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LINEARIZATION OF THE PRODUCT OF JACOBI POLYNOMIALS. III

RICHARD ASKEY AND GEORGE GASPER

In a series of papers [**1**; **2**; **3**; **4**] the operation of linearizing the product of two Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $\alpha, \beta > -1$, has been investigated and the existence of a natural Banach algebra associated with the linearization coefficients has been proven. This was proven for $\alpha + \beta + 1 \geqq 0$ in [**3**] and for a slightly larger region in [**4**]. It was shown in [**4**] that such a Banach algebra does not exist for $-1 < \alpha, \beta < -\frac{1}{2}$. The method used in [**1**; **3**; **4**] was to prove the non-negativity of the expansion coefficients from which the existence of the Banach algebra easily follows. However, as shown in [**4**], the coefficients for a subset of $\alpha \geqq -\frac{1}{2}$, $\alpha + \beta + 1 < 0$ can be negative infinitely often and so a different method must be used for these values of α and β. We now complete the study of the existence of these Banach algebras by considering the remaining cases. For $\alpha > -\frac{1}{2}$ we will show that methods related to those in [**2**] can be used, and for $\alpha = -\frac{1}{2}$ an explicit formula will be given for the coefficients, and estimates of this formula will be used to prove the existence of the Banach algebra. For $-1 < \alpha, \beta < -\frac{1}{2}$ there is a weaker Banach algebra which is easy to obtain (Theorem 2). However, this weaker Banach algebra suffers from the defect of having a maximal ideal space which is larger than the maximal ideal space of the Banach algebras associated with Jacobi polynomials for $\alpha \geqq -\frac{1}{2}$.

We start with the standard type of Banach algebra. For $\alpha, \beta > -1$, the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ may be defined by

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1-x)^{n+\alpha}(1+x)^{n+\beta}]$$

[**7**, (4.3.1)]. These polynomials are orthogonal on $[-1, 1]$ and

$$\int_{-1}^1 P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta\,dx = \frac{\delta_{n,m}}{h_n},$$

where

$$h_n = h_n^{(\alpha,\beta)} = \frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}$$

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[**7**, (4.3.3)]. Then with $\xi(k, m, n)$ defined by

(1) $$\xi(k, m, n) = \int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(x)\,(1-x)^\alpha(1+x)^\beta\,dx,$$

we have

$$P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) = \sum_{k=|n-m|}^{n+m} \xi(k, m, n) h_k P_k^{(\alpha,\beta)}(x).$$

As in [**2**; **4**], in order to establish the existence of the Banach algebras for $\alpha \geqq \beta$, it suffices to show, for $n \geqq m \geqq 1$, that

(2) $$\sum_{k=n-m}^{n+m} |\xi(k, m, n)| h_k P_k^{(\alpha,\beta)}(1) \leqq C P_n^{(\alpha,\beta)}(1) P_m^{(\alpha,\beta)}(1),$$

where $C = C^{(\alpha,\beta)}$ is independent of n and m.
For $\alpha \geqq \beta \geqq -\frac{1}{2}$, (2) was proven in [**2**] and for $\alpha \geqq \beta, \alpha + \beta + 1 \geqq 0$ in [**3**]. Also the results in [**4**] showed that (2) fails for $\alpha < -\frac{1}{2}$ and for $\alpha < \beta$. Thus, in proving (2) for the remaining (α, β) we may assume that $\alpha > \beta, 0 > \alpha \geqq -\frac{1}{2}$, $-\frac{1}{2} > \beta > -1$, since this set contains $\alpha > \beta > -1, \alpha \geqq -\frac{1}{2}, \alpha + \beta + 1 < 0$.

From [**7**, (4.1.1)] and Stirling's formula, it follows that

(3) $$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} \sim n^\alpha$$

and $h_k^{(\alpha,\beta)} \sim k$. Therefore to prove (2) it is sufficient to prove that

(4) $$\sum_{k=n-m}^{n+m} k^{1+\alpha} m^{-\alpha} n^{-\alpha} \left| \int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta\,dx \right| \leqq C,$$
$$n \geqq m \geqq 1,$$

where, as elsewhere, when $k = 0$ it is assumed that k^α is replaced by $(k+1)^\alpha = 1$. For $\alpha > -\frac{1}{2}$ the proof given in [**2**] works if we only consider $\int_0^1$. This is true since the behaviour of $P_n^{(\alpha,\beta)}(x)$ for $0 \leqq x \leqq 1$ and $\alpha \geqq -\frac{1}{2}$ is almost completely controlled by the value of α. Thus for $\alpha > -\frac{1}{2}$ we may restrict ourselves to proving (4) with $\int_{-1}^{1}$ replaced by $\int_{-1}^{0}$. The case $\alpha = -\frac{1}{2}$ will be handled later.

