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ON THE DISTRIBUTION OF THE SEQUENCE $\{nd^*(n)\}$

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ABSTRACT. Let $d^*(n)$ denote the number of unitary divisors of the positive integer n. For $x > 1$, let $B(x)$ denote the number of integers n for which $nd^*(n) \leqq x$. Balasubramanian and Ramachandra proved that there exists a positive constant β such that $B(x) = (\beta + o(1))x/\sqrt{\log x}$. In this note we give an explicit expression for β as an infinite product, namely $\beta = 1/\sqrt{\pi}\prod_p(p-1/2) \big/ \sqrt{p(p-1)} = 0.6189\ldots$, where the product is over all primes p.

Let $d(n)$ denote the number of divisors and $d^*(n)$ the number of unitary divisors of the positive integer n. For $x > 1$, let $A(x)$ denote the number of integers n for which $nd(n) \leqq x$ and let $B(x)$ denote the number of integers n for which $nd^*(n) \leqq x$. Since the value of $nd^*(n)$ determines n, we could define $B(x)$ as the number of integers of the form $nd^*(n)$ not exceeding x. We proved (unpublished) that there exist positive constants c_1 and c_2 such that

$$c_1 x/\sqrt{\log x} < A(x) < c_2 x/\sqrt{\log x}. \tag{1}$$

Balasubramanian and Ramachandra [1] proved that there exist positive constants α and β such that as, $x \longrightarrow \infty$,

$$A(x) = (\alpha + o(1))x/\sqrt{\log x} \text{ and } B(x) = (\beta + o(1))x/\sqrt{\log x}. \tag{2}$$

In fact, they prove a general theorem of which (2) is a special case. However, they do not determine explicitly the values of α and β. The object of this note is to give an expression for β as an infinite product, namely,

$$\beta = \frac{1}{\sqrt{\pi}}\prod_p \frac{p-1/2}{\sqrt{p(p-1)}} = 0.6189\ldots. \tag{3}$$

At the same time we give a proof of the second equation in (2) which is along different lines than the proof in [1]. In what follows all o- and O- estimates refer to $x \longrightarrow \infty$.

Our argument uses the following classical result of Sathe [2], which we formulate as a lemma.

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LEMMA. *Let $\omega(n)$ denote the number of distinct prime divisors of n and let $\rho_m(x)$ denote the number of integers $n \leqq x$ for which $\omega(n) = m$. For $c \geqq 0$ let*

$$f(c) = \frac{1}{\Gamma(c+1)} \prod_p (1 - 1/p)^c (1 + c/p)(1 + c/(p+c)(p-1)).$$

Then, for $0 \leqq c \leqq e$ and $m = (c + o(1)) \log\log x$, the following estimate holds:

$$\rho_m(x) = (f(c) + o(1)) \frac{x(\log\log x)^{m-1}}{(m-1)! \log x}.$$

We shall prove that

$$\beta = f(1/2)/2. \tag{4}$$

It is easy to check that (3) follows from (4). Observe first that if h denotes the number of integers n such that $nd^*(n) \leqq x$ and $d^*(n) > (\log x)^{\log 2}$, and if n' is the largest such integer, then $x \geqq n'd^*(n') > h(\log x)^{\log 2}$, so that $h < x/(\log x)^{\log 2} = o(x/\sqrt{\log x})$. Thus in order to prove (4), we need only consider those n for which $d^*(n) \leqq (\log x)^{\log 2}$, or, since $d^*(n) = 2^{\omega(n)}$, those n for which $\omega(n) \leqq \log\log x$.

