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Properties of the $\mathcal{M}$-Harmonic Conjugate Operator

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Abstract. We define the $\mathcal{M}$-harmonic conjugate operator K and prove that it is bounded on the non-isotropic Lipschitz space and on BMO. Then we show K maps Dini functions into the space of continuous functions on the unit sphere. We also prove the boundedness and compactness properties of $\mathcal{M}$-harmonic conjugate operator with L^p symbol.

1 Introduction

Let B be the unit ball of $\mathbb{C}^n$ with norm $|z| = \langle z, z\rangle^{1/2}$ where $\langle \, , \, \rangle$ is the Hermitian inner product, S be the unit sphere and σ be the rotation-invariant probability measure on S as we follow standard notations of [5] throughout the paper. For $z \in B$, $\xi \in S$, we define the kernel $K(z, \xi)$ by

$$iK(z, \xi) = 2C(z, \xi) - P(z, \xi) - 1$$

where $C(z, \xi) = (1 - \langle z, \xi\rangle)^{-n}$ is the Cauchy kernel and $P(z, \xi) = (1 - |z|^2)^n \cdot |1 - \langle z, \xi\rangle|^{-2n}$ is the invariant Poisson kernel. For each $\xi \in S$, the kernel $K(\, , \xi)$ is $\mathcal{M}$-harmonic. And for all $f \in A(B)$, the ball algebra, such that $f(0)$ is real, the reproducing property of $2C(z, \xi) - 1$ (3.2.5 of [5]) gives

$$\int_S K(z, \xi) \operatorname{Re} f(\xi) \, d\sigma(\xi) = -i\big(f(z) - \operatorname{Re} f(z)\big) = \operatorname{Im} f(z),$$

For that reason we call $K(z, \xi)$ the $\mathcal{M}$-harmonic conjugate kernel.

For $f \in L^1(S)$, we define Kf on S by

$$(Kf)(\zeta) = \lim_{r \to 1} \int_S K(r\zeta, \xi) f(\xi) \, d\sigma(\xi)$$

Since the limit exists almost everywhere (6.2.3 of [5]), Kf is well defined on S and we call Kf the $\mathcal{M}$-harmonic conjugate function of f. For $n = 1$, the definition of Kf is the same as the classical harmonic conjugate function ([1], [2]). Many properties of $\mathcal{M}$-harmonic conjugate function come from those of Cauchy integral and invariant Poisson integral. Indeed the following properties of Kf follow directly from Chapters 5 and 6 of [5].

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1. K is of weak type (1,1) and bounded on $L^p(S)$ for $1 < p < \infty$.
2. If $f \in L^1(S)$, then $Kf \in L^p(S)$ for all $0 < p < 1$.
3. If $f \in L \log L$, then $Kf \in L^1(S)$.
4. If f is in the euclidean Lipschitz space of order α for $0 < \alpha < 1$, then so is Kf.

In this paper, we show additional properties of $\mathcal{M}$-harmonic conjugate operator; we show boundedness of Kf on BMO and on the nonisotropic Lipschitz space, and then we show boundedness and compactness properties of $\mathcal{M}$-harmonic conjugate operator with L^p symbol.

2 $\mathcal{M}$-Harmonic Conjugate Operator

Definition 2.1 Let $Q = Q(\xi, \delta) = \{\eta \in S : d(\xi, \eta) = |1 - \langle \xi, \eta \rangle|^{1/2} < \delta\}$ be a nonisotropic ball of S. The space BMO consists of all $f \in L^1(S)$ satisfying

$$\sup_Q \frac{1}{\sigma(Q)} \int_Q |f - f_Q| \, d\sigma = \|f\|_{\mathrm{BMO}} < \infty,$$

where f_Q is the average of f over Q.