Using $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$ and letting $x = \cos\theta$, we see that we must estimate

$$\sum_{k=n-m}^{n+m} k^{1+\alpha} m^{-\alpha} n^{-\alpha} \left| \int_0^{\pi/2} P_n^{(\beta,\alpha)}(\cos\theta) P_m^{(\beta,\alpha)}(\cos\theta) \cdot P_k^{(\beta,\alpha)}(\cos\theta)(\sin\tfrac{1}{2}\theta)^{2\beta+1}(\cos\tfrac{1}{2}\theta)^{2\alpha+1}\,d\theta \right|.$$

We will use the following properties of $P_n^{(\beta,\alpha)}(\cos\theta)$.

(5) $$|P_n^{(\beta,\alpha)}(\cos\theta)| \leqq A n^\beta, \qquad 0 \leqq \theta \leqq n^{-1},$$

(6) $$|P_n^{(\beta,\alpha)}(\cos\theta)| \leqq A n^{-\frac{1}{2}} \theta^{-\beta-\frac{1}{2}}, \qquad n^{-1} \leqq \theta \leqq \pi/2,$$

(7) $$|P_n^{(\beta,\alpha)}(\cos\theta)| \leqq A n^{-\frac{1}{2}}, \qquad 0 \leqq \theta \leqq \pi/2.$$

These inequalities follow from [7, p. 169] since $\beta < -\frac{1}{2}$.

If $m < n/2$, then $k \sim n$ and using (5) and (7) we have the estimate

$$\sum_{k=n-m}^{n+m} k^{1+\alpha} m^{-\alpha} n^{-\alpha} \int_0^{1/n} n^{\beta} k^{\beta} m^{-\frac{1}{2}} \theta^{2\beta+1}\, d\theta = O(n^{-1} m^{\frac{1}{2}-\alpha}) = O(m^{-\frac{1}{2}-\alpha}) = O(1)$$

since $\alpha > -\frac{1}{2}$. If $n/2 \leqq m \leqq n$, then we use (5) and (7) again to obtain

$$\sum_{k=n-m}^{n+m} k^{1+\alpha} m^{-\alpha} n^{-\alpha} \int_0^{1/n} n^{\beta} k^{-\frac{1}{2}} m^{\beta} \theta^{2\beta+1}\, d\theta = O(n^{-\beta-\frac{3}{2}} m^{\beta-\alpha+1}) = O((m/n)^{\beta-\alpha+1} n^{-\frac{1}{2}-\alpha}).$$

This term is bounded since the restrictions $\alpha < 0$ and $\beta > -1$ imply that $\beta - \alpha + 1 > 0$. Next we integrate from $1/n$ to $1/m$ and use (5), (6), and (7) to obtain

$$\sum_{k=n-m}^{n+m} k^{1+\alpha} m^{-\alpha} n^{-\alpha} \int_{1/n}^{1/m} n^{-\frac{1}{2}} m^{\beta} k^{-\frac{1}{2}} \theta^{\beta+\frac{1}{2}}\, d\theta = O(m^{-\frac{1}{2}-\alpha}) = O(1).$$

We next integrate from $1/m$ to $1/k$ and use (5) and (6) to obtain

$$\sum_{k=n-m}^{n+m} k^{1+\alpha} m^{-\alpha} n^{-\alpha} \int_{1/m}^{1/k} n^{-\frac{1}{2}} m^{-\frac{1}{2}} k^{\beta}\, d\theta = O(m^{\frac{1}{2}-\alpha} n^{\beta-\frac{1}{2}}) = O((m/n)^{\frac{1}{2}-\alpha} n^{\beta-\alpha}) = O(1).$$

Observe that we only needed to estimate this integral if $k < m$. We are then left with

$$(8) \quad \sum_{k=n-m}^{n+m} k^{1+\alpha} m^{-\alpha} n^{-\alpha} \int_{\max(1/m,1/k)}^{\pi/2} P_n^{(\beta,\alpha)}(\cos\theta) \cdot P_m^{(\beta,\alpha)}(\cos\theta) P_k^{(\beta,\alpha)}(\cos\theta)(\sin\tfrac{1}{2}\theta)^{2\beta+1}(\cos\tfrac{1}{2}\theta)^{2\alpha+1}\, d\theta.$$

