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# RINGS SATISFYING CERTAIN CONDITIONS EITHER ON SUBSEMIGROUPS OR ON ENDOMORPHISMS

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**Abstract**

We characterize rings whose multiplicative subsemigroups containing 0 and the additive inverse of each element are subrings. In addition we consider commutative rings for which every non-constant multiplicative endormorphism that preserves additive inverses is a ring endomorphism, and we show that they belong to one of three easily-described classes of rings.

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## 1. Introduction

In this paper we study associative rings $R$ possessing one of the following properties.

($\alpha$) Every (multiplicative) subsemigroup $S$ of $R$ such that $0 \in S$, and such that $a \in S$ if and only if $-a \in S$ (for every $a \in R$), is a subring of $R$.

($\beta$) Every non-constant semigroup endomorphism $\phi$ of $R$ such that $\phi(-a) = -\phi(a)$ (for every $a \in R$) is a ring endomorphism.

Throughout the paper these rings will be called $\alpha$-rings and $\beta$-rings, respectively.

The results here contained (Theorems 2.1 and 3.3) extends Theorem 1 of [9] and Theorem 1 of [11], respectively, which are in turn generalizations of theorems obtained in [4] by Cresp and Sullivan. In addition we observe that a different generalization of the work in [4] and [9] was furnished by Ligh in [8].

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In what follows $R$ will denote an associative ring, the term subsemigroup (subgroup) of $R$ will mean multiplicative subsemigroup (subgroup), and the multiplicative semigroup of $R$ will be denoted as usual by $(R, \cdot)$.

## 2. Subsemigroups

The main result of this section is the following characterization of $\alpha$-rings.

THEOREM 2.1. *A ring $R$ is an $\alpha$-ring if and only if $R$ belongs to one of the following types.*

(i) *$R$ is a finite field of order $2^m = p + 1$, where $p$ is prime and $m$ is a positive integer.*

(ii) *$R$ is a finite field of order $3^m = 2p + 1$, where $p$ is prime and $m$ is a positive integer.*

(iii) *$R$ is a nil ring of order $\leqslant 3$.*

(iv) *$R$ is the ring of order 4 whose additive group is cyclic and generated by $a$ with $a^2 = 2a$.*

The proof of the theorem will utilize the following lemmas, where $2R$ ($3R$) denotes the set $\{2a | a \in R\}$ ($\{3a | a \in R\}$). Moreover, we shall put $[y] = \{y^h | h \in \mathbf{Z}^+\}$ and $-[y] = \{-y^h | h \in \mathbf{Z}^+\}$ $(y \in R)$.

LEMMA 2.2. *If an $\alpha$-ring $R$ has a non-zero idempotent $e$, then either $2R = 0$ or $3R = 0$, and $e$ is the identity of $R$.*

PROOF. Since $R$ is an $\alpha$-ring, the subset $\{0, e, -e\}$ is a subring and contains $2e$. Then, either $2e = 0$ or $3e = 0$. Let $2e = 0$. Then, for every $x \in R$, the subset $-[2x] \cup [2x] \cup \{0, e\}$ is a subring by Property ($\alpha$), so it contains $e + 2x$. Since $e \neq 0$, it is immediate that $e + 2x = e$, whence $2R = 0$. Analogously, if $3e = 0$, then by investigating the subring $-[3x] \cup [3x] \cup \{0, e, -e\}$, we find that $3R = 0$. When $2R = 0$, every subsemigroup of $R$ contains the additive inverses of its own elements; thus $e$ is the identity of $R$ by Lemma 2 of [9]. Now suppose $3R = 0$. Let $x \in R$, and put $a = xe - exe$. Since $a^2 = ea = 0$, $ae = a$, and $R$ is an $\alpha$-ring, the subset $\{0, e, -e, a, -a\}$ is a subring and contains $a + e$. Hence it immediately follows that $a = 0$, that is, $xe = exe$. By a similar argument it is proved that $ex = exe$, so $e$ is a central idempotent. At this point, the subset $-[x - xe] \cup [x - xe] \cup \{0, e, -e\}$ is a subring, by Property ($\alpha$), and it contains $x + e - xe$. Now it is immediate that $x + e - xe = e$, that is, $x = xe$, and that $e$ is the identity of $R$.

