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FRAME-LESS HILBERT C*-MODULES

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Abstract. We show that if A is a compact C*-algebra without identity that has a faithful $*$-representation in the C*-algebra of all compact operators on a separable Hilbert space and its multiplier algebra admits a minimal central projection p such that pA is infinite-dimensional, then there exists a Hilbert A_1-module admitting no frames, where A_1 is the unitization of A. In particular, there exists a frame-less Hilbert C*-module over the C*-algebra $K(\ell^2) + \mathbb{C}I_{\ell^2}$.

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1. Introduction. The classical frame theory for Hilbert spaces has been generalized to the setting of Hilbert C*-modules by M. Frank and D. R. Larson [**8**]. For A being a C*-algebra and being a Hilbert C*-module a set $\{x_i\}_{i\in I}$ of elements of X, where I is an a priori arbitrary index set, is said to be a standard frame for X if the inequality

$$C \cdot \langle x, x\rangle \leq \sum_{i\in I} \langle x, x_i\rangle\langle x_i, x\rangle \leq D \cdot \langle x, x\rangle \tag{1}$$

holds for any $x \in X$ and two fixed positive numbers C, D, where the sum in the middle is supposed to converge w.r.t. the C*-norm of A taking the supremum over the respective sums over all finite subsets of I. They concluded from Kasparov's stabilization theorem that every finitely and every countably generated Hilbert C*-module over a unital C*-algebra has a standard frame. They asked in [**8**, Problem 8.1], for which C*-algebra A, every Hilbert A-module X has a frame? In 2002, D. Bakić and B. Guljaš proved in [**4**] a first affirmative answer: For A being a compact C*-algebra (i.e. admitting a faithful $*$-representation in a C*-algebra of all compact operators $K(H)$ on some Hilbert space H), then every Hilbert A-module X admits a special standard frame $\{x_i\}_{i\in I}$ such that (i) $\langle x_i, x_i\rangle = p_i = p_i^2$ for atomic projections $p_i \in A$, (ii) $\langle x_i, x_j\rangle = 0$ for any $i \neq j$. They called such frames orthonormal bases. Lj. Arambašić proved in 2008 that every full (countably generated) Hilbert A-module X possesses an orthonormal basis if and only if A is $*$-isomorphic to a C*-algebra of compact operators [**1**, Corollaries 6 and 7]. In 2010, Hanfeng Li solved this problem in the commutative unital case to the negative characterizing the unital commutative C*-algebras A such that every Hilbert A-module admits a frame as the finite-dimensional ones [**9**].

The last two results together give the following fact:

COROLLARY 1.1 (cf. [**2**, Theorem 1.4]). *Let I be an infinite set with discrete topology. Then the C*-algebra $A = c_0(I)$ of all converging to zero sequences indexed by I is a compact C*-algebra, and so every Hilbert A-module X admits a standard frame. However, for the unitization $B = A \dotplus \mathbb{C}1_B$, there exists a Hilbert B-module admitting no standard frame.*

M. Amini, M. B. Asadi, G. A. Elliott and F. Khosravi showed in [**2**, Corollary 2.6] in 2017, that every infinite-dimensional nuclear von Neumann algebra A posesses a Hilbert A-module with no standard frame. Moreover, if two C*-algebras A and B are Morita equivalent and A is σ-unital, then the property of A that every Hilbert A-module admits a standard frame inherits to B, cf. [**2**, Theorem 2.4]. Note that the set of compact C*-algebras is closed under Morita equivalence.

In general case, the conjecture is as follows:

CONJECTURE 1.2 (cf. [**2**, Question 1.5]). Every Hilbert C*-module over a C*-algebra A admits a frame if and only if A is a compact C*-algebra.

In the commutative case, Hanfeng Li applied the Serre–Swan theorem. This theorem states that there is a one-to-one correspondence between finitely generated projective modules over a unital commutative C*-algebra C(Ω) and complex vector bundles over Ω [**10**].

In [**7**], G. A. Elliott and K. Kawamura showed that the vector space of bounded uniformly continuous holomorphic sections of every uniform holomorphic Hilbert bundle of dual Hopf type over pure states of a C*-algebra A admits a unique structure of a right Hilbert A-module.

