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Eigenvalues of $-\Delta_p - \Delta_q$ Under Neumann Boundary Condition

Dedicated to Professor Ioan A. Rus on the occasion of his eightieth birthday

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Abstract. The eigenvalue problem $-\Delta_p u - \Delta_q u = \lambda |u|^{q-2}u$ with $p \in (1,\infty)$, $q \in (2,\infty)$, $p \neq q$ subject to the corresponding homogeneous Neumann boundary condition is investigated on a bounded open set with smooth boundary from $\mathbb{R}^N$ with $N \geq 2$. A careful analysis of this problem leads us to a complete description of the set of eigenvalues as being a precise interval $(\lambda_1, +\infty)$ plus an isolated point $\lambda = 0$. This comprehensive result is strongly related to our framework, which is complementary to the well-known case $p = q \neq 2$ for which a full description of the set of eigenvalues is still unavailable.

1 Introduction and Main Result

Our goal in this paper is to investigate the eigenvalue problem

$$\begin{cases} Au := -\Delta_p u - \Delta_q u = \lambda |u|^{q-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu_A} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $p \in (1,\infty)$, $q \in (2,\infty)$, $p \neq q$, $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, and

$$\frac{\partial u}{\partial \nu_A} = \left(|\nabla u|^{p-2} + |\nabla u|^{q-2} \right) \frac{\partial u}{\partial \nu},$$

with ν = the unit outward normal to $\partial\Omega$. The solutions u will be sought in the Sobolev space $W := W^{1,\max\{p,q\}}(\Omega)$, so that the above PDE is satisfied in the distribution sense, and the normal derivative $\frac{\partial u}{\partial \nu_A}$ (associated with operator A) exists in a trace sense (see [3]). Using a Green's formula (see [3, Corollary 2, p. 71]) one can define the eigenvalues of our problem in terms of weak solutions $u \in W$ as follows: $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1.1) if there exists $u_\lambda \in W \setminus \{0\}$ such that

$$\int_\Omega (|\nabla u_\lambda|^{p-2} + |\nabla u_\lambda|^{q-2}) \nabla u_\lambda \nabla v \, dx = \lambda \int_\Omega |u_\lambda|^{q-2} u_\lambda v \, dx, \quad \forall\, v \in W. \tag{1.2}$$

Conversely, if λ is an eigenvalue, then any eigenfunction $u \in W \setminus \{0\}$ corresponding to it satisfies problem (1.1) in the distribution sense. This follows by the same Green's formula.

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In the particular case $q = 2$, the set of eigenvalues for problem (1.1) was completely described in [7] (for $p > 2$) and [4] (for $p \in (1, 2)$). Our goal here is to show that a complete description of the eigenvalue set is also possible for any $q > 2$ and $p \in (1, \infty) \setminus \{q\}$. This general case requires separate analysis, and some difficulties that occur within the new framework have to be overcome.

Note that the case $q = p \neq 2$ has been very much discussed in the literature, but a complete description of the corresponding eigenvalue set is still unavailable (it is only known that, as a consequence of the Ljusternik–Schnirelman theory, there exists a sequence of nonnegative eigenvalues of the corresponding operator; see, *e.g.*, [6]).

Now, choosing $v = u_\lambda$ in (1.2), we infer that no negative λ can be an eigenvalue of problem (1.1). It is also obvious that $\lambda = 0$ is an eigenvalue of this problem (the corresponding eigenfunctions being the nontrivial constants). So we need to investigate the case $\lambda > 0$.

Note that if $\lambda > 0$ is an eigenvalue of (1.1), then testing with $v = 1$ in (1.2) we deduce that

$$\int_\Omega |u_\lambda|^{q-2} u_\lambda \, dx = 0.$$

Thus, the eigenfunctions corresponding to positive eigenvalues of problem (1.1) belong to the nonempty, symmetric, closed cone

$$C := \Big\{ v \in W : \int_\Omega |v|^{q-2} v \, dx = 0 \Big\}.$$

Remark It is easy to see that $C \setminus \{0\} \neq \varnothing$. Indeed, one can simply choose $u = u_1 - u_2$, where u_1, u_2 are nonnegative test functions having supports in two disjoint balls included in Ω such that $\int_\Omega u_1^{q-1}\, dx = \int_\Omega u_2^{q-1}\, dx$. More specifically, let $x_1, x_2 \in \Omega$ be two different interior points of Ω. Then there exists an $\epsilon > 0$ small enough such that the balls $B_\epsilon(x_1)$, $B_\epsilon(x_2)$ are included in Ω and $B_\epsilon(x_1) \cap B_\epsilon(x_2) = \varnothing$. Consider the functions u_i, $i = 1, 2$,

$$u_i(x) := \begin{cases} e^{1/(|x - x_i|^2 - \epsilon^2)}, & x \in B_\epsilon(x_i), \\ 0, & x \in \Omega \setminus B_\epsilon(x_i). \end{cases}$$