Now we apply an asymptotic formula for $P_n^{(\beta,\alpha)}(\cos\theta)$. It suffices to use

$$(9) \quad (\sin\tfrac{1}{2}\theta)^{\beta+\frac{1}{2}}(\cos\tfrac{1}{2}\theta)^{\alpha+\frac{1}{2}} P_n^{(\beta,\alpha)}(\cos\theta) = (\pi n)^{-\frac{1}{2}}\cos(N\theta+\gamma) + O(n^{-\frac{3}{2}}\theta^{-1}), \qquad 1/n \leqq \theta \leqq \pi/2,$$

[7, Theorem 8.21.13] where $N = n + (\alpha+\beta+1)/2$ and $\gamma = -(\beta+1/2)\pi/2$. Using (9) in (8) leads to the estimation of

$$\sum_{k=n-m}^{n+m} k^{\alpha+\frac{1}{2}} m^{-\alpha-\frac{1}{2}} n^{-\alpha-\frac{1}{2}} \int_{\max(1/m,1/k)}^{\pi/2} \cos(N\theta+\gamma) \cdot \cos(M\theta+\gamma)\cos(K\theta+\gamma)(\sin\tfrac{1}{2}\theta)^{-\beta-\frac{1}{2}}(\cos\tfrac{1}{2}\theta)^{-\alpha-\frac{1}{2}}\, d\theta$$
$$+ \sum_{k=n-m}^{n+m} k^{\alpha+\frac{1}{2}} m^{-\alpha-\frac{1}{2}} n^{-\alpha-\frac{1}{2}} \int_{\max(1/m,1/k)}^{\pi/2} \theta^{-\beta-\frac{3}{2}}[m^{-1}+k^{-1}]\, d\theta.$$

The error terms are bounded by

$$\sum_{k=n-m}^{n+m} k^{\alpha+\frac{1}{2}} m^{-\alpha-\frac{1}{2}} n^{-\alpha-\frac{1}{2}}[m^{-1}+k^{-1}] = O(m^{-\alpha-\frac{1}{2}}) = O(1).$$

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Since $\beta < -\frac{1}{2}$, $(\sin \frac{1}{2}\theta)^{-\beta-\frac{1}{2}} (\cos \frac{1}{2}\theta)^{-\alpha-\frac{1}{2}} = g(\theta)$ is a bounded function of bounded variation for $0 \leqq \theta \leqq \pi/2$. The boundedness allows us to consider $\int_0^{\pi/2}$ (the same argument as for the error terms) and the bounded variation allows us to conclude that

$$\int_0^{\pi/2} \cos(N\theta + \gamma) \cos(M\theta + \gamma) \cos(K\theta + \gamma) g(\theta)\, d\theta = O\left(\frac{1}{|N \pm M \pm K|}\right).$$

This leads to the estimate

$$\sum_{k=n-m}^{n+m} k^{\alpha+\frac{1}{2}} m^{-\alpha-\frac{1}{2}} n^{-\alpha-\frac{1}{2}} |N \pm M \pm K|^{-1} = O\left(m^{-\alpha-\frac{1}{2}} \sum_{k=n-m}^{n+m} |N \pm M \pm K|^{-1}\right)$$
$$= O(m^{-\alpha-\frac{1}{2}} \log m) = O(1),$$

which completes the proof of (2) for $\alpha > -\frac{1}{2}$, $\alpha \geqq \beta$.

For the remaining case $\alpha = -\frac{1}{2}$, $-1 < \beta < -\frac{1}{2}$, we use the following formula of Dougall which is given in [**7**, p. 390, Problem 84] for ultraspherical polynomials $P_n^{(\lambda)}(x)$.

$$(10) \quad \int_{-1}^{1} P_k^{(\lambda)}(x) P_m^{(\lambda)}(x) P_n^{(\lambda)}(x) (1 - x^2)^{\lambda-\frac{1}{2}}\, dx$$
$$= \frac{\alpha_{s-k}\alpha_{s-m}\alpha_{s-n}}{\alpha_s} \int_{-1}^{1} [P_s^{(\lambda)}(x)]^2 (1 - x^2)^{\lambda-\frac{1}{2}}\, dx$$

for $\lambda > -\frac{1}{2}$, $\lambda \neq 0$, provided that $k + m + n = 2s$ is even and a triangle with sides k, m, n exists, i.e., $|n - m| \leqq k \leqq n + m$. Here

$$\alpha_k = \binom{k + \lambda - 1}{k} = \frac{(\lambda)_k}{k!} = \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)\Gamma(\lambda)}.$$