We denote $f \in \mathrm{lip}_\alpha$ the nonisotropic Lipschitz space of order α $(0 < \alpha < 2)$ if

$$\sup_{\xi, \eta \in S} \frac{|f(\xi) - f(\eta)|}{d(\xi, \eta)^\alpha} = \|f\|_{\mathrm{lip}_\alpha} < \infty.$$

BMO and lip_α become Banach spaces provided that we identify functions which differ by a constant. The next lemma, using similar idea as [4], tells that we can regard BMO as the limit of lip_α as α decreases to zero.

Lemma 2.2 *Let $f \in L^1(S)$ and $0 < \alpha \leq 2$, then the norm $\|f\|_{\mathrm{lip}_\alpha}$ is equivalent to*

$$\sup_Q \frac{1}{\sigma(Q)^{1+\alpha/2n}} \int_Q |f - f_Q| \, d\sigma.$$

Proof Suppose that $f \in \mathrm{lip}\,\alpha$. Let $Q = Q(\xi, \delta)$, then since $\sigma(Q) \approx \delta^{2n}$, we have

$$\begin{aligned} |f(\xi) - f_Q| &\leq \frac{1}{\sigma(Q)} \int_Q |f(\xi) - f(\eta)| \, d\sigma(\eta) \\ &\leq \|f\|_{\mathrm{lip}_\alpha} \frac{1}{\sigma(Q)} \int_Q d(\xi, \eta)^\alpha \, d\sigma(\eta) \\ &\leq C \|f\|_{\mathrm{lip}_\alpha} \sigma(Q)^{\alpha/2n}. \end{aligned}$$

Thus

$$\sup_Q \frac{1}{\sigma(Q)^{1+\alpha/2n}} \int_Q |f - f_Q| \, d\sigma \leq C \|f\|_{\mathrm{lip}_\alpha}.$$

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Conversely, suppose

$$\sup_Q \frac{1}{\sigma(Q)^{1+\alpha/2n}} \int_Q |f - f_Q| \, d\sigma \le C.$$

Fix $\xi, \eta \in S$. Let $\delta = 2|1 - \langle \xi, \eta \rangle|^{1/2}$ and $Q = Q(\xi, \delta)$. Then we get

$$|f(\xi) - f(\eta)| \le |f(\xi) - f_Q| + |f_Q - f(\eta)| = I + II.$$

We will only estimate I, since the estimate of II is identical. Inductively, choose a sequence of nonisotropic balls $\{Q_k\}$ such that $k = 1, 2, 3, \dots,$

$$Q_k \searrow \{\xi\} \quad \text{as} \quad k \to \infty,$$

$$\sigma(Q_k) = \frac{1}{2}\sigma(Q_{k-1}),$$

$$Q_0 = Q.$$

Then

$$I \le |f(\xi) - f_{Q_k}| + \sum_{j=1}^{k} |f_{Q_j} - f_{Q_{j-1}}| = I_1 + I_2.$$

As $k \to \infty$, I_1 converges to 0 for almost all ζ. So it suffices to estimate I_2. Observe that

$$\begin{aligned} I_2 &\le \sum_{j=1}^{k} \frac{1}{\sigma(Q_j)} \int_{Q_j} |f - f_{Q_{j-1}}| \, d\sigma \\ &\le 2 \sum_{j=1}^{k} \frac{1}{\sigma(Q_{j-1})} \int_{Q_{j-1}} |f - f_{Q_{j-1}}| \, d\sigma \\ &\le 2C \sum_{j=1}^{k} \sigma(Q_{j-1})^{\alpha/2n} \\ &= 2C\sigma(Q)^{\alpha/2n} \sum_{j=1}^{k} \frac{1}{2^{j\alpha/2n}}. \end{aligned}$$

Since $\delta = 2|1 - \langle \xi, \eta \rangle|^{1/2}$, we have $I_2 \le Cd(\xi, \eta)^\alpha$. Thus we have $|f(\xi) - f(\eta)| \le Cd(\xi, \eta)^\alpha$ for almost all ξ, η. Since f is a representation of some equivalent class in $L^1(S)$, we can redefine f so that

$$|f(\xi) - f(\eta)| \le Cd(\xi, \eta)^\alpha \quad (\xi, \eta \in S).$$

Therefore the proof is complete. ■

Theorem 2.3 *K is bounded on* lip_α $(0 < \alpha < 1)$, *and on* BMO.