**Lemma 2.3.** *If $R$ is an $\alpha$-ring with identity, and $2R = 0$, then $R$ is a finite field of order $2^m = p + 1$, where $p$ is prime and $m$ is a positive integer.*

**Proof.** If $2R = 0$, every subsemigroup of $R$ contains the additive inverses of its own elements. Therefore the statement follows from Theorem 2 of [4].

**Lemma 2.4.** *If $R$ is an $\alpha$-ring with identity, and $3R = 0$, then $R$ is a periodic field.*

**Proof.** Let $e$ be the identity of $R$. For every $x \in R \setminus 0$, the subset $-[x] \cup [x] \cup \{0, e, -e\}$ is a subring by Property ($\alpha$), so it contains $x + e$. Hence it easily follows that $x = x^2 f(x)$ for some polynomial $f(\lambda) \in \mathbf{Z}[\lambda]$. Thus $R$ is commutative by a well-known theorem of Herstein [6], it is periodic by a theorem of Chacron [1, Proposition 2], and $(R, \cdot)$ is union of groups [3, Theorem 4.3]. Furthermore, the only idempotents of $R$ are 0 and $e$ by Lemma 2.2; thus we may immediately conclude that $R$ is a periodic field.

**Lemma 2.5.** *Let $R$ be an $\alpha$-field. If $3R = 0$, then $R$ has a unique element of order 2, and every finite subgroup of even order has order $2p$ with $p$ prime $\geqslant 1$.*

**Proof.** Let $e$ be the identity of $R$. Since $3e = 0$ implies $-e \neq e$, and $(-e)^2 = e$, it follows that $R$ contains an element of order 2. Let $f$ be any element of $R$ having order 2; since $R$ is an $\alpha$-field, the subset $H = \{0, e, -e, f, -f\}$ is a subring and, obviously, a finite field. Thus $H \setminus 0$ is a finite cyclic group, with a unique element of order 2. Hence $f = -e$. Now let $G$ be a finite subgroup of $R$ having even order $2rs$ with $r > 1$, $s > 1$. Then $G$ contains $-e$, whence $-x = -ex \in G$ for every $x \in G$. So, by Property ($\alpha$), $G \cup 0$ is a subfield of $R$. From this and from $3G = 0$, it follows that $|G \cup 0| = 3^j$ for some positive integer $j$. Therefore

$$(1) \qquad 2rs = 3^j - 1 \qquad (j > 1),$$

Moreover $G$, being an abelian group, contains a subgroup $A$ of order $2r$ and a subgroup $B$ of order $2s$. The same argument employed above for $G$ shows that $A \cup 0$ and $B \cup 0$ are subfields of $G \cup 0$, of orders $3^h$ and $3^k$, respectively, ($h$, $k$ positive integers). Then we have

$$(2) \qquad 2r = 3^h - 1, \qquad 2s = 3^k - 1, \qquad (h, k > 1),$$

and, using relations (2) in (1), we deduce that

$$3^j = 2rs + 1 = \frac{(3^h - 1)(3^k - 1)}{2} + 1 = \frac{3^{h+k} - 3^h - 3^k + 3}{2}.$$

This is a contradiction, since $h, k, j > 1$. So $G$ must have order $2p$ with $p$ prime $\geqslant 1$.

LEMMA 2.6. *Let $R$ be an $\alpha$-field. If $3R = 0$, then $R$ is a finite field of order $3^m = 2p + 1$ with $p$ prime, $p \geqslant 1$ and $m$ a positive integer.*