In this paper, we study Hilbert C*-modules over a C*-algebra $A = K(H) \dotplus \mathbb{C}I_H$, where $K(H)$ is the C*-algebra of compact operators on a separable infinite-dimensional Hilbert space H, and we give a partial affirmative response to the above conjecture. Indeed, we have applied the Elliott–Kawamura approach and concluded the following result:

THEOREM 1.3. *If $A = K(\ell^2) \dotplus \mathbb{C}I_{\ell^2}$, then there exists a Hilbert A-module that possess no frames.*

2. Holomorphic Hilbert bundle. Let A be a C*-algebra, $\hat{A}$ the spectrum of A and $P(A)$ be the set of pure states of A. In general, $P(A)$ is not compact, in this case, we consider $P_0(A) = P(A) \cup \{0\}$. However, we set $P_0(A) = P(A)$, when $P(A)$ is compact.

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We use the notations $\pi = [f]$ and $f = (\pi, e)$, whenever $\pi : A \longrightarrow B(H_\pi)$ is a member of $\hat{A}$ and $e = h \otimes h$ for some unit vector $h \in H_\pi$ and f is the pure state $f(\cdot) = \langle \pi(\cdot)h, h\rangle$.

In this case, the unitary equivalence class of f (as a set) is equal to

$$R_1(H_\pi) := \{e \in B(H_\pi) : \text{e is a rank one projection}\}.$$

The set $R_1(H_\pi)$ has a natural holomorphic manifold structure that is independent of the chosen representative element in each equivalence class in $P(A)$ [**7**]. Therefore, we can identify $P(A)$ as the disjoint union of projective spaces, i.e.

$$P(A) = \bigcup_{\pi \in \hat{A}} \{\pi\} \times R_1(H_\pi).$$

Then, $P_0(A)$ has a natural holomorphic manifold structure and it has a natural uniform structure determined by the semi-norms arising from evaluation at the elements of A.

In [**7**], G. A. Elliott and K. Kawamura introduced the notion of (locally trivial) holomorphic Hilbert bundle over pure states of a C*-algebra. They also introduced the notion of (not necessarily locally trivial) uniform holomorphic Hilbert bundle of dual Hopf type as a direct sum of holomorphic Hilbert bundles which are dual Hopf bundles (cf. [**7**, p. 4850]). In fact, we set

$$\mathcal{H} = \{B(H_\pi, K_\pi)e\}_{(\pi \in \hat{A} \cup \{0\}, e \in R_1(H_\pi))},$$

where K_π is a Hilbert space, for all $\pi \in \hat{A}$. If $X(\mathcal{H})$, the vector space of bounded uniformly continuous holomorphic sections of $\mathcal{H}$, exhausting fibres, then the pair $(\mathcal{H}, X(\mathcal{H}))$ is a uniformly continuous holomorphic Hilbert bundle of dual Hopf type. In this case, for any $S \in X(\mathcal{H})$ and any $\pi \in \hat{A}$, there exists an operator $S_\pi \in B(H_\pi, K_\pi)$ such that

$$S((\pi, e)) = S_\pi e \qquad (e \in R_1(H_\pi)).$$

As shown in [**7**], $X(\mathcal{H})$ is a Hilbert A-module. In fact, for any $S, T \in X(\mathcal{H})$, the A-valued inner product is defined by S^*T, where

$$S^*(\pi, e) = eS_\pi^* \in eB(K_\pi, H_\pi), \qquad \text{for all } (\pi, e) \in P_0(A).$$

Since $(S_\pi^* T_\pi)_{\pi \in \hat{A}} \in \prod_{\pi \in \hat{A}}(B(H_\pi))$ is uniformly continuous, we can consider S^*T belongs to A, by a result by L. G. Brown [**5**].