Let $I = [0, \log\log x]$, and for $k \in I$, let $B_k^x = \{n : nd^*(n) \leqq x, \omega(n) = k + 1\}$. By the observations in the preceding paragraph, we need to show

$$\sum_{k \in I} |B_k^x| = (f(1/2)/2 + o(1))x/\sqrt{\log x}. \tag{5}$$

For $k \in I$, define c_k by $k = c_k \log\log x$. Then by the lemma,

$$\sum_{k \in I} |B_k^x| = (f(1/2)/2) + o(1)) \frac{x}{\log x} S$$

where

$$S = \sum_{k \in I} f(c_k) \frac{(\log\log x)^k}{2^k k!}$$

In order to prove (5) we need to show that

$$S = (f(1/2) + o(1))\sqrt{\log x}. \tag{6}$$

Let M, H, N, L and T be defined as follows:

$$\begin{aligned} M &= 1/2 \log\log x \\ H &= \sqrt{\log\log x \log\log\log x} \\ N &= M - H \\ L &= M + H \\ T &= \sqrt{\log\log\log x} \end{aligned} \tag{7}$$

Write $S = S_1 + S_2 + S_3$, where the intervals of summation for S_1, S_2 and S_3 are respectively, $[0, N]$, (N, L) and $[L, 2M]$.

We estimate S_1 as follows: Since f is continuous and therefore bounded on [0, 1] and since the largest term in the sum is the last one, we have

$$\begin{aligned} S_1 &\ll M(2M)^N/2^N[N]! \\ &\ll M^{N+1}/(N/e)^N N^{1/2}, \text{ by Stirling's formula,} \\ &\ll M^{1/2}(1-H/M)^{H-M}e^{M-H} \\ &\ll \sqrt{M\log x}\exp(-(1+o(1))H^2/M) \\ &\ll \sqrt{\log x}M^{-3/2+o(1)} \\ &= o(\sqrt{\log x}), \end{aligned}$$

where, in making the estimations, we used (7) in various places.

A similar argument shows that $S_3 = o\sqrt{\log x})$, since the largest term in S_3 is the first one and the number of terms is at most M.

It remains to estimate S_2. By Stirling's formula and the fact that for k in (N, L) the estimates $k = (1+o(1))M$ and $c_k = 1/2 + o(1)$ hold, we have

$$S_2 = \frac{f(1/2)+o(1)}{\sqrt{2\pi M}} \sum_{N<k<L} (eM/k)^k. \tag{8}$$

Define r_t by $r_t = t - M$. We may write

$$(eM/k)^k = \sqrt{\log x}e^{r_k}G, \text{ where } G = (1+r_k/M)^{-k}.$$

Then

$$\log G = k\sum_{\nu=1}^{\infty}\frac{(-1)^\nu}{\nu}(r_k/M)^\nu = -r_k - r_k^2/2M$$
$$+ O(\log\log\log x)^{3/2}/(\log\log x)^{1/2},$$

so that

$$G = (1+o(1))\exp(-r_k - r_k^2/2M).$$

It follows that

$$(eM/k)^k = (1+o(1))\sqrt{\log x}g(k)$$

where

$$g(t) = \exp(-r_t^2/2M) = \exp(-(t-M)^2/2M).$$

Thus (8) may be written as

$$S_2 = f(1/2)+o(1))\sqrt{(\log x/2\pi M}\sum_{N<k<L} g(k). \tag{9}$$

From the Euler-Maclaurin summation formula, and the fact that the summands in (9) are symmetrical about M, we get

$$\sum_{N<k<M} g(k) = g(N) + g(M) + 2\int_N^M g(t)dt + 2\int_N^M g'(t)(t-[t]-1/2)dt. \tag{10}$$

The substitution $z\sqrt{2M} = t - M$ gives ($T = \sqrt{\log\log\log x}$)

$$2\int_N^M g(t)dt = 2\sqrt{2M}\int_{-T}^0 \exp(-z^2)dz = (1+o(1)\sqrt{2\pi M}.$$

The second integral in (10) is $O(\sqrt{\log\log\log x})$ and $g(N)$ and $g(M)$ are bounded independently of x. It now follows from (8) that (6) and hence also (4) holds.

We have not been able to determine the value of α by the method of this note.

References

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