These are test functions (see, *e.g.*, [2, p. 108]), and thus they belong to the Sobolev space W. Obviously, $u\colon \Omega \to \mathbb{R}$ defined by

$$u(x) = u_1(x) - u_2(x), \quad \forall\, x \in \Omega,$$

belongs to $C \setminus \{0\}$. Of course, tu also belongs to $C \setminus \{0\}$ for all $t \in \mathbb{R} \setminus \{0\}$.

The main result of this paper is the following theorem.

Theorem 1.1 *Assume $p \in (1, \infty)$, $q \in (2, \infty)$ and $p \neq q$. Then the eigenvalue set of problem (1.1) is precisely $\{0\} \cup (\lambda_1, +\infty)$, where*

$$\lambda_1 := \inf_{v \in C \setminus \{0\}} \frac{\int_\Omega |\nabla v|^q \, dx}{\int_\Omega |v|^q \, dx}. \tag{1.3}$$

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2 Proof of Theorem 1.1

As pointed out before, problem (1.1) cannot have negative eigenvalues, while $\lambda = 0$ is an eigenvalue of this problem. In what follows we investigate the case $\lambda > 0$.

For the rest of the proof, we start by introducing some notation and recalling some well-known results. For each $r > 1$, define

$$C_r := \Big\{ v \in W^{1,r}(\Omega) : \int_\Omega |v|^{r-2} v \, dx = 0 \Big\}.$$

Note that $C = C_q$ only if $q > p$; otherwise (*i.e.*, if $q < p$), C is a proper subset of C_q.

Consider the eigenvalue problem

$$\begin{cases} -\Delta_r u = \lambda |u|^{r-2} u & \text{in } \Omega, \\ |\nabla u|^{r-2} \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $r > 1$. Define

$$\lambda_1^N(r) := \inf_{v \in C_r \setminus \{0\}} \frac{\int_\Omega |\nabla v|^r \, dx}{\int_\Omega |v|^r \, dx}.$$

We know from [5, Theorem 6.2.29] that if $r \geq 2$, then $\lambda = \lambda_1^N(r)$ is the lowest positive eigenvalue of problem (2.1). In particular, we deduce that $\lambda_1 = \lambda_1^N(q) > 0$ if $q > 2$, $1 < p < q$ and $\lambda_1 \geq \lambda_1^N(q) > 0$ if $2 < q < p$.

Further, define

$$\nu_1 := \inf_{v \in C \setminus \{0\}} \frac{\frac{1}{q} \int_\Omega |\nabla v|^q \, dx + \frac{1}{p} \int_\Omega |\nabla v|^p \, dx}{\frac{1}{q} \int_\Omega |v|^q \, dx}.$$

It is easy to check that

$$\lambda_1 = \nu_1. \tag{2.2}$$

Indeed, note that for each $u \in C \setminus \{0\}$ and each $t > 0$, we have

$$\nu_1 \leq \frac{\frac{1}{p} \int_\Omega |\nabla (tu)|^p \, dx + \frac{1}{q} \int_\Omega |\nabla (tu)|^q \, dx}{\frac{1}{q} \int_\Omega |tu|^q \, dx} = \frac{q t^{p-q}}{p} \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^q \, dx} + \frac{\int_\Omega |\nabla u|^q \, dx}{\int_\Omega |u|^q \, dx}.$$

Thus, letting $t \to 0$ if $p > q$ and $t \to \infty$ if $p < q$, and then passing to infimum in the right-hand side, we get $\nu_1 \leq \lambda_1$. On the other hand, for all $u \in C \setminus \{0\}$, we have

$$\frac{\frac{1}{p} \int_\Omega |\nabla u|^p \, dx + \frac{1}{q} \int_\Omega |\nabla u|^q \, dx}{\frac{1}{q} \int_\Omega |u|^q \, dx} \geq \frac{\int_\Omega |\nabla u|^q \, dx}{\int_\Omega |u|^q \, dx} \geq \lambda_1,$$

which implies $\nu_1 \geq \lambda_1$. Consequently, (2.2) holds true.

2.1 The Nonexistence Part

We have the following two claims.

Claim 1 *There is no eigenvalue of problem* (1.1) *in* $(0, \lambda_1)$.

