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An Isospectral Deformation on an Infranil-Orbifold

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Abstract. We construct a Laplace isospectral deformation of metrics on an orbifold quotient of a nilmanifold. Each orbifold in the deformation contains singular points with order two isotropy. Isospectrality is obtained by modifying a generalization of Sunada's theorem due to DeTurck and Gordon.

1 Introduction

A Riemannian orbifold (see [11, 12]) is a mildly singular generalization of a Riemannian manifold. For example, the quotient space of a Riemannian manifold under an isometric, properly discontinuous group action is a Riemannian orbifold [16]. First defined in 1956 by I. Satake, orbifolds have proven useful in many settings including the theory of 3-manifolds, symplectic geometry, and string theory.

The local structure of a Riemannian orbifold is given by the orbit space of a Riemannian manifold under the isometric action of a finite group. If a point p in the manifold is fixed under a nontrivial group action, the corresponding element of the orbit space $\bar{p}$ is called a *singular point* of the orbifold. The *isotropy type* of a point $\bar{p}$ in the orbit space is the isomorphism class of the isotropy group of a point p in the manifold that projects to $\bar{p}$ under the quotient. The *singular set* of an orbifold is the set of all singular points of the orbifold.

The tools of spectral geometry can be transferred to the setting of Riemannian orbifolds by exploiting the well-behaved local structure of these spaces (see [3, 14]). Given a smooth function f on an orbifold O, the Laplacian of f is computed by taking the Laplacian of lifts of f in the orbifold's local coverings. As in the manifold setting, the eigenvalue spectrum of the Laplace operator of a compact Riemannian orbifold is a sequence $0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty$ where each eigenvalue has finite multiplicity. We say that two orbifolds are *isospectral* if their Laplace spectra agree.

In this note we show that the formulation of Sunada's Theorem found in [4] can be used to obtain isospectral deformations on Riemannian orbifolds with nontrivial singular sets. We prove this fact in Section 2 by observing that the proof of Theorem 2.7 in [4] does not require that the action of the discrete subgroup Γ be free. In Section 3 we display an example of an isospectral deformation of metrics on an orbifold quotient of a nilmanifold.

The only other known examples of non-manifold isospectral deformations on orbifolds were recently obtained by Sutton using a blend of the torus action method

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and the Sunada technique [15]. Other examples of non-manifold isospectral orbifolds include pairs with boundary in [1] and [2]; isospectral flat 2-orbifolds that are not conjugate (in terms of lengths of closed geodesics) [6]; a $(2m)$-manifold isospectral to a $(2m)$-orbifold on m-forms [7]; pairs of isospectral orbifolds for which the maximal isotropy groups have different orders [10]; and arbitrarily large finite families of isospectral orbifolds [13].

2 Isospectral Deformations on Orbifolds

In this section we observe that the generalization of Sunada's method found in [4] can be further generalized to include isospectral deformations of metrics on orbifolds. In what follows we will assume that G is a Lie group with simply connected identity component G_0. We let Γ be a discrete subgroup of G such that $G = \Gamma G_0$ and $(G_0 \cap \Gamma)\backslash G_0$ is compact.

Given an automorphism $\Phi\colon G \to G$, we say that Φ is an *almost-inner automorphism* if, for each $x \in G$, there exists an element $a \in G$ such that $\Phi(x) = axa^{-1}$. More generally, if $\Phi\colon G \to G$ is an automorphism such that for each $\gamma \in \Gamma$ there exists $a \in G$ satisfying $\Phi(\gamma) = a\gamma a^{-1}$, we say that Φ is an *almost-inner automorphism of G relative to* Γ. We denote the set of all almost-inner automorphisms of G (resp. almost-inner automorphisms of G with respect to Γ) by $\mathrm{AIA}(G)$ (resp. $\mathrm{AIA}(G;\Gamma)$).

We have the following theorem.