3. Frame existence problem. THEOREM 3.1. *Suppose that A is a C*-algebra, $f_0 \in P(A)$, $\pi_0 = [f_0]$, H_{π_0} is a separable Hilbert space and W is a countable subset of $P(A)$ such that $f_0 \in \overline{W} \setminus W$. If there exists a uniform holomorphic Hilbert bundle of dual Hopf type $\mathcal{H} = (B(H_\pi, K_\pi)e_\pi)_{(\pi, e_\pi) \in P_0(A)}$ such that for any $\pi \in [W]$, K_π is separable and K_{π_0} is non-separable, then the Hilbert A-module $X(\mathcal{H})$ possess no frames.*

Proof. Assume that $\{S_j\}_{j \in J}$ is a frame for $X(\mathcal{H})$. Hence, there exist positive numbers C, D such that for any section $S \in X(\mathcal{H})$, the following inequality holds

$$CS^*S \leq \sum_{j \in J} S^* S_j S_j^* S \leq DS^*S.$$

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Hence, for every $\pi \in \hat{A}$, $e_\pi \in R_1(H_\pi)$ and $S \in X(\mathcal{H})$, we have

$$CS^*S((\pi, e_\pi)) \leq \sum_{j\in J} S^*S_jS_j^*S(\pi, e_\pi)) \leq DS^*S((\pi, e_\pi)),$$

so

$$Ce_\pi S_\pi^* S_\pi e_\pi \leq \sum_{j\in J} e_\pi S_\pi^* S_{j\pi} e_\pi S_{j\pi}^* S_\pi e_\pi \leq De_\pi S_\pi^* S_\pi e_\pi .$$

In particular, for any non-zero element $x_\pi \in H_\pi$, we have

$$C\|S_\pi(x_\pi)\|^2 \leq \sum_{j\in J} | \langle S_\pi(x_\pi), S_{j\pi}(x_\pi)\rangle |^2 \leq D\|S_\pi(x_\pi)\|^2.$$

Since bounded holomorphic sections exhaust fibres, so for any $y_\pi \in K_\pi$, there exists a section $S \in X(\mathcal{H})$ such that $S_\pi(x_\pi) = y_\pi$. Thus,

$$C\|y_\pi\|^2 \leq \sum_{j\in J} | \langle y_\pi, S_{j\pi}(x_\pi) |^2 \leq D\|y_\pi\|^2. \tag{2}$$

According to Inequality 2, for all $\pi \in \hat{A}$, $0 \neq x_\pi \in H_\pi$ and $0 \neq y_\pi \in K_\pi$, the following set has to be countable:

$$F_{x_\pi, y_\pi} := \{j \in J : \langle y_\pi, S_{j\pi}(x_\pi)\rangle \neq 0\}.$$

In particular, if $\pi \in [W]$, then K_π is separable and so it has a countable orthonormal basis as E_π. Hence, for each $\pi \in [W]$, the following set has to be countable

$$F_{\pi, x_\pi} := \{j \in J : S_{j\pi}(x_\pi) \neq 0\} = \bigcup_{y_\pi \in E_\pi} \{j \in J : \langle y_\pi, S_{j\pi}(x_\pi)\rangle \neq 0\}.$$

Consequently, if we write $W = \{(\pi_n, e_n) : n \in \mathbb{N}\}$, then $F = \bigcup_{n\in\mathbb{N}} F_{\pi_n, x_n}$ is a countable set, where for any $n \in \mathbb{N}$, $x_n \in H_{\pi_n}$ and $e_n = x_n \otimes x_n$. Also, we use the notation $f_0 = (\pi_0, e_0)$, where $e_0 = x_0 \otimes x_0$ for some unit vector $x_0 \in H_{\pi_0}$.

For each $j \in F$, $\mathrm{Im}(S_{j\pi_0})$ is a separable space, since H_{π_0} is separable. Then, $K_0 = \langle \bigcup_{j\in F} \mathrm{Im}(S_{j\pi_0})\rangle$ is a separable subspace of the non-separable Hilbert space K_{π_0}; hence, there exists a unit element $y_{\pi_0} \in K_{\pi_0}$ that is orthogonal to K_0. Then for any $j \in F$, $S_{j\pi_0}^*(y_{\pi_0}) = 0$.