Assume by contradiction that there exists a $\lambda \in (0, \lambda_1)$ that is an eigenvalue of (1.1), with $u_\lambda \in C \setminus \{0\}$ the corresponding eigenfunction. Using (1.3) and the definition relation (1.2) with $v = u_\lambda$, we derive

$$0 < (\lambda_1 - \lambda) \int_\Omega |u_\lambda|^q \, dx \le \int_\Omega |\nabla u_\lambda|^q \, dx - \lambda \int_\Omega |u_\lambda|^q \, dx$$
$$\le \int_\Omega |\nabla u_\lambda|^p \, dx + \int_\Omega |\nabla u_\lambda|^q \, dx - \lambda \int_\Omega |u_\lambda|^q \, dx = 0.$$

This contradiction shows that Claim 1 holds true.

Claim 2 *$\lambda = \lambda_1$ is not an eigenvalue of problem* (1.1).

Assume the contrary, *i.e.*, there exists $u_{\lambda_1} \in C \setminus \{0\}$ such that (1.2) holds true with $\lambda = \lambda_1$. Letting $v = u_{\lambda_1}$ in (1.2), we get

$$\int_\Omega |\nabla u_{\lambda_1}|^p \, dx + \int_\Omega |\nabla u_{\lambda_1}|^q \, dx = \lambda_1 \int_\Omega |u_{\lambda_1}|^q \, dx.$$

From this equality and the definition of λ_1, one gets

$$\int_\Omega |\nabla u_{\lambda_1}|^p \, dx + \lambda_1 \int_\Omega |u_{\lambda_1}|^q \, dx \le \int_\Omega |\nabla u_{\lambda_1}|^p \, dx + \int_\Omega |\nabla u_{\lambda_1}|^q \, dx = \lambda_1 \int_\Omega |u_{\lambda_1}|^q \, dx,$$

which yields

$$\int_\Omega |\nabla u_{\lambda_1}|^p \, dx = 0 \Longrightarrow \nabla u_{\lambda_1} = 0 \quad \text{a.e. in } \Omega.$$

By Weyl's regularity lemma, $u_{\lambda_1} \in C^\infty(\Omega)$, so u_{λ_1} is a constant function. This combined with the fact that $u_{\lambda_1} \in C$ implies $u_{\lambda_1} = 0$, contradiction. So Claim 2 holds true.

2.2 The Existence Part

Let us first recall the following theorem (Lagrange multiplier rule) (see, *e.g.*, [10, Thm. 3.3.3, p. 179] or [8, Thm. 2.2.10, p. 76]), which will play a key role in our analysis.

Lemma 2.1 *Let X and Y be real Banach spaces and let $f\colon D \to \mathbb{R}$, $h\colon D \to Y$ be C^1 functions on the open set $D \subset X$. If y is a local solution of the minimization problem*

$$\text{(P)} \qquad \min f(x), \quad h(x) = 0,$$

and $h'(y)$ is a surjective operator, then there exists $y^* \in Y^*$ such that

$$f'(y) + y^* \circ h'(y) = 0, \tag{2.3}$$

where Y^* stands for the dual of Y.

Our purpose in this subsection is to prove the following claim.

Claim 3 *Every $\lambda \in (\lambda_1, \infty)$ is an eigenvalue of problem* (1.1).

In order to prove Claim 3, let us fix a $\lambda > \lambda_1$ and define $I_\lambda\colon W \to \mathbb{R}$ by

$$I_\lambda(u) := \frac{1}{q} \int_\Omega |\nabla u|^q \, dx + \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \frac{\lambda}{q} \int_\Omega |u|^q \, dx.$$

Standard arguments can be used to deduce that $I_\lambda \in C^1(W \setminus \{0\}, \mathbb{R})$ (actually, $I_\lambda \in C^1(W, \mathbb{R})$ if $2 < q < p$) with the derivative given by

$$\langle I_\lambda'(u), \phi \rangle = \int_\Omega |\nabla u|^{q-2} \nabla u \nabla \phi \, dx + \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi \, dx - \lambda \int_\Omega |u|^{q-2} u \phi \, dx,$$

for all $u \in W \setminus \{0\}$ (actually, all $u \in W$ if $2 < q < p$) and all $\phi \in W$. Thus, we note that λ is an eigenvalue of problem (1.1) if and only if I_λ possesses a nontrivial critical point. Further, we split the discussion into two cases: $1 < p < q$, $q > 2$, and $2 < q < p$, respectively.

2.2.1 The Case $1 < p < q$, $q > 2$

In this case, $C = C_q$, $W = W^{1,q}(\Omega)$ and $\lambda_1 = \lambda_1^N(q)$.