PROOF. Let $e$ be the identity of $R$. If $R = \{0, e, -e\}$, we have $|R| = 3$ and the statement is true. Otherwise, we have $R \setminus \{0, e, -e\} \neq \varnothing$. Let $x, y \in R \setminus \{0, e, -e\}$ and let $X = \langle x, -e \rangle$ and $Y = \langle y, -e \rangle$ be the subgroups generated by $x$, $-e$ and by $y$, $-e$, respectively. By Lemma 2.4, $X$ and $Y$ have finite orders, which, moreover, are even numbers, since $-e \in X \cap Y$. Consequently, $|X| = 2p$, $|Y| = 2q$ and $|X \cap Y| = 2r$ for some primes $p, q > 1$ and $r \geqslant 1$, in view of Lemma 2.5. Suppose $X \neq Y$. Since $2r$ divides both $2p$ and $2q$, we have either $r = 1$, or $r = p = q$. In the first case $XY$ is a subgroup of order $2pq$, in contradiction to Lemma 2.5. In the second we have $X = X \cap Y = Y$, which is another contradiction. Thus $X = Y$, whence $R \setminus 0 = X$. At this point, we may conclude that $|R| = 2p + 1$. Moreover, $3R = 0$ implies that $|R| = 3^m$ for some positive integer $m$, which proves the statement.

REMARK 2.7. The primes of the form $2^m - 1$ which appear in Lemma 2.3 are the well-known Mersenne primes, where $m$ is necessarily prime. Analogously, it is easily verifiable that, if $p$ is a prime and $m$ a positive integer satisfying the condition $3^m = 2p + 1$, then $m$ must be prime. In fact, suppose $m > 1$ and put $m = hq$ with $q$ prime $> 1$ and $h$ positive integer. Then $2p = 3^{hq} - 1 = (3^h - 1)(3^{h(q-1)} + \cdots + 3^h + 1)$, whence $3^h - 1 = 2$. Thus $h = 1$, and $m = q$ is prime. Pairs $(m, p)$ satisfying the above condition do actually exist; we include a small table of such pairs

| $m$ | 1 | 3 | 7 | 13 | $\cdots$ |
|---|---|---|---|---|---|
| $p$ | 1 | 13 | 1093 | 797161 | $\cdots$ |

LEMMA 2.8. *Let $R$ be an $\alpha$-ring without non-zero idempotents. Then, for every $x \in R$, either $x^2 = 0$ or $x^2 = 2x$.*

PROOF. Let $|R| > 1$, and let $x \in R \setminus 0$. The subset $H = -[x] \cup [x] \cup \{0\}$ is a subring by Property ($\alpha$) and contains $x - x^2$. Since $x \neq x^2$, we have either $x - x^2 = x^h$ or $x - x^2 = -x^h$ for some positive integer $h$. If $h > 1$, we have $x = x^2 f(x) = xf(x)x$ for some polynomial $f(\lambda) \in \mathbf{Z}[\lambda]$, and $xf(x)$ is a non-zero idempotent, which is a contradiction. Thus, for every $x \in R$, we have either $x^2 = 0$ or $x^2 = 2x$.

In what follows we shall denote by $R^2$ the set $\{xy | x, y \in R\}$.

LEMMA 2.9. *Let $R$ be an $\alpha$-ring without non-zero idempotents. Then either $R^2 = 0$ and $|R| \leqslant 3$, or $R$ is the ring of order 4 whose additive group is cyclic generated by an element $a$ satisfying the relation $a^2 = 2a$.*

PROOF. Let $x \in R$ with $x^2 = 0$. Then the subset $\{0, x, -x\}$ is a subring by Property ($\alpha$), and it contains $2x$. Hence it easily follows that either $2x = 0$ or $3x = 0$. Next, let $y \in R$ with $y^2 \neq 0$. Then from Lemma 2.8 it follows that $y^2 = 2y$ and $(-y)^2 = -2y$, whence $y^2 = 2y = -2y$, and also $y^3 = 2y^2 = 4y = 0$. At this point we may distinguish two cases:

(1) $R$ satisfies the identity $x^2 = 0$. Then, for every $x \in R$, we have either $2x = 0$ or $3x = 0$. Since the subsets $H = \{x \in R | 2x = 0\}$ and $K = \{x \in R | 3x = 0\}$ are additive subgroups of $R = H \cup K$, we must have either $R = H$ or $R = K$. In the first case, every subsemigroup of $R$ contains the zero and the additive inverses of its own elements. So, $R^2 = 0$ and $|R| \leqslant 2$ follows from Theorem 1 of [4]. If $R = K$, let us suppose $|R| > 1$ and let $x, y \in R \setminus 0$. Then we have $0 = (x + y)^2 = xy + yx$, whence $xyx = 0$. Therefore, the subset $\{0, x, -x, xy, -xy\}$ is a subring by Property ($\alpha$), and it contains $x + xy$. This implies that $xy = 0$. Hence, the subset $\{0, x, -x, y, -y\}$ is also a subring by Property ($\alpha$), and it contains $x + y$. Now it is immediate that either $y = x$ or $y = -x$. Thus $R = \{0, x, -x\}$ and this implies that $R^2 = 0$ and $|R| = 3$.

(2) $R$ contains an element $y$ such that $y^2 \neq 0$. Then $4y = 0$ and, for every $w \in R$, we must have either $4w = 0$ or $3w = 0$. Repeating the argument used in (1), we see that $4w = 0$ for every $w \in R$. Now, for every $x \in R \setminus 0$ with $x^2 = 0$, we have $2x = 0$. Consequently the subset $\{0, x, 2y\}$ is a subring by Property ($\alpha$), and it contains $x + 2y$. Hence it easily follows that $x = 2y$, and that $2y$ is the unique element of $R$ with index of nilpotence 2. Now, for every $z \in R$ with $z^2 \neq 0$, we have $z^2 = 2z = 2y$. Hence, $(yz)^2 \neq 0$ implies that $(yx)^2 = 2yz = z^3 = 0$, which is a contradiction. So we must have $(yz)^2 = 0$, whence $yz = 2y$. In the same way we find that $zy = 2y$; thus the subset $\{0, y, -y, z, -z, 2y\}$ is a subring by Property ($\alpha$), and it contains $y + z$. At this point it is immediate that either $z = y$ or $z = -y$. So $R = \{0, y, -y, 2y\}$ is the ring of order 4 described in the lemma.

PROOF OF THEOREM 2.1. From the preceding lemmas we immediately deduce that every $\alpha$-ring belongs to one of the types listed in the statement. The converse is immediately verifiable.

REMARK 2.10. Let $R$ be a field of order $3^m (m \geqslant 1)$ with identity $e$. Since $3e = 0$, we have $2e \in R \setminus \{0, e\}$; thus $R$ has a subsemigroup containing the zero which is not a subring. Next, let $R$ be a nil ring of order 3. Since the additive group of $R$ is cyclic, we have $R = \{0, a, -a\}$ and $a^2 = 0$. Hence the subset $\{0, a\}$ is a subsemigroup of $R$ containing the zero, but it is not a subring. Finally, let $R$ be the ring of order 4 with the additive group generated by an element $a$ satisfying the relation $a^2 = 2a$. It is immediate that the subset $\{0, a, 2a\}$ is a subsemigroup of $R$ but not a subring. That being stated, let $R$ be a ring all of whose subsemigroups containing the zero are subrings. Obviously, $R$ is a $\alpha$-ring, and consequently it is one of the rings listed in the statement of Theorem 2.1. But from the above it follows that if $|R| > 2$, then $R$ is necessarily a field of order $2^m = p + 1$, where $p$ is a prime [9, Theorem 1].

## 3. Endomorphisms

The purpose of this section is to describe commutative $\beta$-rings. We recall that an ideal $I$ of a ring $R$ is said to be *completely prime* if $a, b \in R$, $ab \in I$ imply $a \in I$ or $b \in I$. $R$ is *completely prime* if the zero ideal is a completely prime ideal in $R$.

LEMMA 3.1. *Let $R$ be a $\beta$-ring. If $I$ is a proper, completely prime ideal of $R$, then $I = 0$.*

The proof is analogous to that of [7, Lemma 1].

In what follows we shall use the terminology of [10].