Theorem 2.1 ([4]) *Let G, G_0, and Γ be as above with G_0 nilpotent, and let $\Phi \in \mathrm{AIA}(G;\Gamma)$. Suppose that G acts effectively and properly discontinuously on the left by isometries on a Riemannian manifold (M, g) and that Γ acts freely on M with $\Gamma\backslash M$ compact. Then, letting g denote the submersion metric, $(\Phi(\Gamma)\backslash M, g)$ is isospectral to $(\Gamma\backslash M, g)$.*

The proof of Theorem 2.1 is based on work by Donnelly in [5] concerning the existence of a heat kernel on a manifold M that admits a properly discontinuous (but not necessarily free) action by a group Γ. Donnelly shows that if $\Gamma\backslash M$ is compact, then there exists a unique heat kernel on M. Furthermore, Donnelly gives the following relationship between the heat kernels on M and on $\Gamma\backslash M$.

Theorem 2.2 ([5]) *Let Γ act properly discontinuously on M with compact quotient $\overline{M} = \Gamma\backslash M$. Suppose that F is a fundamental domain for $\Gamma\backslash M$. If $\bar{x}, \bar{y} \in \overline{M}$, then set*

$$\overline{E}(t, \bar{x}, \bar{y}) = \sum_{\gamma\in\Gamma} E(t, x, \gamma \cdot y),$$

where $x, y \in F$, $\bar{x} = \pi(x)$, and $\bar{y} = \pi(y)$. If E is the heat kernel of M, the sum on the right converges uniformly on $[t_1, t_2] \times F \times F$, $0 < t_1 \le t_2$, to the heat kernel on $\overline{M}$.

Notice that since the action of Γ need not be free, the quotient space $\overline{M}$ may not be a manifold.

Theorem 2.1 relies on the fact that two manifolds (M_1, g_1) and (M_2, g_2) are isospectral if and only if they have the same heat trace, *i.e.*,

$$\int_{M_1} E_1(t, x, x)\, dx = \int_{M_2} E_2(t, x, x)\, dx,$$

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where E_i denotes the heat kernel on M_i. In particular, the proof uses Theorem 2.2 to pull the heat trace back from the quotient $\Gamma\backslash M$ to the cover M in order to use combinatorial arguments to reexpress the heat trace on $\Gamma\backslash M$. The new expression of the heat trace makes it evident that, when comparing the heat trace of $(\Gamma\backslash M, g)$ with the heat trace of $(\Phi(\Gamma)\backslash M, g)$, if certain volumes (which depend only on Γ and $\Phi(\Gamma)$) are equal, then the respective heat traces are equal. DeTurck and Gordon show that when Φ is an almost-inner automorphism, these volumes are in fact equal, and hence $(\Gamma\backslash M, g)$ and $(\Phi(\Gamma)\backslash M, g)$ are isospectral.

We note that, as with Theorem 2.2, the proof of Theorem 2.1 does not rely on the freeness of the action of Γ on M. Therefore we make the following generalization of Sunada's theorem.

Theorem 2.3 *Suppose that G, G_0, and Γ are as above and G_0 is nilpotent. Suppose that G acts effectively and properly discontinuously on the left by isometries on (M, g) with $\Gamma\backslash M$ compact. Let $\Phi \in \mathrm{AIA}(G; \Gamma)$. Then, letting g denote the submersion metric, the quotient orbifolds $(\Gamma\backslash M, g)$ and $(\Phi(\Gamma)\backslash M, g)$ are isospectral.*

3 Examples

Now we apply Theorem 2.3 to give an example of a nontrivial isospectral deformation on an orbifold. We first note the following.