A careful analysis shows that I_λ is not coercive on W, and consequently, we cannot use the Direct Method in the Calculus of Variations in order to determine critical points of I_λ. Our idea (inspired by [1, Section 2.3.3]) will be to consider the restriction of I_λ to the Nehari-type manifold defined by

$$\begin{aligned} \mathcal{N}_\lambda &:= \left\{ u \in C_q \setminus \{0\} : \langle I_\lambda'(u), u \rangle = 0 \right\} \\ &= \left\{ u \in C_q \setminus \{0\} : \int_\Omega |\nabla u|^q \, dx + \int_\Omega |\nabla u|^p \, dx = \lambda \int_\Omega u^q \, dx \right\}. \end{aligned}$$

In fact, this is a natural idea since any possible eigenfunction corresponding to λ is necessarily an element of $\mathcal{N}_\lambda$. Note that for all $v \in \mathcal{N}_\lambda$, functional $I_\lambda(v)$ has the following expression

$$\begin{aligned} I_\lambda(v) &= \frac{1}{q} \int_\Omega |\nabla v|^q \, dx + \frac{1}{p} \int_\Omega |\nabla v|^p \, dx - \frac{\lambda}{q} \int_\Omega |v|^q \, dx \\ &= -\frac{1}{q} \int_\Omega |\nabla v|^p \, dx + \frac{1}{p} \int_\Omega |\nabla v|^p \, dx = \frac{q-p}{pq} \int_\Omega |\nabla v|^p \, dx. \end{aligned}$$

Consequently, denoting

$$m_\lambda := \inf_{w \in \mathcal{N}_\lambda} I_\lambda(w),$$

we have $m_\lambda \geq 0$.

In what follows the proof of Claim 3 is done in several steps.

Step 1. $\mathcal{N}_\lambda \neq \varnothing$. Indeed, since $\lambda > \lambda_1^N(q)$, it follows by the definition of $\lambda_1^N(q)$ that there exists $v_\lambda \in C_q \setminus \{0\}$ for which

$$\int_\Omega |\nabla v_\lambda|^q \, dx < \lambda \int_\Omega |v_\lambda|^q \, dx.$$

Then there exists $t > 0$ such that $t v_\lambda \in \mathcal{N}_\lambda$, *i.e.*,

$$t^q \int_\Omega |\nabla v_\lambda|^q \, dx + t^p \int_\Omega |\nabla v_\lambda|^p \, dx = \lambda t^q \int_\Omega |v_\lambda|^q \, dx.$$

This is obvious when

$$t = \left(\frac{\lambda \int_\Omega |v_\lambda|^q \, dx - \int_\Omega |\nabla v_\lambda|^q \, dx}{\int_\Omega |\nabla v_\lambda|^p \, dx} \right)^{1/(p-q)}.$$

Note that we have also used the fact that C_q is a cone. If $w \in C_q$, then $tw \in C_q$ for all $t > 0$.

Step 2. Every minimizing sequence for I_λ *on* $\mathcal{N}_\lambda$ *is bounded in* $W^{1,q}(\Omega)$. Let $\{u_n\}$ be a minimizing sequence in $\mathcal{N}_\lambda$, *i.e.*,

$$(2.4)\quad 0 < \lambda \int_\Omega |u_n|^q \, dx - \int_\Omega |\nabla u_n|^q \, dx = \int_\Omega |\nabla u_n|^p \, dx \to \frac{pq}{q-p} m_\lambda, \quad \text{as } n \to \infty.$$

Assume by contradiction that $\{u_n\}$ is unbounded in $W^{1,q}(\Omega)$, so a subsequence of it, again denoted $\{u_n\}$, converges in the norm of $W^{1,q}(\Omega)$ to ∞. Then by (2.4) it follows that $\int_\Omega |u_n|^q \, dx \to \infty$ and $\int_\Omega |\nabla u_n|^q \, dx \to \infty$ as well. Set $v_n := \frac{u_n}{\|u_n\|_{L^q(\Omega)}}$. Since $\int_\Omega |\nabla u_n|^q \, dx < \lambda \int_\Omega |u_n|^q \, dx$, we deduce that $\int_\Omega |\nabla v_n|^q \, dx < \lambda$ for all n. Thus, $\{v_n\}$ is bounded in $W^{1,q}(\Omega)$. It follows that there exists $v_0 \in W^{1,q}(\Omega)$ such that $v_n \rightharpoonup v_0$ in $W^{1,q}(\Omega)$ (hence in $W^{1,p}(\Omega)$ as well) and $v_n \to v_0$ in $L^q(\Omega)$. In particular, this last convergence implies that $v_0 \in C_q$ (*cf.* Lebesgue's Dominated Convergence Theorem).