LEMMA 3.2. *Let $R$ be a $\beta$-ring. If $(R, \cdot)$ is a semilattice of archimedean semigroups, then either $R$ is completely prime or $R$ is a nil ring.*

PROOF. From [2, Theorem A and Theorem 1.3] it follows either that $(R, \cdot)$ is archimedean, or that it contains a proper, completely prime semigroup ideal $I$. (We remark that in [2] the term "prime" stands for "completely prime".) In the first case $R$ is obviously a nil ring. In the second case, let $\phi$ be the map of $R$ into $R$ defined by $\phi(x) = 0$ for $x \in I$, and $\phi(x) = x$ for $x \in R \setminus I$. It is easily seen that $\phi$ is a non-constant semigroup endomorphism of $R$, and that $x \in I$ if and only if $-x \in I$. Hence $\phi(-x) = -\phi(x)$ and, since $R$ is a $\beta$-ring, $\phi$ is a ring endomorphism whose kernel is $I$. Thus $I$ is a ring ideal. Hence $I = 0$ by Lemma 3.1, and so $R$ is completely prime.

Now we are able to state the following.

THEOREM 3.3. *A commutative $\beta$-ring belongs to one of the following types*

(i) *$R$ is a ring of order $\leqslant 3$;*

(ii) *$R$ is the ring of order 4 whose additive group is cyclic generated by $a$ with $a^2 = 2a$;*

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(iii) *$R = R^2$ is the direct sum of a ring $P$ satisfying the identities $x^2 = 2x = 0$ and a ring $Q$ satisfying the identities $x^3 = 3x = 0$.*

PROOF. Let $\chi$ be the map of $R$ in $R$ defined by $\chi(a) = a^3$ for every $a \in R$. If $\chi$ is non-constant, then, since $R$ is a commutative $\beta$-ring, $\chi$ is a ring endomorphism. Then $R$ satisfies the identity

$$(3) \qquad (a+b)^3 = a^3 + b^3.$$

If $\chi$ is constant, we have $a^3 = \chi(a) = \chi(0) = 0$, and (3) continues to hold. Analogously, utilizing the map $\psi$ defined by $\psi(a) = a^5$ $(a \in R)$, we obtain the identity

$$(4) \qquad (a+b)^5 = a^5 + b^5.$$

Now we recall that, since $R$ is commutative, $(R, \cdot)$ is a semilattice of archimedean semigroups [3, Theorem 4.13]; consequently $R$ is either completely prime or a nil ring, by Lemma 3.2. In the first case, if $|R| > 1$, we obtain from (3) that $3a + 3b = 0$ for every $a, b \in R \setminus 0$. Hence, when $a = b$, it follows that $6a = 0$. Utilizing these relations in (4), we find that $a^3 - a^2b - ab^2 + b^3 = 0$. Replacing $a$ by $-a$ and summing the two relations, we obtain $2a^2b - 2b^3 = 0$, whence $2a^2 = 2b^2$. Moreover, $3a + 3b = 0$ implies that $9a^2 = 9b^2$, whence $a^2 = b^2$. Now if $a + b \neq 0$, then $(a+b)(a-b) = 0$ implies that $a = b$. Therefore, either $R = \{0, a\}$ or $R = \{0, a, -a\}$. Now let us suppose that $R$ is a nil ring. If $R^2 \subset R$, let $a$ be an element of $R \setminus R^2$, $h$ the least positive integer such that $a^{2h} = 0$, and $J = R \setminus \{a, -a\}$. Let $\phi$: $R \to R$ be defined by $\phi(a) = a^h$, $\phi(-a) = -a^h$ and $\phi(x) = 0$ otherwise. Then $\phi$ is non-constant, and it is a ring endomorphism by Property $(\beta)$. Therefore, for every $x \in J$, we have $\phi(a + x) = a^h \neq 0$. Hence, either $a + x = a$ or $a + x = -a$, whence either $x = 0$ or $x = -2a$. So we have $R = \{0, a, -a, -2a\}$. At this point, either $|R| \leqslant 3$ or the elements $0, a, -a, -2a$ are distinct; in this case the additive group of $R$ is cyclic, and $-2a = 2a$. Moreover, we cannot have $a^2 = 0$, since then we would have $h = 1$, whence $\phi(2a) = 2\phi(a) = 2a$, and $2a \in \{0, a, -a\}$, which is a contradiction. Thus $a^2 \neq 0$, and it is immediate that $a^2 = 2a$. Finally, we have to examine the case $R = R^2$. First, let us verify that $x^3 = 3x^2 = 6x = 0$ for every $x \in R$. In fact, putting $b = a^2$ in (3) and (4), we find that $3a^4 + 3a^5 = 0$ and $5a^6 + 10a^7 + 10a^8 + 5a^9 = 0$, whence $3a^4 = 5a^6 = 0$, and consequently $a^6 = 0$. Then $(a+b)^6 = 0$ for every $a, b \in R$ and, using again $3a^4 = a^6 = 0$, we obtain $20a^3b^3 = 0$. Moreover, from (3) it follows that $3a^2b + 3ab^2 = 0$, whence $3a^3b^3 = -3a^4b^2 = 0$. Therefore $a^3b^3 = 0$, and this, in view of the fact that $R = R^2$, implies that $x^3 = 0$ for every $x \in R$. Now $3a^2b + 3ab^2 = 0$ implies that $3a^2b^2 = 0$, whence $3x^2 = 0$ for every $x \in R$. Finally, $3(a+b)^2 = 0$ implies that $6ab = 0$, that is, $6x = 0$ for every $x \in R$. That being stated, we let $P = \{x \in R \mid 2x = 0\}$, and we let $Q = \{x \in R \mid 3x = 0\}$.