Lemma 3.1 *Suppose that G is a Lie group and that Γ is a uniform discrete subgroup of G. Suppose that G acts on M on the left by isometries. If Φ is an automorphism of G and G acts on M in such a way that there exists a diffeomorphism Ψ of M satisfying $\Psi(a \cdot x) = \Phi(a) \cdot \Psi(x)$ for all $a \in G$ and $x \in M$, then $(\Gamma\backslash M, \Psi^* g)$ is isometric to $(\Phi(\Gamma)\backslash M, g)$.*

Proof First, notice that if g is a metric on M and $\Psi\colon M \to M$ is a diffeomorphism, then, by design, $\Psi\colon (M, \Psi^* g) \to (M, g)$ is an isometry. Furthermore, if G acts on (M, g) by isometries, then $\Phi(\Gamma)$, which is a subgroup of G, also acts on (M, g) by isometries. Since $\Psi(a \cdot x) = \Phi(a) \cdot \Psi(x)$ for all $a \in G$ and $x \in M$, Γ acts on $(M, \Psi^* g)$ by isometries. Thus we may consider the Riemannian manifolds $(\Phi(\Gamma)\backslash M, g)$ and $(\Gamma\backslash M, \Psi^* g)$, where g and $\Psi^* g$ denote submersion metrics.

Consider the map $\bar{\Psi}\colon (\Gamma\backslash M, \Psi^* g) \to (\Phi(\Gamma)\backslash M, g)$ given by

$$\bar{\Psi}(\bar{p}) = \pi_{\Phi(\Gamma)} \circ \Psi \circ \pi_\Gamma^{-1}(\bar{p}),$$

where $\pi_{\Phi(\Gamma)}$ and π_Γ denote the natural projection maps. Since $\Psi(a \cdot x) = \Phi(a) \cdot \Psi(x)$ for all $a \in G$ and $x \in M$, this map is well defined and bijective. By the definitions of the submersion metric and pullback metric, $\bar{\Psi}$ is an isometry. ■

Applying Theorem 2.3 in conjunction with Lemma 3.1 will allow us to produce an isospectral deformation on a fixed orbifold $\Gamma\backslash M$. Theorem 2.3 gives isospectral metrics on two distinct orbifolds $\Gamma\backslash M$ and $\Phi(\Gamma)\backslash M$. We will ultimately use Lemma 3.1 to convert to a pair of isospectral metrics on a fixed orbifold, $\Gamma\backslash M$.

In [4, Appendix B], K. B. Lee translates Theorem 2.1 to the setting of infranilmanifolds. For a group G we have that $\mathrm{Aut}(G) \ltimes G$ acts on G by $(\phi, g) \cdot h = g\phi(h)$.

Consider the case when G is a simply connected nilpotent Lie group and Γ is a uniform discrete subgroup of G. Take Π to be a finite extension of Γ in $\mathrm{Aut}(G) \ltimes G$. If the action of Π on G is free, then $\Pi\backslash G$ is an infranilmanifold. Lee observes that by setting Γ, G_0, and G from Theorem 2.1 equal to Π, G, and ΠG, and assuming that the action of Π on G is free, we can find isospectral deformations on infranilmanifolds. We note that a priori, the action of Π on G need not be free. Thus by working in this setting we introduce the possibility of finding isospectral orbifold quotients of G.

Lee gives a specific example to illustrate his case. His example is based on a similar example found in [8].

Let G be the Lie group $\{(x_1, x_2, y_1, y_2, z_1, z_2) | x_i, y_i, z_i \in \mathbb{R}\}$, where group multiplication is defined by

$$(x_1, \dots, z_2)(x_1', \dots z_2') \\ = (x_1 + x_1', \dots, y_2 + y_2', z_1 + z_1' + x_1 y_1' + x_2 y_2', z_2 + z_2' + x_1 y_2').$$

Suppose that Γ is the integer lattice in G and define $\Phi_t : G \to G$ by

$$\Phi_t(x_1, x_2, y_1, y_2, z_1, z_2) = (x_1, x_2, y_1, y_2, z_1, z_2 + t y_2),$$

where $t \in [0, 1)$. In the original example Gordon and Wilson show that each Φ_t is an almost-inner automorphism so, applying Lemma 3.1 (with $\Psi = \Phi_t$), the family Φ_t, $t \in [0, 1)$, gives rise to an isospectral deformation on $\Gamma\backslash G$. They also show that the deformation is nontrivial.