Dividing (2.4) by $\|u_n\|^p_{L^q(\Omega)}$ we get

$$\int_\Omega |\nabla v_n|^p \, dx \to 0 \quad \text{as } n \to \infty.$$

Next, since $v_n \rightharpoonup v_0$ in $W^{1,p}(\Omega)$, we infer that

$$\int_\Omega |\nabla v_0|^p \, dx \le \liminf_{n\to\infty} \int_\Omega |\nabla v_n|^p \, dx = 0,$$

and consequently v_0 is a constant function. In fact, from $v_0 \in C_q$ we see that $v_0 = 0$. It follows that $v_n \to 0$ in $L^q(\Omega)$, which contradicts the fact that $\|v_n\|_{L^q(\Omega)} = 1$ for all n.

Consequently, $\{u_n\}$ must be bounded in $W^{1,q}(\Omega)$.

Step 3. $m_\lambda := \inf_{w\in\mathcal{N}_\lambda} I_\lambda(w) > 0$. Assume by contradiction that $m_\lambda = 0$. Let $\{u_n\} \subset \mathcal{N}_\lambda$ be a minimizing sequence, *i.e.*,

$$(2.5)\qquad 0 < \lambda \int_\Omega |u_n|^q \, dx - \int_\Omega |\nabla u_n|^q \, dx = \int_\Omega |\nabla u_n|^p \, dx \to 0, \quad \text{as } n \to \infty.$$

By Step 2 we know that $\{u_n\} \subset C_q$ is bounded in $W^{1,q}(\Omega)$. It follows that there exists $u_0 \in W^{1,q}(\Omega)$ such that (on a subsequence, again denoted $\{u_n\}$) one has $u_n \rightharpoonup u_0$ in $W^{1,q}(\Omega)$ (hence in $W^{1,p}(\Omega)$) and $u_n \to u_0$ in $L^q(\Omega)$. Therefore, $u_0 \in C_q$ and

$$\int_\Omega |\nabla u_0|^p \, dx \le \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^p \, dx = 0,$$

and consequently $u_0 = 0$. Thus, we have proved that $u_n \rightharpoonup 0$ in $W^{1,q}(\Omega)$.

Now set $v_n := u_n / \|u_n\|_{L^q(\Omega)}$. Since $\int_\Omega |\nabla u_n|^q \, dx < \lambda \int_\Omega |u_n|^q \, dx$, we have $\int_\Omega |\nabla v_n|^q \, dx < \lambda$ for all n. Thus, $\{v_n\} \subset C_q$ is bounded in $W^{1,q}(\Omega)$. It follows that there exists $v_0 \in C_q$ such that $v_n \rightharpoonup v_0$ in $W^{1,q}(\Omega)$ and $v_n \to v_0$ in $L^q(\Omega)$.

Dividing (2.5) by $\|u_n\|^p_{L^q(\Omega)}$, we get

$$\int_\Omega |\nabla v_n|^p \, dx = \|u_n\|^{q-p}_{L^q(\Omega)} \Big[\lambda - \int_\Omega |\nabla v_n|^q \, dx \Big] \to 0 \quad \text{as } n \to \infty.$$

Next, since $v_n \rightharpoonup v_0$ in $W^{1,p}(\Omega)$, we infer that

$$\int_\Omega |\nabla v_0|^p \, dx \le \liminf_{n\to\infty} \int_\Omega |\nabla v_n|^p \, dx = 0,$$

and consequently v_0 is a constant function. In fact, $v_0 = 0$, since $v_0 \in C_q$. Thus, $v_n \to 0$ in $L^q(\Omega)$, which contradicts the fact that $\|v_n\|_{L^q(\Omega)} = 1$ for all n.

Consequently, m_λ is positive, as asserted.

Step 4. There exists $u \in \mathcal{N}_\lambda$ such that $I_\lambda(u) = m_\lambda$. Let $\{u_k\} \subset \mathcal{N}_\lambda$ be a minimizing sequence, *i.e.*, $I_\lambda(u_k) \to m_\lambda$ as $k \to \infty$.

By Step 2 $\{u_k\}$ is bounded in $W^{1,q}(\Omega)$. Thus, there exists $u \in C_q$ such that u_k converges weakly in $W^{1,q}(\Omega)$ and strongly in $L^q(\Omega)$ to u.