Proof To show the bounedeness of K on lip_α, by Lemma 2.2 and the triangle inequality, it suffices to show that that for every $f \in \mathrm{lip}_\alpha$ there is a constant $\lambda = \lambda(Q, f)$ such that

$$\frac{1}{\sigma(Q)^{1+\alpha/2n}} \int_Q |Kf(\eta) - \lambda| \, d\sigma(\eta) \le C(\alpha) \|f\|_{\mathrm{lip}_\alpha} \tag{2.1}$$

where $C(\alpha)$ is a constant, independent of Q and f.

For each $Q = Q(\xi_Q, \delta)$, we write

$$\begin{aligned} f(\eta) &= \big(f(\eta) - f_Q \big) \chi_{2Q}(\eta) + \big(f(\eta) - f_Q \big) \chi_{S \setminus 2Q}(\eta) + f_Q \\ &= f_1(\eta) + f_2(\eta) + f_Q. \end{aligned}$$

Since $K f_Q = 0$, we have

$$Kf = Kf_1 + Kf_2.$$

Define

$$g(z) = \int_S \big(2C(z, \xi) \ - 1 \big) f_2(\xi) \, d\sigma(\xi).$$

Then it is continuous on $B \cup Q$. By setting $\lambda = -ig(\xi_Q)$ in (2.1), we shall prove the theorem. The integral in (2.1) is estimated as

$$\begin{aligned} \int_Q |Kf(\eta) + ig(\xi_Q)| \, d\sigma(\eta) &\le \int_Q |Kf_1| \, d\sigma + \int_Q |Kf_2 + ig(\xi_Q)| \, d\sigma \\ &= I_1 + I_2. \end{aligned}$$

Estimate of I_1: By Hölder's inequality we get

$$\begin{aligned} \frac{1}{\sigma(Q)} \int_Q |Kf_1| \, d\sigma &\le \left(\frac{1}{\sigma(Q)} \int_Q |Kf_1|^2 \, d\sigma \right)^{1/2} \\ &\le \left(\frac{1}{\sigma(Q)} \int_S |Kf_1|^2 \, d\sigma \right)^{1/2} \le \frac{C}{\sigma(Q)^{1/2}} \|f_1\|_2, \end{aligned}$$

since K is bounded on $L^2(S)$. Now by replacing f_1 by $(f - f_Q)\chi_{2Q}$, we get

$$\begin{aligned} \|f_1\|_2 &= \left(\int_{2Q} |f - f_Q|^2 \, d\sigma \right)^{1/2} \\ &\le \left(\int_{2Q} |f - f_{2Q}|^2 \, d\sigma \right)^{1/2} + \sigma(2Q)^{1/2} |f_{2Q} - f_Q|. \end{aligned}$$

Further, using Lemma 2.2 and triangle inequalities, we see

$$\frac{1}{\sigma(Q)} \int_Q |Kf_1| \, d\sigma \le C_1 \sigma(Q)^{\alpha/2n} \|f\|_{\mathrm{lip}_\alpha} \left(1 + 2^{2n} \left(\frac{\sigma(2Q)}{\sigma(Q)} \right)^{1/2} \right).$$

Estimate of I_2: Since $f_2 \equiv 0$ on $2Q$, we have

$$I_2 = \int_Q |f_2 + iKf_2 - g(\xi_Q)| \, d\sigma$$
$$\leq \int_{S\setminus 2Q} 2|f_2(\eta)| \int_Q |C(\xi,\eta) - C(\xi_Q,\eta)| \, d\sigma(\xi) \, d\sigma(\eta).$$