On the other hand, for any $j \in J \setminus F$, we have $S_{j\pi_0}(x_0) = 0$, since $(\pi_0, e_0) \in \overline{W}$ and S_j is continuous. Thus, for any $j \in J$, we have $\langle y_{\pi_0}, S_{j\pi_0}(x_0)\rangle = 0$. By (2), y_{π_0} is equal to zero, that is a contradiction. Therefore, the Hilbert A-module $X(\mathcal{H})$ admits no frames. □

4. $K(\ell^2) \dot{+} \mathbb{C}I_{\ell^2}$. In the following, we consider $A = K(H) \dot{+} \mathbb{C}I_H$, where H is a separable infinite-dimensional Hilbert space. Also, let $\{h_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for H and $e_n = h_n \otimes h_n$, for all $n \in \mathbb{N}$.

We recall that $\hat{A} = \{\pi_0, \pi_1\}$, where $\pi_1 = id$ and $\pi_0(T + \lambda I_H) = \lambda$, for every $T \in K(H)$ and $\lambda \in \mathbb{C}$. Thus, we can consider

$$P(A) = (\{\pi_1\} \times R_1(H)) \cup \{(\pi_0, 1)\}.$$

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Note that in this case, $P(A)$ is a compact Hausdorff space and also $(\pi_0, 1) \in \overline{W} \setminus W$, where $W = \{(\pi_1, e_n) : n \in \mathbb{N}\}$.

THEOREM 4.1. *There exists a uniform holomorphic vector bundle of dual Hopf type over $P(A)$ satisfying the conditions of Theorem 3.1.*

Proof. Hanfeng Li showed in [**9**, Lemma 2.1], that there exists an uncountable set $\mathcal{F}$ of injective maps from $\mathbb{N}$ to $\mathbb{N}$ such that for any distinct $f, g \in \mathcal{F}$, $f(n) \neq g(n)$ for all but finitely many $n \in N$, and $f(n) \neq g(m)$ for all $n \neq m$.

Let $K_{\pi_1} = \ell^2$ with the standard basis $\{z_n\}_{n\in\mathbb{N}}$ and K_{π_0} be a non-separable Hilbert space with an orthonormal basis $\{h_f\}_{f\in\mathcal{F}}$ indexed by $\mathcal{F}$. For each $f \in \mathcal{F}$, consider the isometry $u_f : H \longrightarrow \ell^2$, given by $u_f(h_n) = z_{f(n)}$ for all $n \in \mathbb{N}$. Also, we consider $v_f : \mathbb{C} \longrightarrow K_{\pi_0}$ by $v_f(\lambda) = \lambda h_f$.

Now, we can define $S_f : P(A) \longrightarrow (\bigcup_{e\in R_1(H)} B(H, \ell^2)e) \cup (B(\mathbb{C}, K_{\pi_0})1)$ by

$$S_f((\pi, e)) = \{\begin{matrix} u_f e & \pi = \pi_1 \\ v_f 1 & \pi = \pi_0 \end{matrix}$$

Set $V = \{\sum_{i=1}^{n} \lambda_i S_{f_i} : n \in \mathbb{N}, \lambda_i \in \mathbb{C}, f_i \in \mathcal{F}\}$. We claim that the function $(\pi, e) \mapsto \|S(\pi, e)\|$ is continuous on $P(A)$ for every $S \in V$.

For this, we note that if $S = \sum_{i=1}^{m} \lambda_i S_{f_i} \in V$, then there is a finite subset J of $\mathbb{N}$ such that $f_i(n) \neq f_j(n)$, for all $n \in J^c$ and $i \neq j$. Hence, if $e = x \otimes x$, for some unit vector $x \in H$, then we have

$$\begin{aligned} \|S(\pi_1, e)\|^2 &= \|\sum_{i=1}^{m} \lambda_i u_{f_i}(x)\|^2 = \|\sum_{i=1}^{m} \lambda_i \left(\sum_{n=1}^{\infty} \langle x, h_n\rangle z_{f_i(n)}\right)\|^2 \\ &= \|\sum_{i=1}^{m}\sum_{n\in J} \lambda_i \langle x, h_n\rangle z_{f_i(n)}\|^2 + \sum_{i=1}^{m} \|\sum_{n\in J^c} \lambda_i \langle x, h_n\rangle z_{f_i(n)}\|^2 \\ &= \|\sum_{i=1}^{m}\sum_{n\in J} \lambda_i \langle x, h_n\rangle z_{f_i(n)}\|^2 + \sum_{i=1}^{m} |\lambda_i|^2 \left(1 - \|\sum_{n\in J} \langle x, h_n\rangle z_{f_i(n)}\|^2\right). \end{aligned}$$

