

## TRANSVERSALS, INDIVISIBLE PLEXES AND PARTITIONS OF LATIN SQUARES

JUDITH EGAN

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A  $k$ -plex in a latin square of order  $n$  is a selection of  $kn$  entries in which each row, column and symbol occurs exactly  $k$  times. A 1-plex is also called a transversal. A  $k$ -plex is indivisible if it contains no  $c$ -plex for  $0 < c < k$ . Two or more plexes are parallel if no two of them share an entry. This thesis is the first major study giving general results on indivisible  $k$ -plexes for  $k > 1$ .

For  $n \notin \{2, 6\}$ , the existence of a latin square of order  $n$  with a partition into 1-plexes was shown by Bose *et al.* [1]. A natural extension of the result to  $k$ -plexes is the following. We prove that if  $k$  is a proper divisor of  $n$ , then there exists a latin square of order  $n$  composed of parallel indivisible  $k$ -plexes.

Define  $\kappa(n)$  to be the largest integer  $k$  such that some latin square of order  $n$  contains an indivisible  $k$ -plex. A conjecture by Rodney (see [3, p. 143]) says that every latin square contains a 2-plex, which implies that  $\kappa(n) < n$  for all  $n > 2$ . We show that for all  $n > 2$ , there exists a latin square of order  $n$  with two parallel indivisible  $\lfloor n/2 \rfloor$ -plexes. A corollary is that  $\kappa(n) \geq \lfloor n/2 \rfloor$  for all  $n > 2$ . We report on extensive computations of the indivisible partitions of the latin squares of order  $n \leq 9$ . Up to order eight we count the number of indivisible partitions of every type.

Due to Bose *et al.* [1], and Finney [10] for  $n = 6$ , we know that there exists a latin square for each order  $n > 2$  which possesses a  $k$ -plex for all  $0 \leq k \leq n$ . Wanless [12] showed that there are latin squares of every even order which contain no odd plexes, where an odd plex is a  $k$ -plex such that  $k$  is odd. It has been conjectured by Rodney (see [4]), and by Wanless [12], that every latin square has a set of  $\lfloor n/2 \rfloor$  parallel 2-plexes which implies that every latin square of odd order has a  $k$ -plex for each

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$0 \leq k \leq n$ . We show that among the latin squares of even order there are many other possibilities concerning the existence of odd  $k$ -plexes. We prove that for all even  $n > 2$ , there exists a latin square of order  $n$  which has no  $k$ -plex for any odd  $k < \lfloor n/4 \rfloor$ , but does have a  $k$ -plex for every other  $k \leq \frac{1}{2}n$ .

A result by Wanless and Webb [13] is that, for all  $n > 3$ , there exists a latin square of order  $n$  with at least one entry not in any transversal. Such a latin square is called a confirmed bachelor and illustrates a restriction on the transversals in a latin square. In our study of transversals we consider bachelor latin squares in greater detail and other types of restrictions that might occur. We give a concise alternative proof of the result by Wanless and Webb. A main result is that for all even  $n \geq 10$ , except perhaps if  $n$  is a power of 2, there exists a latin square of order  $n$  that possesses a transversal, but every transversal coincides on a single entry. A theorem by Wanless, which was prompted by our data, shows that there exist arbitrarily large latin squares of odd order in which the proportion of entries not in a transversal is asymptotic to one ninth. We report on computations of parallel transversals in the latin squares of order nine. Thus we prove that the above mentioned conjectures by Rodney and Wanless are true for order nine, and that every latin square of order nine has at least three parallel transversals. Our computations suggest that the constructions which prove our theorems illustrate rare behaviour in large latin squares. The computations also give further evidence in support of a conjecture by Ryser [11] (see [3, p. 143]) that every latin square of odd order has a transversal.

Many of the theorems in this thesis rely on a simple but powerful lemma [7, 9]. The lemma gives a necessary condition for the existence of  $k$ -plexes. To show the existence of  $k$ -plexes we use constructive techniques. It remains open to establish a sufficient condition for the existence of  $k$ -plexes in latin squares.

The main results of this thesis can be found in [2, 5–8].

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JUDITH EGAN, School of Mathematical Sciences, Monash University,  
Victoria 3800, Australia  
e-mail: [judith.egan@monash.edu](mailto:judith.egan@monash.edu)