Using (1), (3), (10), $P_n^{(\lambda)}(1) = (2\lambda)_n/n!$, and

$$\frac{P_{2n}^{(\lambda)}(x)}{P_{2n}^{(\lambda)}(1)} = \frac{P_{2n}^{(\alpha,\alpha)}(x)}{P_{2n}^{(\alpha,\alpha)}(1)} = \frac{P_n^{(\alpha,-\frac{1}{2})}(2x^2 - 1)}{P_n^{(\alpha,-\frac{1}{2})}(1)}$$
$$= (-1)^n \frac{P_n^{(-\frac{1}{2},\alpha)}(1 - 2x^2)}{P_n^{(\alpha,-\frac{1}{2})}(1)}, \quad \lambda = \alpha + \tfrac{1}{2},$$

(see [**7**, pp. 59, 81]) we obtain, for $\alpha = -\frac{1}{2}$,

$$\xi(k, m, n) = \frac{A(\beta)(-1)^{n+m+k}(2n)!(2m)!(2k)!\Gamma(n + \beta + 1)}{\Gamma(2n + 2\beta + 1)\Gamma(2m + 2\beta + 1)\Gamma(2k + 2\beta + 1)n!}$$
$$\cdot \frac{\Gamma(m + \beta + 1)\Gamma(k + \beta + 1)\Gamma(k + m - n + \beta + \frac{1}{2})}{m!k!(k + m - n)!}$$
$$\cdot \frac{\Gamma(k + n - m + \beta + \frac{1}{2})\Gamma(n + m - k + \beta + \frac{1}{2})\Gamma(n + m + k + 2\beta + 1)}{(k + n - m)!(n + m - k)!\Gamma(n + m + k + \beta + \frac{3}{2})},$$

where $A(\beta)$ is independent of k, m, n. Then using $\Gamma(n+a)/\Gamma(n+b) \sim n^{a-b}$ it is easy to see that

$$\sum_{k=n-m}^{n+m} |\xi(k,m,n)| h_k P_k^{(-\frac{1}{2},\beta)}(1)[P_n^{(-\frac{1}{2},\beta)}(1) P_m^{(-\frac{1}{2},\beta)}(1)]^{-1}$$

is bounded by

$$m^{\frac{1}{2}-\beta} \sum_{k=n-m}^{n+m} k^{\frac{1}{2}-\beta}[(k+m-n+1)(k+n-m+1)(n+m-k+1)]^{\beta-\frac{1}{2}}$$
$$= O\left(m^{\frac{1}{2}-\beta} \sum_{k=n-m}^{n+m} [(k+m-n+1)(n+m-k+1)]^{\beta-\frac{1}{2}}\right)$$
$$= O\left(\sum_{k=n-m}^{n} (k+m-n+1)^{\beta-\frac{1}{2}}\right) + O\left(\sum_{k=n}^{n+m} (n+m-k+1)^{\beta-\frac{1}{2}}\right) = O(1)$$

since $\beta < -\frac{1}{2}$. This concludes the proof of (2) for $\alpha \geqq \beta$, $\alpha \geqq -\frac{1}{2}$, and yields the following best possible result.

THEOREM 1. *Let* $R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$ *and*

$$R_n^{(\alpha,\beta)}(x) R_m^{(\alpha,\beta)}(x) = \sum_{k=|n-m|}^{n+m} \mu(k,m,n) t(k) R_k^{(\alpha,\beta)}(x),$$

where

$$\mu(k,m,n) = \int_{-1}^{1} R_n^{(\alpha,\beta)}(x) R_m^{(\alpha,\beta)}(x) R_k^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta\, dx,$$

$$\frac{1}{t(k)} = \int_{-1}^{1} [R_k^{(\alpha,\beta)}(x)]^2 (1-x)^\alpha (1+x)^\beta\, dx.$$

When $\alpha \geqq \beta > -1$ *and* $\alpha \geqq -\frac{1}{2}$, *we have*

$$\sum_{k=|n-m|}^{n+m} |\mu(k,m,n)| t(k) \leqq C,$$

where C *is independent of* n *and* m, *and if*

$$||a||_1 = \sum_{n=0}^{\infty} |a(n)| t(n) < \infty, \qquad ||b||_1 = \sum_{n=0}^{\infty} |b(n)| t(n) < \infty,$$

$$(a * b)(n) = \sum_{m=0}^{\infty} \sum_{k=|n-m|}^{n+m} a(k) b(m) \mu(k,m,n) t(k) t(m),$$

then $*$ *is a commutative and associative operation and*

$$||a * b||_1 \leqq C ||a||_1 ||b||_1.$$

If $\beta > \alpha > -1$ *and* $\beta \geqq -\frac{1}{2}$, *then we have similar results with* $R_n^{(\alpha,\beta)}(x)$ *replaced by* $P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(-1)$.

