cap C$), and let $\zeta \in \Gamma$ be interior to C. Then, under inversion in c, the image of that part of each disc $\Omega_r(\zeta)$ $(0 < r < 1)$ which lies exterior to C again lies in $\Omega_r(\zeta)$.

Proof. Let $L(z) = i\dfrac{\zeta + z}{\zeta - z}$. Then $L(z)$ maps $h_r(\zeta)$ onto a straight line parallel to the real axis and c onto a semi-circle $L(c)$ with diameter on the real axis. The inversion in c corresponds to inversion in $L(c)$, and the assertion is evident.

THEOREM 12. *There exists a function $f(z)$ holomorphic and bounded in D such that almost every point $\zeta \in \Gamma$ is a horocyclic Meier point of f, while the set of Meier points of f has measure zero.*

Proof. We shall prove that the function $f(z)$ constructed by Jenkins in [13] has the required properties.

Let d be the domain obtained from the unit disc $|w| < 1$ by inserting at each point $e^{i(m/n)\pi}$ a radial slit of length $1/\sqrt{n}$ where m, n are integers, $n > 0$, $|m| \leqslant n$, and the fraction m/n is in its lowest terms.

We obtain from the domain d a Riemann surface R by the following construction. For each slit s_j $(j = 1, 2, \cdots)$ let d_j be a domain obtained from d by reflection in the diameter bearing s_j. Then we cross-join d_j to d along s_j and the corresponding slit on d_j. For each d_j, let the remaining boundary slits of d_j be denoted by s_{jk} $(k = 1, 2, \cdots; k \neq j)$, where s_{jk} corresponds to s_k. For each d_j and each slit s_{jk} on d_j, let the domain d_{jk} be obtained from d_j by reflection in the diameter bearing s_{jk}. We cross-join d_{jk} to d_j along s_{jk} and the corresponding slit on d_{jk} for each admissible value of k. For each d_{jk}, let the remaining boundary slits of d_{jk} be denoted by s_{jkl} $(l = 1, 2, \cdots; l \neq k, l \neq j)$, where s_{jkl} corresponds to s_{jk}.

Continuing this process, we obtain a Riemann surface R which has no relative boundary over $|w| < 1$. Evidently R is simply connected and of hyperbolic type so that there exists a function $w = f(z)$ which maps D in a one-to-one conformal manner onto the surface R. We assume that f carries the origin $z = 0$ onto the point of d covering the origin $w = 0$.

The surface is invariant under the following transformations. Let d' and d'' be two sheets of R cross-joined along the slit s. Select any point p' in d', and let p'_w denote the point in $|w| < 1$ covered by p'. Let p''_w denote the point in $|w| < 1$ obtained from p'_w by reflection in the diameter which contains the radial segment covered by s. With p' we associate the point p'' in d'' which covers p''_w. Under such an association d' is transformed into d'' and conversely, while the slit s is fixed. Any sheet d^* attached to d' is transformed into a sense-reversed (with respect to the diameter bearing the slit along which it is cross-joined to d') replica of itself attached to d'', and any sheet d^{**} attached to d'' is transformed into a sense-reversed (with respect to the diameter bearing the slit along which it is cross-joined to d'') replica of itself attached to d', etc. We may extend such a mapping to the points on the cross-joins by continuity to obtain, for each choice of d', d'' and s, a transformation which leaves R invariant. Note that the slit s is the only pointwise fixed subset of R.

Each corresponding transformation in D is an anti-conformal transformation of D onto itself, and thus must be the conjugate of a linear transformation. Since each transformation on R fixes pointwise a slit s, the transformation in D fixes pointwise an arc in D with its endpoints on Γ. The conjugate of a linear transformation carrying D onto itself can leave such an arc pointwise fixed only if the arc lies on a circle orthogonal to Γ and the mapping in question is inversion in that circle.

We can now give a geometric description of $f(z)$. In the mapping $f(z)$ of D onto R, the subset of D mapped onto the initial sheet d of R is a subdomain δ of D bounded by a countable set of open arcs c_j $(j=1,2,\cdots)$ on circles orthogonal to Γ (one for each slit s_j; $j=1,2,\cdots$) together with a set H on Γ. Since the length of an arc (in D) of a circle orthogonal to Γ is for a suitable constant, say K^*, less than K^* times the length of the arc on Γ which the circle intercepts, the boundary of δ is a rectifiable Jordan curve. If Φ denotes a one-to-one conformal mapping of the disc $|Z|<1$ onto d, then $f^{-1}(\Phi(z))$ maps $|Z|<1$ in a one-to-one conformal manner onto δ. The boundary of d consists of Γ_w: $|w|=1$ and the enumerable collection of slits $s_1, s_2, \cdots$. Due to the choice of the lengths of the slits $s_1, s_2, \cdots$, no Stolz triangle with a vertex on Γ_w can be completely contained in d. According to a theorem of Lavrentieff [14, Theorem 1], the set of points on $|Z|=1$ mapped onto Γ_w by Φ, say E, must be of measure zero. Since the domain δ has a rectifiable boundary and H is the image under $f^{-1}(\Phi(z))$ of the set E of measure zero, H is of measure zero by the Riesz theorem [24, p. 49].