By the above pieces of information we deduce that

$$I_\lambda(u) \leq \liminf_{k\to\infty} I_\lambda(u_k) = m_\lambda. \tag{2.6}$$

Since $u_k \in \mathcal{N}_\lambda$ for all k, we have

$$\int_\Omega |\nabla u_k|^q \, dx + \int_\Omega |\nabla u_k|^p \, dx = \lambda \int_\Omega |u_k|^q \, dx, \quad \forall k. \tag{2.7}$$

If $u = 0$, then it follows by (2.7) that u_k converges strongly to 0 in $W^{1,q}(\Omega)$ (and consequently in $W^{1,p}(\Omega)$). Thus,

$$0 < \lambda \int_\Omega |u_k|^q \, dx - \int_\Omega |\nabla u_k|^q \, dx = \int_\Omega |\nabla u_k|^p \, dx \to 0, \quad \text{as } k \to \infty.$$

Next, arguing as in the proof of Step 3, we are led to a contradiction. Consequently, $u \in C_q \setminus \{0\}$.

Now, letting $k \to \infty$ in (2.7), we deduce

$$\int_\Omega |\nabla u|^q \, dx + \int_\Omega |\nabla u|^p \, dx \leq \lambda \int_\Omega |u|^q \, dx.$$

If we have equality here, then $u \in \mathcal{N}_\lambda$, and everything is done. Assume the contrary, *i.e.*,

$$\int_\Omega |\nabla u|^q \, dx + \int_\Omega |\nabla u|^p \, dx < \lambda \int_\Omega |u|^q \, dx. \tag{2.8}$$

Let $t > 0$ be such that $tu \in \mathcal{N}_\lambda$, *i.e.*,

$$t = \Big(\frac{\lambda \int_\Omega |u|^q \, dx - \int_\Omega |\nabla u|^q \, dx}{\int_\Omega |\nabla u|^p \, dx} \Big)^{1/(p-q)}.$$

From (2.8) and our condition $p < q$, one can infer that $t \in (0,1)$. Finally, since $tu \in \mathcal{N}_\lambda$ with $t \in (0,1)$ we have

$$\begin{aligned} 0 < m_\lambda \leq I_\lambda(tu) &= \frac{t^p}{p} \int_\Omega |\nabla u|^p \, dx + \frac{t^q}{q} \int_\Omega |\nabla u|^q \, dx - \lambda \frac{t^q}{q} \int_\Omega |u|^q \, dx \\ &= \frac{t^p}{p} \int_\Omega |\nabla u|^p \, dx - \frac{t^p}{q} \int_\Omega |\nabla u|^p \, dx \\ &\leq t^p \liminf_{k\to\infty} I_\lambda(u_k) = t^p m_\lambda < m_\lambda, \end{aligned}$$

which is impossible. Hence, relation (2.8) cannot be valid, and consequently we must have $u \in \mathcal{N}_\lambda$, and thus $I_\lambda(u) = m_\lambda$ (see (2.6)).

Step 5. The proof of the theorem is concluded. Let $u \in \mathcal{N}_\lambda \setminus \{0\}$ be the minimizer found in Step 4. In fact u is a solution of the minimization problem $\min_{w\in W\setminus\{0\}} I_\lambda(w)$, under restrictions

$$h_1(w) := \int_\Omega |\nabla w|^q\, dx + \int_\Omega |\nabla w|^p\, dx - \lambda \int_\Omega |w|^q\, dx = 0, \tag{2.9}$$

$$h_2(w) := \int_\Omega |w|^{q-2} w\, dx = 0. \tag{2.10}$$

Now Lemma 2.1 (Lagrange multiplier rule) comes into play. We choose $X = W$, $Y = \mathbb{R}^2$, $D = W \setminus \{0\}$, $f = I_\lambda$, $h = (h_1, h_2)$. Obviously, the dual Y^* can be identified with $\mathbb{R}^2$. All the conditions from the statement of Lemma 2.1 are met, including the surjectivity condition on $h'(u)$, which means that for any pair $(\zeta_1, \zeta_2) \in \mathbb{R}^2$, there is a $w \in W$ such that $\langle h_1'(u), w\rangle = \zeta_1$, $\langle h_2'(u), w\rangle = \zeta_2$. Indeed, choosing $w = au + b$ with $a, b \in \mathbb{R}$ in these equations, we obtain a linear algebraic system in a and b:

$$aq \int_\Omega |\nabla u|^q\, dx + ap \int_\Omega |\nabla u|^p\, dx - \lambda a q \int_\Omega |u|^q\, dx = \zeta_1,$$
$$b(q-1) \int_\Omega |u|^{q-2}\, dx = \zeta_2,$$

which yields

$$a(p-q) \int_\Omega |\nabla u|^p\, dx = \zeta_1, \quad b(q-1) \int_\Omega |u|^{q-2}\, dx = \zeta_2.$$