By Lemma 6.1.1 of [5], we get an upper bound such that

$$I_2 \leq C_2 \delta \sigma(Q) \int_{S\setminus 2Q} \frac{|f_2(\eta)|}{|1 - \langle \eta, \xi_Q \rangle|^{n+1/2}} \, d\sigma(\eta). \tag{2.2}$$

where C_2 is an absolute constant. Let $\eta \in S \setminus 2Q$. Then

$$|f(\eta) - f_Q| \leq \frac{1}{\sigma(Q)} \int_Q |f(\eta) - f(\xi)| \, d\sigma(\xi)$$
$$\leq C_3 \|f\|_{\mathrm{lip}_\alpha} \frac{1}{\sigma(Q)} \int_Q d(\eta,\xi)^\alpha \, d\sigma(\xi).$$

Since $\xi \in Q$, by the triangle inequality we have

$$d(\eta,\xi)^\alpha \leq C_4 \big(d(\eta,\xi_Q)^\alpha + d(\xi_Q,\xi)^\alpha \big)$$
$$\leq C_4 \big(d(\eta,\xi_Q)^\alpha + \delta^\alpha \big).$$

Thus

$$|f(\eta) - f_Q| \leq C_5 \|f\|_{\mathrm{lip}_\alpha} \big(d(\eta,\xi_Q)^\alpha + \delta^\alpha \big)$$

where the constant C_5 depends on α. The integral of (2.2) is bounded as follows

$$\int_{S\setminus 2Q} \frac{|f_2(\eta)|}{|1 - \langle \eta, \xi_Q \rangle|^{n+1/2}} \, d\sigma(\eta) \leq C_5 \|f\|_{\mathrm{lip}_\alpha} \int_{S\setminus 2Q} \frac{|1 - \langle \eta, \xi_Q \rangle|^{\alpha/2} + \delta^\alpha}{|1 - \langle \eta, \xi_Q \rangle|^{n+1/2}} \, d\sigma(\eta).$$

Since $0 < \alpha < 1$, the direct calculation as in 6.1.3 of [5] shows that the right hand side of the above is less than or equal to

$$C' \frac{1}{1-\alpha} \delta^{\alpha - 1} \|f\|_{\mathrm{lip}_\alpha}$$

where the constant C' is independent of δ and f. Therefore, there is a constant C'' depending on α such that

$$I_2 = C'' \delta^{2n+\alpha} \|f\|_{\mathrm{lip}_\alpha}.$$

Thus the proof of boundedness on lip_α is complete.

Boundedness of K on BMO can be shown by the same way as on lip_α, once we use the fact that if $f \in \mathrm{BMO}$ and $1 < p < \infty$, then

$$\sup_Q \Big(\frac{1}{\sigma(Q)} \int_Q |f - f_Q|^p \, d\sigma \Big)^{1/p} \leq C_p \|f\|_{\mathrm{BMO}},$$

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and

$$|f_{2^kQ} - f_Q| \le 2^{2n} k \|f\|_{\mathrm{BMO}}$$

for every positive integer k. ■

We define the modulus of continuity ω_φ of a function ω_φ on S by

$$\omega_\varphi(t) = \sup\{|\varphi(\xi) - \varphi(\eta)| : |\xi - \eta| \le t\}.$$

We say that φ is a Dini function if

$$\int_0^\alpha \omega_\varphi(t) \frac{dt}{t} < \infty$$

for some $\alpha > 0$.

Proposition 2.4 *If f is a Dini function, then Kf is continuous on S.*

Proof It is enough to show that the function

$$F(z) = \int_S f(\xi) K(z, \xi)\, d\sigma(\xi)$$

is uniformly continuous on B.

First, we extend f to a continuous function on $\bar{B}$, in such a way that $f(z) = f(z/|z|)$ for $1/2 \le |z| \le 1$. And then we define $G(z, \xi) = \big(f(\xi) - f(z)\big) K(z, \xi)$ for $z \in \bar{B}, \xi \in S, z \ne \xi$.

Let $z = r\eta$ for $\frac{1}{2} \le r \le 1$ and $\eta \in S$.