In his example, Lee defines $\alpha \in \mathrm{Aut}(G) \ltimes G$ by

$$\alpha(x_1, x_2, y_1, y_2, z_1, z_2) = (x_1, x_2, -y_1, -y_2, -z_1, -z_2 + \tfrac{1}{2})$$

and lets $\Pi = \Gamma \cup \alpha\Gamma$. Since α commutes with Φ_t for all t, we can extend each Φ_t to an element $\tilde{\Phi}_t$ of $\mathrm{AIA}(\Pi G; \Pi)$. If g is a ΠG-invariant metric on G, then for each t, $(\tilde{\Phi}_t(\Pi)\backslash G, g)$ is isospectral to $(\Pi\backslash G, g)$.

Lee implicitly assumed that the action of Π on G is free. However, we can see by closer inspection that the action of Π on G is not free. For example, any point of the form $(x_1, x_2, 0, 0, 0, \frac{1}{4})$ is fixed by $\alpha \in \Pi$. In fact the set of all fixed points of the action of Π on G is:

$$\{(x_1, x_2, y_1, y_2, z_1, z_2) \in \mathbb{R}^6 \mid x_1, x_2 \in \mathbb{R}, y_1, y_2, z_1 \in \tfrac{1}{2}\mathbb{Z}, z_2 = \tfrac{n}{2} + \tfrac{1}{4}\},$$

where n is any integer. The isotropy group of a point in this set has the form

$$\{1, (\phi, (0, 0, 2y_1, 2y_2, 2z_1, 2z_2))\},$$

where $\phi(x_1, x_2, y_1, y_2, z_1, z_2) = (x_1, x_2, -y_1, -y_2, -z_1, -z_2)$. So we see that $\Pi\backslash G$ is an orbifold containing singular points with $\mathbb{Z}_2$ isotropy type. Thus Lee's example is an illustration of Theorem 2.3. After applying Lemma 3.1 with $\Psi = \Phi_t$ and $\Phi = \tilde{\Phi}_t$, we have an isospectral deformation of metrics on the *orbifold* $\Pi\backslash G$.

This example is a nontrivial deformation. Indeed, suppose that $\tau : (\Pi\backslash G, g) \to (\Pi\backslash G, \Phi_t^* g)$ is an isometry. Then because G is simply connected and Π is discrete, G is the universal cover of $\Pi\backslash G$. Thus τ lifts to an isometry, also called τ from (G, g) to $(G, \Phi_t^* g)$. Since G is a nilpotent Lie group, τ must be an element of $\mathrm{Aut}(G) \ltimes G$ (see [9]). Furthermore, because τ is a lift, we have that $\tau \circ \Pi \circ \tau^{-1} = \Pi$ within the transformation group $\mathrm{Aut}(G) \ltimes G$. On the other hand, G is normal in $\mathrm{Aut}(G) \ltimes G$, so conjugation by τ maps G to itself. Therefore, conjugation by τ leaves Γ invariant. This implies that τ must descend to an isometry $\tau\colon (\Gamma\backslash G, g) \to (\Gamma\backslash G, \Phi_t^* g)$. However, from [8] we know that no such isometry can exist. Thus $(\Pi\backslash G, g)$ cannot be isometric to $(\Pi\backslash G, \Phi_t^* g)$.

Note that Lee's example can be modified to produce examples of isospectral deformations on manifolds. For example, suppose that we define $\beta : G \to G$ by

$$\beta(x_1, x_2, y_1, y_2, z_1, z_2) = (x_1, x_2, y_1, y_2, -z_1, z_2 + \tfrac{1}{2}).$$

Letting $\Pi' = \Gamma \cup \beta\Gamma$, we see that since β^2 is simply translation by $(0, 0, 0, 0, 0, 1)$, Π' is a finite extension of Γ. Since β commutes with the maps Φ_t defined above, we can extend each Φ_t to an element $\tilde{\Phi}_t$ of $\mathrm{AIA}(\Pi' G; \Pi')$. Finally by direct computation we can see that the action of Π' on G has no fixed points.