It is immediate that $P$ and $Q$ are ring ideals of $R$, that $P \cap Q = 0$ and that, for every $x \in R$, we have $x = 7x = 3x + 4x$, with $3x \in P$ and $4x \in Q$. Therefore $R$ is the direct sum of $P$ and $Q$. For every $x \in P$, we have $2x^2 = 0$ and, since we have shown that $3x^2 = 0$, we may conclude that $x^2 = 0$. So the proof is complete.

REMARK 3.4. It is immediate that the rings of types (i) and (ii) described in the statement of Theorem 3.3 are $\beta$-rings. As regards the rings of type (iii), Duncan and Macdonald have shown in [5] that rings like $P$ (called *power rings* in [11]) do exist. A similar argument can be used to show that rings like $Q$ also exist. At this point, the existence of rings satisfying condition (iii) is assured. But we do not know whether they are $\beta$-rings, and it is still not known whether power rings are $\varepsilon'$-rings.

REMARK 3.5. From Theorem 3.3 we may easily deduce Theorem 1 of [11]. In fact, if $R$ is a ring of order 3, we have necessarily $R = \{0, a, -a\}$, and it is easily seen that the map $\phi$ defined by $\phi(0) = 0$, and $\phi(a) = \phi(-a)$ is a non-constant semigroup endomorphism which does not preserve addition. The same result may be obtained when $R$ is the ring of type (ii), with the map $\phi$ defined by $\phi(0) = 0$, $\phi(a) = \phi(-a) = a$, and $\phi(2a) = 2a$. Further, let us suppose that $R$ is a commutative ring with the property ($\varepsilon'$) introduced in [11]. Obviously, $R$ is a $\beta$-ring and, if $|R| > 2$, it follows from the above that $R$ is of type (iii). Now the function $\phi$ defined by $\phi(x) = x^2$ for every $x \in R$ is a ring endomorphism by property ($\varepsilon'$), and it induces in $R$ the identity $(a + b)^2 = a^2 + b^2$, whence $2ab = 0$. Since $R = R^2$, this implies that $2x = 0$ for every $x \in R$. Thus $Q = 0$, and $R = P$ is a power ring.

REMARK 3.6. In Theorem 3.3 the hypothesis of commutativity may be weakened. Following the terminology used for semigroups, we say that a ring $R$ is *medial* if $abcd = acbd$ for every $a, b, c, d \in R$. A medial ring need not be commutative, as is shown by the ring of real square matrices of the form $\left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right]$. The authors have proved that Theorem 3.3 continues to hold if the word "commutative" is replaced by "medial", but the proof is here omitted.

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