The function $f(z)$ defined on D can be thought of as the continuation of $f(z)$ defined on δ. If we reflect δ in each of the arcs c_j $(j=1,2,\cdots)$ and continue this process, we sweep out the domain D while the corresponding transformations on R completely cover R as the image of d. The images of H under these successive inversions have measure zero. Thus, their enumerable union K has measure zero.

We shall show that $C_\Omega(f,\zeta)=\{w\colon |w|\leqslant 1\}$ for each point $\zeta\in\Gamma-K$. Then, since $|f(z)|\leqslant 1$, $C(f,\zeta)=\{w\colon |w|\leqslant 1\}$ for each point $\zeta\in\Gamma-K$ (and hence for each point $\zeta\in\Gamma$). Since f has a radial limit almost everywhere, the set of Meier points of f is of measure zero. By Theorem 10, $\Pi_{T_\omega}(f,\zeta)$ $=C_\Omega(f,\zeta)$ for almost every point $\zeta\in\Gamma$, so that

$$C(f,\zeta)=C_\Omega(f,\zeta)=\Pi_{T_\omega}(f,\zeta)\subseteq\Pi_\omega(f,\zeta)$$

for almost every point $\zeta\in\Gamma-K$. Thus $\Pi_\omega(f,\zeta)=C(f,\zeta)$ for almost every

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point $\zeta \in \Gamma$, and the set of horocyclic Meier points of f is of measure 2π as asserted.

If $\zeta \in \Gamma - K$, then ζ is not an endpoint of any arc c_j $(j = 1, 2, \cdots)$ nor is ζ an endpoint of the reflection of any such arc. So there exists a sequence c_j, c_{jk}, c_{jkl}, $\cdots$ of arcs on circles orthogonal to Γ such that ζ lies interior to each such circle. These arcs correspond under f to cross-joins s_j, s_{jk}, s_{jkl}, $\cdots$ on R, where d and d_j are cross-joined along s_j, etc. Also, if $\delta_j \subset D$ is the domain obtained from δ by reflection in c_j, then f carries δ_j onto d_j, etc.

Now if $C_\Omega(f, \zeta) \neq \{w: |w| \leqslant 1\}$, then there exists a point w_0, $|w_0| < 1$, and a closed neighborhood $N(w_0)$ of w_0 contained in $\{w: |w| \leqslant 1\}$ such that $N(w_0)$ has area $\eta > 0$ and

$$N(w_0) \cap C_\Omega(f, \zeta) = \phi.$$

Since $f(\delta) = d$, we can choose the disc $\Omega_r(\zeta)$ so large that

$$\text{area}\,[f(\delta \cap \Omega_r(\zeta))] > \pi - \eta/2.$$

Hence, we must have

$$f(\delta \cap \Omega_r(\zeta)) \cap N(w_0) \neq \phi.$$

Now let $\delta_j^* \subset \delta_j$ be the reflection of $\delta \cap \Omega_r(\zeta)$ in c_j. Then $f(\delta_j^*) \subset f(\delta_j) = d_j$. As previously stated, f in δ_j is the continuation of f in δ by reflection in the arc c_j. The corresponding transformation on R between d and d_j preserves area so that, since $f(\delta_j^*)$ is the image of $f(\delta \cap \Omega_r(\zeta))$ under this transformation on R,

$$\text{area}\, f(\delta_j^*) = \text{area}\, f(\delta \cap \Omega_r(\zeta)).$$

Now $\delta_j^* \subset \delta_j$ and by Remark 10, $\delta_j^* \subset \Omega_r(\zeta)$. Thus, $\delta_j^* \subset \delta_j \cap \Omega_r(\zeta)$, so that

$$\text{area}\, f(\delta_j \cap \Omega_r(\zeta)) > \text{area}\, f(\delta_j^*) = \text{area}\, f(\delta \cap \Omega_r(\zeta)) > \pi - \eta/2.$$

Thus

$$f(\delta_j \cap \Omega_r(\zeta)) \cap N(w_0) \neq \phi.$$

Proceeding in this fashion we obtain the sequence of domains

$$\delta \cap \Omega_r(\zeta),\ \delta_j \cap \Omega_r(\zeta),\ \delta_{jk} \cap \Omega_r(\zeta),\ \cdots$$

which converges to ζ, while the image under f of each such domain intersects $N(w_0)$. Since $N(w_0)$ is closed and bounded, there exists a point in $N(w_0)$ which belongs to $C_{\Omega_r(\zeta)}(f, \zeta)$. Thus,

$$C_\Omega(f, \zeta) \cap N(w_0) \neq \phi,$$

which contradicts our assumption that this intersection is empty. This completes the proof of the theorem.

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