Then there is a constant c_n, depending only on n, such that

$$|G(z, \xi)| \le c_n \omega(|\eta - \xi|) |C(\eta, \xi)| = c_n |F_\eta(\xi)|,$$

since $P(z, \xi) \le 2^n |C(z, \xi)| \le 2^{2n} |C(\eta, \xi)|$. From 6.5.2 of [5],

$$\int_S F_\eta(\xi)\, d\sigma(\xi) < \infty$$

and the family $\{F_\eta \mid \eta \in S\}$ is unitary invariant. Hence $\{G(z,) \mid z \in \bar{B}\}$ is uniformly integrable. Then we apply the Vitali's theorem to get that function

$$H(z) = \int_S G(z, \xi)\, d\sigma(\xi)$$

is continuous on $\bar{B}$. However, for $z \in B$, $F(z) = H(z)$ since

$$\int_S K(z, \xi)\, d\sigma(\xi) = 0.$$

Therefore, F is uniformly continuous on B and this completes the proof. ■

3 $\mathcal{M}$-Harmonic Conjugate Operator With Symbol

Definition 3.1 Let $1 \le p, q \le \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. For $\varphi \in L^q(S)$, we define the operator K_φ on $L^p(S)$ by $(K_\varphi f)(\xi) = K(\varphi f)(\xi)$ for $\xi \in S$.

Theorem 3.2

(a) *For $\varphi \in L^q(S)$ $(1 < q < \infty)$, K_φ is bounded on $L^p(S)$ if and only if $\varphi \in L^\infty(S)$.*
(b) *For $\varphi \in L^2(S)$, K_φ is compact on $L^2(S)$ if and only if $\varphi = 0$.*

Proof First we prove (a). Suppose $\varphi \in L^\infty$. If $f \in L^p$, then $\varphi f \in L^p$. Since K is bounded on $L^p(S)$, it is obvious that K_φ is bounded on $L^p(S)$. Conversely, suppose the operator K_φ is bounded on L^p. Now we define for $z \in B$ and $\xi \in S$,

$$P_z(\xi) = \frac{(1-|z|^2)^{\frac{n}{2}}}{(1-\langle z, \xi\rangle)^n}.$$

Then for each $z \in B$, $\bar{P}_z \in A(S)$ and $P_z(\xi)\bar{P}_z(\xi) = P(z, \xi)$. By Proposition 1.4.10 of [5],

$$\|P_z\|_p^p \le C(1-|z|^2)^{(1-p/2)n}.$$

Thus Hölder's inequality yields

$$\begin{aligned} \Big| \int_S K_\varphi P_z \bar{P}_z \, d\sigma \Big| &\le \|K_\varphi P_z\|_p \, \|P_z\|_q \\ &\le \|K_\varphi\| \, \|P_z\|_p \, \|P_z\|_q \\ &\le C\|K_\varphi\|, \end{aligned}$$

where C is an absolute constant. Note that from the theorem of Koranyi and Vagi [3] (Theorem 6.3.1 of [5]) we have

$$\int_S \Big| \int_S K(r\xi, \zeta) g(\zeta) \, d\sigma(\zeta) \Big|^q \, d\sigma(\xi) \le C_q \|g\|_q^q$$

for every $g \in L^q(S)$. Write $z = t\eta$ for $\eta \in S$ and for $0 < t < 1$. Thus there is a constant c_q such that

$$\begin{aligned} &\int_S \Big| \bar{P}_{t\eta}(\xi) \int_S K(r\xi, \zeta) P_{r\eta}(\zeta) \varphi(\zeta) \, d\sigma(\zeta) \Big|^q \, d\sigma(\xi) \\ &\qquad \le \Big(\frac{1+t}{1-t}\Big)^{nq/2} \int_S \Big| \int_S K(r\xi, \zeta) P_{r\eta}(\zeta) \varphi(\zeta) \, d\sigma(\zeta) \Big|^q \, d\sigma(\xi) \\ &\qquad \le c_q \Big(\frac{1+t}{1-t}\Big)^{nq/2} \|P_{t\eta}\varphi\|_q^q \\ &\qquad \le c_q \Big(\frac{1+t}{1-t}\Big)^{nq} \|\varphi\|_q^q. \end{aligned}$$