Notice that the manifold $\Pi'\backslash G$ is nonorientable. Indeed, if $\Pi'\backslash G$ were orientable, it would possess a nonvanishing orientation form. This form would have to lift to a Π'-invariant nonvanishing orientation form on G. However, the fact that the determinant of the Jacobian of $\beta \in \Pi'$ is negative makes this impossible.

On the other hand, suppose that

$$\gamma(x_1, x_2, y_1, y_2, z_1, z_2) = (-x_1, x_2, -y_1, y_2, z_1 + \tfrac{1}{2}, z_2 + \tfrac{1}{2}).$$

Then we see that γ^2 is translation by $(0, 0, 0, 0, 1, 1)$. Letting Π'' be the group generated by Γ and γ, and using the same reasoning as above, we find an isospectral deformation on the orientable manifold $\Pi''\backslash G$.

Thus we have isospectral deformations of metrics on manifolds. The proof that the deformations are nontrivial is identical to the one given above.

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References

[1] P. Bérard and D. Webb, *On ne peut pas entendre l'orientabilité d'une surface.* C. R. Acad. Sci. Paris Sér. I Math. **320**(1995), no. 5, 533–536.

[2] P. Buser, J. Conway, P. Doyle, and K.-D. Semmler, *Some planar isospectral domains.* Internat. Math. Res. Notices **1994**, no. 9.

[3] Y.-J. Chiang, *Spectral geometry of V-manifolds and its application to harmonic maps.* In: Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990), Proc. Sympos. Pure Math., 54, Part 1, American Mathematical Society, Providence, RI, 1993, pp. 93–99.

[4] D. M. DeTurck and C. S. Gordon, *Isospectral deformations II. Trace formulas, metrics and potentials.* With an appendix by K. B. Lee, Comm. Pure Appl. Math. **42**(1989), no. 8, 1067–1095. doi:10.1002/cpa.3160420803

[5] H. Donnelly, *Asymptotic expansions for the compact quotients of properly discontinuous group actions.* Illinois J. Math. **23**(1979), no. 3, 485–496.

[6] P. Doyle and J. Rossetti, *Isospectral hyperbolic surfaces having matching geodesics.* New York J. Math. **14**(2008), 193–204.

[7] C. S. Gordon and J. Rossetti, *Boundary volume and length spectra of Riemannian manifolds: what the middle degree Hodge spectrum doesn't reveal.* Ann. Inst. Fourier **53**(2003), no. 7, 2297–2314.

[8] C. S. Gordon and E. N. Wilson, *Isospectral deformations of compact solvmanifolds.* J. Differential Geom. **19**(1984), no. 1, 241–256.

[9] ———, *Isometry groups of Riemannian solvmanifolds.* Trans. Amer. Math. Soc. **307**(1988), no. 1, 245–269. doi:10.2307/2000761

[10] J. P. Rosetti, D. Schueth, and M. Weilandt, *Isospectral orbifolds with different maximal isotropy orders.* Ann. Glob. Anal. Geom. **34**(2008), no. 4, 351–366. doi:10.1007/s10455-008-9110-3

[11] I. Satake, *On a generalization of the notion of manifold.* Proc. Nat. Acad. Sci. U.S.A. **42**(1956), 359–363. doi:10.1073/pnas.42.6.359

[12] P. Scott, *The geometries of* 3*-manifolds.* Bull. London Math. Soc. **15**(1983), no. 5, 401–487. doi:10.1112/blms/15.5.401

[13] N. Shams, E. Stanhope, and D. L. Webb, *One cannot hear orbifold isotropy type.* Arch. Math. **87**(2006), no. 4, 375–384.

[14] E. Stanhope, *Spectral bounds on orbifold isotropy.* Ann. Global Anal. Geom. **27**(2005), no. 4, 355–375. doi:10.1007/s10455-005-1584-7

[15] C. J. Sutton, *Equivariant isospectrality and isospectral deformations of metrics on spherical orbifolds.* http://arxiv.org/abs/math/0608557.

[16] W. P. Thurston, *The geometry and topology of three-manifolds.* Lecture notes, Princeton University, Princeton, NJ, 1979.

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