Now, if a net $\{(\pi_1, e_\alpha)\}_{\alpha\in I}$ is convergent to (π_1, e) (or $(\pi_0, 1)$) and for every $\alpha \in I$, $e_\alpha = x_\alpha \otimes x_\alpha$ for some unit vector $x_\alpha \in H$, then $|\langle x_\alpha, y\rangle| \to |\langle x, y\rangle|$ (or $|\langle x_\alpha, y\rangle| \to 0$), for all $y \in H$. Consequently, for every $f \in S$ and $y_1, \cdots, y_N \in H$, we have

$$\|\sum_{n=1}^{N} \langle x_\alpha, y_n\rangle z_{f(n)}\| (= \left(\sum_{n=1}^{N} |\langle x_\alpha, y_n\rangle|^2\right)^{\frac{1}{2}}) \to \|\sum_{n=1}^{N} \langle x, y_n\rangle z_{f(n)}\|$$

$$\left(\text{or } \|\sum_{n=1}^{N} \langle x_\alpha, y_n\rangle z_{f(n)}\| \to 0\right).$$

Thus, $\|S(\pi_1, e_\alpha)\| \to \|S(\pi_1, e)\|$ (or $\|S(\pi_1, e_\alpha)\| \to \|S(\pi_0, 1)\|$). This proves the claim.

Therefore, V is a linear space of bounded holomorphic sections with uniformly continuous norm and it exhausts each fibre. Now, as mentioned in [**7**], by Zorn's lemma, we can extend it to a linear space $X(\mathcal{H})$ of the bounded holomorphic sections with

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uniformly continuous norm, maximal with this property, and exhausting each fibre. Clearly, $X(\mathcal{H})$ satisfies the conditions of Theorem 3.1. □

The following results can be obtained from Theorems 3.1 and 4.1.

COROLLARY 4.2. *The C*-algebra* $K(\ell^2) \dotplus \mathbb{C}I_{\ell^2}$ *has a frame-less Hilbert module.*

COROLLARY 4.3. *Let A be a compact C*-algebra without identity that has a faithful* $*$*-representation in the C*-algebra of all compact operators on a separable Hilbert space. Suppose, the multiplier algebra of A has a minimal central projection p such that pA is infinite-dimensional. Denote the C*-algebra* $A \dotplus \mathbb{C}1_A$ *by* A_1*, i.e. the unitization of A. Then, for* A_1*, there exists a Hilbert* A_1*-module admitting no frames.*

Proof. Any compact C*-algebra A has the form $A = c_0 - \sum_\alpha \oplus K(H_\alpha)$, where the symbol $K(H_\alpha)$ denotes the C*-algebra of all compact operators on some Hilbert space H_α, and the c_0-sum is either a finite block-diagonal sum or a block-diagonal sum with a c_0-convergence condition on the C*-algebra components $K(H_\alpha)$. The c_0-sum may possess arbitrary cardinality. This kind of C*-algebras has been precisely characterized by W. Arveson in [**3**, Section I.4, Theorem I.4.5]. The sort of compact C*-algebras A in the supposition forces all Hilbert spaces H_α to be separable or finite-dimensional, and at least one of the Hilbert spaces H_α has to be infinite-dimensional, say H_β.

Suppose, the minimal central projection $p \subset Z(M(A))$ maps A to one of its infinite block-diagonal direct summands $K(H_\beta)$, i.e. $pA = K(H_\beta)$. The same projection p applied to the C*-algebra A_1 yields $pA_1 = pA \dotplus \mathbb{C}p1_{pA_1}$. By the above corollary, there exists a Hilbert pA_1-module X that admits no frames. Since p is central, X is a Hilbert A_1-module, too. The property of X not to admit any frame does not change. □

REMARK 4.4. The complementary case of compact C*-algebras to that one treated in the corollary is the one of non-unitary compact C*-algebras for which any of the infinitely many direct summands are finite-dimensional (but, may be, of arbitrary large dimension). It remains open. In the same manner, the analogous assertion can be proved for more general compact C*-algebras A provided Theorem 4.1 can be reproved for non-separable Hilbert spaces H.

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