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Since the last term of the above inequalities is independent of r, the integrand of the first term of the above is uniformly integrable. By applying Vitali's theorem and Fubini's theorem

$$\begin{aligned}\int_S (K_\varphi P_{t\eta})\bar{P}_{t\eta}\, d\sigma &= \int_S \lim_{r\nearrow 1}\int_S K(r\xi,\zeta)P_{t\eta}(\zeta)\varphi(\zeta)\, d\sigma(\zeta)\bar{P}_{t\eta}(\xi)\, d\sigma(\xi)\\ &= \lim_{r\nearrow 1}\int_S\int_S K(r\xi,\zeta)P_{t\eta}(\zeta)\varphi(\zeta)\, d\sigma(\zeta)\bar{P}_{t\eta}(\xi)\, d\sigma(\xi)\\ &= \lim_{r\nearrow 1}\int_S P_{t\eta}(\zeta)\varphi(\zeta)\int_S K(r\xi,\zeta)\bar{P}_{t\eta}(\xi)\, d\sigma(\xi)\, d\sigma(\zeta)\\ &= \int_S P_{t\eta}(\zeta)\varphi(\zeta)\big(i(1-t^2)^{n/2} - i\bar{P}_{t\eta}(\zeta)\big)\, d\sigma(\zeta).\end{aligned}$$

Thus

$$\Big|\int_S (K_\varphi P_{t\eta})\bar{P}_{t\eta}\, d\sigma\Big| = \Big|\int_S P_{t\eta}(\zeta)\varphi(\zeta)\big(-(1-t^2)^{n/2} + \bar{P}_{t\eta}(\zeta)\big)\, d\sigma(\zeta)\Big| \le C\|K_\varphi\|.$$

Since $C\|K_\varphi\|$ is a constant independent of t, taking $t \nearrow 1$, by the reproducing property of the invariant Poisson integral, we have $|\varphi(\eta)| \le C\|K_\varphi\|$ at almost all η. Therefore φ is bounded and this proves (a). Now we will prove (b). Pick $f \in L^2(S)$. Choose a sequence of polynomial $\{g_k\}$ such that $\|g_k - \bar{f}\|_2$ converges to zero. Then

$$\Big|\int_S P_z\bar{f}\, d\sigma - \int_S P_z g_k\, d\sigma\Big| \le C\|\bar{f} - g_k\|_2$$

converges to zero uniformly on z. Thus

$$\begin{aligned}\lim_{|z|\to 1}\int_S P_z\bar{f}\, d\sigma &= \lim_{|z|\to 1}\lim_{k\to\infty}\int_S P_z g_k\, d\sigma\\ &= \lim_{k\to\infty}\lim_{|z|\to 1}\int_S P(z,\xi)g_k(\xi)\frac{(1-\langle \xi, z\rangle)^n}{(1-|z|^2)^{\frac{n}{2}}}\, d\sigma(\xi)\\ &= \lim_{k\to\infty}\lim_{|z|\to 1}(1-|z|^2)^{\frac{n}{2}} g_k(z) = 0,\end{aligned}$$

which means P_z converges to zero weakly as $|z| \to 1$. From (a)

$$\Big|\int_S (K_\varphi P_{t\eta})\bar{P}_{t\eta}\, d\sigma\Big| = \Big|\int_S P_{t\eta}(\zeta)\varphi(\zeta)\big(-(1-t^2)^{n/2} + \bar{P}_{t\eta}(\zeta)\big)\, d\sigma(\zeta)\Big|.$$

Since K_φ is compact, the left hand side converges to zero as $t \to 1$. And the righthand side converges to $\varphi(\eta)$. This completes the proof. ■

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