Using the same argument given in [**6**] for $\alpha \geqq \beta \geqq -\frac{1}{2}$, we see that the maximal ideal space of this Banach algebra is isomorphic to the closed interval

$[-1, 1]$. For the Fourier-Gelfand transform of $a(n)$ see [**4**] and for results which are dual to those above see [**5**].

As was pointed out in [**4**], Theorem 1 fails for $-1 < \alpha, \beta < -\frac{1}{2}$. However, there is still a Banach algebra which can be defined for these values of α and β. Let $c > 1$ be a fixed real number and set

$$\gamma(k; m, n) = \frac{\xi(k, m, n)P_k^{(\alpha,\beta)}(c)}{P_n^{(\alpha,\beta)}(c)P_m^{(\alpha,\beta)}(c)}.$$

Then we will show that

$$\sum_{k=|n-m|}^{n+m} |\gamma(k; m, n)|h_k \leqq A$$

for a constant A independent of n and m. Using

$$P_n^{(\alpha,\beta)}(c) \cong (c-1)^{-\alpha/2}(c+1)^{-\beta/2}[(c+1)^{\frac{1}{2}} + (c-1)^{\frac{1}{2}}]^{\alpha+\beta} \cdot (2\pi n)^{-\frac{1}{2}}(c^2-1)^{-\frac{1}{4}}[c + (c^2-1)^{\frac{1}{2}}]^{n+\frac{1}{2}}$$

[**7**, (8.21.9)], and $|P_n^{(\alpha,\beta)}(x)| = O(n^{-\frac{1}{2}})$, i.e. (7), we find from (1) that

$$\gamma(k; m, n) = O(k^{-1}d^{k-n-m}), \qquad d = c + (c^2-1)^{\frac{1}{2}} > 1;$$

and thus

$$\sum_{k=|n-m|}^{n+m} |\gamma(k; m, n)|h_k = O\left(\sum_{k=|n-m|}^{n+m} d^{k-n-m}\right) = O(1).$$

In a standard fashion this leads to the following theorem.

THEOREM 2. *Let $-1 < \alpha, \beta < -\frac{1}{2}$ and define $||a||_1 = \sum_{n=0}^{\infty}|a(n)|h_n$. If $||a||_1 < \infty$, $||b||_1 < \infty$, and*

$$(a \# b)(n) = \sum_{m=0}^{\infty} \sum_{k=|n-m|}^{n+m} a(k)b(m)\gamma(n; m, k)h_k h_m,$$

then $\#$ is a commutative and associative operation and

$$||a \# b||_1 \leqq A||a||_1 \, ||b||_1.$$

Also if

(11)
$$f(x) = \sum a(n)\, h_n\, P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(c),$$
$$g(x) = \sum b(n)\, h_n\, P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(c),$$
$$h(x) = \sum (a \# b)(n) h_n\, P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(c),$$

then

$$h(x) = f(x)\, g(x).$$

Following the argument in [**6**] we see that the Fourier-Gelfand transform of $a(n)$ is given by (11) and the maximal ideal space is isomorphic to the set of complex z for which

$$|z + (z^2-1)^{\frac{1}{2}}| \leqq c + (c^2-1)^{\frac{1}{2}},$$

where $(z^2-1)^{\frac{1}{2}}$ is chosen so that $|z + (z^2-1)^{\frac{1}{2}}| \geqq 1$. This is an ellipse with foci at ± 1 and the ends of its major axis at $z = \pm c$.

References

1. R. Askey, *Linearization of the product of orthogonal polynomials*, pp. 223–228 in *Problems in analysis* (Princeton Univ. Press, Princeton, N.J., 1970).
2. R. Askey and S. Wainger, *A dual convolution structure for Jacobi polynomials*, pp. 25–36 in *Orthogonal expansions and their continuous analogues*, Proc. Conference, Edwardsville, Illinois, 1967 (Southern Illinois Univ. Press, Carbondale, Illinois, 1968).
3. G. Gasper, *Linearization of the product of Jacobi polynomials*. I, Can. J. Math. *22* (1970), 171–175.
4. ——— *Linearization of the product of Jacobi polynomials*. II, Can. J. Math. *22* (1970), 582–593.
5. ——— *Positivity and the convolution structure for Jacobi series*, Ann. of Math. (to appear).
6. S. Igari and Y. Uno, *Banach algebra related to the Jacobi polynomials*, Tôhoku Math. J. *21* (1969), 668–673.
7. G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloq. Publ., Vol. 23 (Amer. Math. Soc., Providence, R.I., 1967).

Mathematics Centre,
Amsterdam, The Netherlands;
University of Wisconsin,
Madison, Wisconsin;
University of Toronto,
Toronto, Ontario