Thus, a and b can be uniquely determined, hence $h'(u)$ is surjective, as asserted. Consequently, Lemma 2.1 is applicable to our minimization problem. Specifically, there exist some constants $c, d \in \mathbb{R}$ such that (see equation (2.3)):

$$\begin{aligned}\Big[\int_\Omega |\nabla u|^{q-2} \nabla u \nabla \phi\, dx + \int_\Omega |\nabla u|^{p-2} \nabla u \nabla\phi\, dx - \lambda \int_\Omega |u|^{q-2} u\phi\, dx \Big] \\ + c\Big[q \int_\Omega |\nabla u|^{q-2}\nabla u \nabla\phi\, dx + p \int_\Omega |\nabla u|^{p-2} \nabla u\nabla\phi\, dx - q\lambda \int_\Omega |u|^{q-2} u\phi\, dx\Big] \\ + d(q-1)\int_\Omega |u|^{q-2}\phi\, dx = 0, \quad \text{for all } \phi \in W^{1,q}(\Omega).\end{aligned}$$

Testing with $\phi = 1$ above, we deduce

$$-q\lambda \int_\Omega |u|^{q-2} u\, dx - cq\lambda \int_\Omega |u|^{q-2} u\, dx + d(q-1)\int_\Omega |u|^{q-2}\, dx = 0,$$

which, in view of (2.10), yields $d = 0$.

Next, testing with $\phi = u$ above and using (2.9), we deduce

$$c(p-q)\int_\Omega |\nabla u|^p\, dx = 0,$$

which implies $c = 0$. Therefore, for all $\phi \in W^{1,q}(\Omega)$,

$$\int_\Omega |\nabla u|^{q-2} \nabla u\nabla\phi\, dx + \int_\Omega |\nabla u|^{p-2}\nabla u \nabla \phi\, dx - \lambda \int_\Omega |u|^{q-2} u\phi\, dx = 0,$$

i.e., λ is an eigenvalue of problem (1.1).

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2.2.2 The Case $2 < q < p$

Obviously, in this case, $W = W^{1,p}(\Omega)$ and $C \subset C_q$.

Fortunately, under our assumption ($2 < q < p$) I_λ is a coercive functional as shown next. We will conclude the proof of Claim 3 in three steps.

Step 1. I_λ is coercive, i.e.,

$$\lim_{\|u\|_{W^{1,p}(\Omega)}\to\infty,\ u\in C}\Big(\frac{1}{p}\int_\Omega|\nabla u|^p\,dx+\frac{1}{q}\int_\Omega|\nabla u|^q\,dx-\frac{\lambda}{q}\int_\Omega|u|^q\,dx\Big)=\infty.$$

Define $\alpha, \beta, \gamma: C \to \mathbb{R}$ by

$$\alpha(u):=\int_\Omega|\nabla u|^p\,dx,\quad \beta(u):=\int_\Omega|\nabla u|^q\,dx,\quad \gamma(u):=\int_\Omega|u|^q\,dx,$$

so that

$$I_\lambda(u)=\frac{1}{p}\alpha(u)+\frac{1}{q}\beta(u)-\frac{\lambda}{q}\gamma(u).$$

In order to go further, note that since $q \in (2, p)$, the standard norm on $W^{1,p}(\Omega)$, *i.e.*,

$$\|u\|_{W^{1,p}(\Omega)}=\|\nabla u\|_{L^p(\Omega)}+\|u\|_{L^p(\Omega)},$$

is equivalent to the following norm (see [2, Remark 15, p. 286]):

$$|\|u\||_{W^{1,p}(\Omega)}=\|\nabla u\|_{L^p(\Omega)}+\|u\|_{L^q(\Omega)}.$$

Thus, $\|u\|_{W^{1,p}(\Omega)}\to\infty$ if and only if $|\|u\||_{W^{1,p}(\Omega)}\to\infty$.

On the other hand, by the definition of λ_1 we have

$$\lambda_1\gamma(u)\le\beta(u),\quad \forall\, u\in C.$$

Then, since the estimates

$$\frac{1}{p}\alpha(u)+\frac{1}{q}\beta(u)\ge\frac{1}{p}\big(\alpha(u)+\beta(u)\big)\ge\frac{1}{p}\min\{1,\lambda_1\}\big[\alpha(u)+\gamma(u)\big],$$

hold true, we deduce that

$$\lim_{\|u\|_{W^{1,p}(\Omega)}\to\infty,\ u\in C}\frac{1}{p}\alpha(u)+\frac{1}{q}\beta(u)=\infty. \tag{2.11}$$

Further, Hölder's inequality yields

$$\beta(u)\le|\Omega|^{(p-q)/p}\alpha(u)^{q/p},\quad \forall\, u\in W^{1,p}(\Omega).$$

Combining this estimate with relation (2.11), we get

$$\lim_{\|u\|_{W^{1,p}(\Omega)}=\infty,\ u\in C}\alpha(u)\to\infty.$$

Using again Hölder's inequality, we have

$$I_\lambda(u)\ge\frac{1}{p}\alpha(u)+\frac{1}{q}\beta(u)-\frac{\lambda}{\lambda_1}|\Omega|^{(p-q)/p}\alpha(u)^{q/p}.$$

Since $q \in (2, p)$, we infer that the term in the right-hand side of the above inequality blows up as $\|u\|_{W^{1,p}(\Omega)}\to\infty$. The conclusion of this step is now clear.

Step 2. Functional I_λ has a global minimum point over C, say $\theta_\lambda \in C$, such that $I_\lambda(\theta_\lambda) < 0$.

Indeed, by Step 1 we know that I_λ is coercive. On the other hand, C is a weakly closed subset of the Banach space W, and for any $u \in C$ and any sequence (u_m) in C such that u_m converges weakly to u in W, we have $I_\lambda(u) \leq \liminf_{m\to\infty} I_\lambda(u_m)$. Then we can apply [9, Theorem 1.2] in order to obtain the existence of a global minimum point of I_λ, say $\theta_\lambda \in C$, *i.e.*, $I_\lambda(\theta_\lambda) = \min_C I_\lambda$. Using the fact that $\lambda_1 = \nu_1$ (see relation (2.2)), we deduce that for any $\lambda > \lambda_1$ there exists $w_\lambda \in C$ such that $I_\lambda(w_\lambda) < 0$, so $I_\lambda(\theta_\lambda) \leq I_\lambda(w_\lambda) < 0$. In particular, this shows that $\theta_\lambda \not\equiv 0$.

Step 3. We conclude the proof of Theorem 1.1.

Let $\theta_\lambda \in C$ be the minimizer found in Step 2, *i.e.*, $I_\lambda(\theta_\lambda) = \min_{w\in C} I_\lambda(w)$. Thus, θ_λ is actually a solution of the minimization problem $\min_{w\in W} I_\lambda(w)$, under restriction

$$h(w) := \int_\Omega |w|^{q-2} w \, dx = 0.$$

Lemma 2.1 is again applicable, with $X = W$, $Y = \mathbb{R}$, $D = W$, $f = I_\lambda$, $h\colon W \to \mathbb{R}$ as defined above, and $y := \theta_\lambda$. It is easily seen that all the conditions of Lemma 2.1 are fulfilled, including the fact that $h'(\theta_\lambda)$ is surjective. Therefore, there exists a constant $a \in \mathbb{R}$ such that (*cf.* (2.3))

$$\Big[\int_\Omega |\nabla\theta_\lambda|^{p-2}\nabla\theta_\lambda\nabla\phi \, dx + \int_\Omega |\nabla\theta_\lambda|^{q-2}\nabla\theta_\lambda\nabla\phi \, dx - \lambda\int_\Omega |\theta_\lambda|^{q-2}\theta_\lambda\phi \, dx\Big] + a(q-1)\int_\Omega |\theta_\lambda|^{q-2}\phi \, dx = 0, \quad \forall \phi \in W^{1,p}(\Omega).$$

Testing with $\phi = 1$ above, we deduce

$$a(q-1)\int_\Omega |\theta_\lambda|^{q-2} \, dx = 0,$$

which yields $a = 0$. Thus, for all $\phi \in W^{1,p}(\Omega)$,

$$\int_\Omega |\nabla\theta_\lambda|^{p-2}\nabla\theta_\lambda\nabla\phi \, dx + \int_\Omega |\nabla\theta_\lambda|^{q-2}\nabla\theta_\lambda\nabla\phi \, dx - \lambda\int_\Omega |\theta_\lambda|^{q-2}\theta_\lambda\phi \, dx = 0,$$

i.e., λ is an eigenvalue of problem (1.1).

Final comments

(a) In view of [7, Theorem 1.1] and [4, Theorem 1], our present result (Theorem 1.1) extends to the more general case $p \in (1, \infty)$, $q \in [2, \infty)$, $p \neq q$ with the same conclusion.

(b) If $1 < p < q$ and $q \geq 2$, then λ_1 defined by (1.3) is the first positive eigenvalue of $-\Delta_q$ with Neumann boundary condition, *i.e.*, $\lambda_1 = \lambda_1^N(q)$. On the other hand, if $2 \leq q < p$, then C is a proper subset of C_q, and we have $\lambda_1 \geq \lambda_1^N(q)$. It seems that, in fact, $\lambda_1 > \lambda_1^N(q)$. This is an open problem.

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